Università di Pisa

Dipartimento di Matematica

Corso di Laurea Magistrale in Matematica

Tesi di Laurea

Q.I. RIGIDITY FOR LATTICES OF HYPERBOLIC ISOMETRIES

Relatore

Prof. **Roberto Frigerio**

Controrelatore

Prof. **Bruno Martelli**

Candidato

Marco Barberis

Anno accademico 2016/2017

C'era una volta un fama che ballava tregua e ballava provala davanti alla vetrina di un negozio pieno di cronopios e di speranze. Le più irritate erano le speranze sempre pronte a far di tutto perché i famas non ballino tregua e provala, ma spera, che è il ballo più in voga presso i cronopios e le speranze.

[...]

-Fama,- dissero le speranze, -non ballare tregua, e neppure provala davanti a questo negozio.

Ma il fama ballava e rideva, e così mortificava le speranze.

Allora le speranze si lanciarono sul fama e lo malmenarono. Lo lasciarono a terra vicino a uno steccato, e il fama mandava lamenti, immerso nel suo sangue e nella sua tristezza.

I cronopios si avvicinarono furtivi, questi oggetti verdi e umidi. Attorniarono il fama e si misero a compatirlo, dicendogli:

-Cronopio cronopio cronopio.

E il fama capiva, e la sua solitudine era meno amara.

Da *Usanze dei Famas*,
in *Storie di Cronopios e di Famas* di Julio Cortazar.

Contents

Introduction

The main purpose of this dissertation is to introduce the quasi isometric rigidity property for classes of groups, and then prove that it holds for lattices in the Lie groups of hyperbolic isometries.

This work is part of the area of geometry known as geometric group theory. The underlying idea under geometric group theory, name which has been introduced by Gromov in the '80s although the matter was already on the radar of researchers, is to study groups with geometric techniques. This is in some sense the converse of a common situation in geometry and topology, where the goal is to define groups conveying properties of spaces (*e.g.* homotopy and homology groups, mapping class groups) and use our knowledge about group theory to study geometrical objects. In geometric group theory this situation in reversed: first a natural metric on finitely generated groups, arising just from their algebraic structure, is defined, and then groups are seen as metric spaces and studied with geometric techniques. Since the metric only depends on the algebraic structure it is preserved under some algebraic equivalences (which include but do not restrict to isomorphism, since some information is lost during the process): the ultimate goal of our theory is to understand what pieces of information about algebraic properties of groups can be inferred just from the knowledge of their geometrical structure.

Metrics on groups are not defined up to isometric equivalence, but only up to a coarser equivalence relation, quasi isometry. Thus the weakest algebraic relation that implies quasi isometry is not the isomorphism but the so called virtual isomorphism: this is due to the fact that in forgetting the algebraic structure to focus on the metrical viewpoint much information is lost. Anyway it would be a surprising result if the quasi isometric information about a group were enough to determine its virtual isomorphism class uniquely. This is in fact not generally the case, and that is the reason why the concept of quasi isometric rigidity is defined. A class of groups is said to be quasi isometrically rigid if a group quasi isometric to a member of the class is virtually isomorphic to a possibly different member of the class. The ultimate goal in this field is to find the smallest quasi isometrically rigid classes, since they convey the maximal amount of algebraic information that can be deduced from the geometric structure.

It has been proven by Pekka Tukia in 1986 that the class of cocompact hyperbolic lattices (in dimension at least 3) is quasi isometrically rigid, and this is the best possible result, since those groups are all quasi isometric to each other. Regarding noncocompact lattices, Richard Schwartz has proven in 1996 that (in dimension at least 3 again) each of those lattices is quasi isometrically rigid by itself. Throughout this thesis we will prove both results, whose proofs will turn out, in a slightly surprising manner, to be very different in nature.

In Chapter 1 we will introduce the most common definitions and some basic results of geometric group theory, such as Milnor-Svarc Theorem, and give a more formal statement of the quasi rigidity property we have just discussed. It will be moreover presented an example of groups which are quasi isometric to each other but not virtually isomorphic. More technical material is contained in the three appendices, which present definitions and statements we will refer to throughout the course of the thesis.

Chapter 2 is devoted to the proof of Tukia's Theorem about uniform lattices. The underlying idea of the proof is that a cocompact lattice, which is quasi isometric to the whole hyperbolic space, can be regarded as a group of homeomorphisms of the ideal boundary of the space. This leads to the definition and the study of quasiconformal mappings, which will be a tool heavily used throughout the proofs of both rigidity theorems; the first section of the chapter is devoted to the study of the properties of these maps. In the second section of the chapter some other technical tools, such as measurable conformal structures, are developed, and in the end the proof of Tukia's Theorem is given. The chapter ends with some remarks about the two-dimensional case.

Chapter 3 contains the proof of Schwartz Rigidity Theorem. The study of noncocompact hyperbolic lattices will be based on the thick-thin decomposition, which lets us to view them as cocompact lattices of isometries of a so called truncated hyperbolic space. In the first section of the chapter the behaviour of quasi isometries between truncated hyperbolic spaces is studied, as it will be the main tool to prove the rigidity theorem. In the second section the proof of Schwartz Rigidity Theorem is given, along with a proof of the well-known Mostow Rigidity Theorem, which follows in a rather straightforward manner from the work we will have already done, and represents an impressive result by itself.

Chapter 1

Basic facts about geometric group theory

1.1 Metrics on groups

1.1.1 Definition of the metrics

Geometric group theory, a branch of geometry whose name is due to Gromov, has the goal of studying groups via geometric techniques. The study of groups arising naturally from topological or geometric objects has been one of the most influential ideas in geometry and has led to incredible results. Geometric group theory tries to reverse this paradigm by endowing groups with a natural metrical structure arising just from their algebraic properties, but studying this new way to deal with algebraic information with geometric tools. The first thing we want to do is endow a group, or a graph associated to it (we will later see that there will be no difference) with a metric structure. For technical reasons that will be clear in few pages we will just deal with finitely generated group, and each group in this thesis will be assumed to be finitely generated, unless otherwise stated. This leads us to our first definition.

Definition 1.1 (Cayley graph)**.** Given a finitely generated group G with generators $S = \{g_1, \dots, g_s\}$ we define the *Cayley graph* associated to G as the graph Γ_G with vertices

$$V(\Gamma_G) = G$$

given by elements of the group, and unoriented edges

$$E(\Gamma_G) = \{[g, gs] \text{ with } g \in G \text{ and } s \in S\}\,.$$

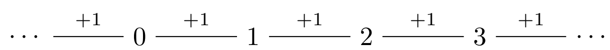
We can endow a Cayley graph with a metric assigning length 1 to all the edges and using the minimal path-length for the distance between any two points in the graph.

The metric on a Cayley graph can be restricted to the set of vertices, which can be identified with the group G, giving rise to the so call *word-metric*.

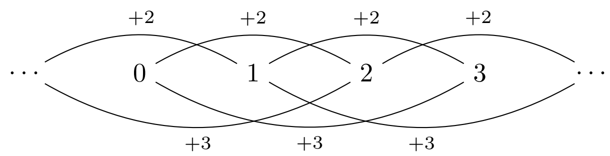
Definition 1.2 (Word-metric)**.** Let G be a finitely generated group with generating set $\{g_1, \dots, g_s\}$. The *word-metric* on G arising from S is the metric given, for any $g, g' \in G$, by:

$$d_S(g, g') = \min \left\{ k \in \mathbb{N} \text{ s.t. it exists } h = \prod_{i=1}^{k} g_i^{\pm 1} \text{ with } gh = g' \right\}.$$

It is immediately clear from the definition that such a metric depends on the generating set. Indeed it is trivial to check that in both definition the only points at unit distance from the identity $e \in G$ are the generators and their inverses: this shows immediately that, changing the generating set, we will get, in general, non isometric spaces. In the case of Cayley graphs this behaviour is very clear, as we can even get non homeomorphic graphs: for instance this is the Cayley graph for $\mathbb{Z}$ associated to the generating set $S = \{1\}$



while this is the one associated to the generator $S = \{2, 3\}$



and the latter is far from being even simply connected.

1.1.2 Quasi isometries

Since the two metrics we have defined in the previous subsection arise very naturally from the algebraic structure of the groups we are inclined to think that they have to be the right choice for the study we want to pull off, even if they do not seem to be well defined. To ensure the good definition of the metric the whole point will be to consider our metric spaces no longer up to isometry but up to a substantially weaker equivalence relation, quasi isometry.

Definition 1.3 (Quasi isometry)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \longrightarrow Y$ is said to be *coarse Lipschitz* is there exist positive real constants (L, A) such that, for all $x, x' \in X$, it holds

$$d_Y(f(x), f(x')) \leq L d_X(x, x') + A.$$

Moreover, if the inequality

$$\frac{1}{L} d_X(x, x') - A \leq d_Y(f(x), f(x'))$$

holds, the map f is said to be a *quasi isometric embedding.* An (L, A)-quasi isometric embedding $f : X \longrightarrow Y$ is said to be a *quasi isometry* if there exists another (L, A)-quasi isometric embedding $f' : Y \longrightarrow X$, called *quasi inverse*, such that

$$\mathrm{dist}(f' \circ f, \mathrm{Id}_X) < +\infty \text{ and } \mathrm{dist}(f \circ f', \mathrm{Id}_Y) < +\infty.$$

Quasi isometries, as it appears from the definition, are in general very far from being continuous, neither they have to be bijective at all. It is trivial to check, for instance, that all finite-diameter metric spaces are indeed quasi isometric, and cannot be quasi isometric to a non finite-diameter one. Moreover this easily show that two quasi isometric spaces are very far from being homeomorphic, and it is also true that it is possible to endow the same topological space with two non quasi isometric distance functions (*e.g.* Euclidean and hyperbolic spaces): topology, indeed, will play a little role in our arguments.

The following proposition is almost trivial.

Proposition 1.4. *Let G be a finitely generated group. Then any two word-metrics on it are quasi isometric.*

Proof. Let S, S' be two generating sets corresponding to two different word-metrics. Than the identity map $\mathrm{Id}_G : (G, d_S) \longrightarrow (G, d_{S'})$ is L-Lipschitz where

$$L = \max_{g \in S} \{d_{S'}(g, e)\}$$

which is finite since the generating set is finite. The quasi inverse of this map is clearly the identity itself. □

The previous proposition in very easy but does not hold any longer if we remove the assumption on G to be finitely generated: this is the main reason why geometric group theory only deals with such groups.

Another very easy fact is that the inclusion $G \subseteq \Gamma_G$ of a group into its Cayley graph is indeed a quasi isometry: from this coarse viewpoint, we will no longer need to distinguish groups or they Cayley graphs. Both approaches have their pros and cons: dealing with word-metrics on groups is usually simpler, as they are countable sets, but Cayley graphs carry a stronger geometric structure, as they are connected (and indeed geodesic) spaces.

Given a metric space X it appears natural to try to consider the set of quasi isometries of the space on itself. Differently from the usual case of isometries, this set is not naturally endowed with a group structure, but this can be achieved with little work, once again using, as in the definition of quasi inverse, the idea that maps lying at finite distance from each other can be identified.

Definition 1.5. Let (X, D) be a metric space. The set

$$QI(X) \stackrel{def}{=} \{f : X \longrightarrow X \text{ q.i.}\}/_{\sim}$$

under the equivalence relation given by

$$f \sim g \text{ if } \operatorname{dist}(f, g) < +\infty$$

and endowed with the composition is a group, called *quasi isometry group*.

We naturally have a map $\operatorname{Isom}(X) \longrightarrow QI(X)$ which, in general, is neither injective nor surjective. A counterexample to injectivity is given by any finite-diameter set with nontrivial isometries, such S^1. A counterexample to surjectivity is given by hyperbolic spaces $\mathbb{H}^{n+1}$, where isometries of the hyperbolic space are in bijection to Moebius transformations of the n-sphere at infinity, while quasi isometries induce a strictly wider set of maps, called quasi conformal, as we will see in detail in Subsection 2.1.3. In special cases, anyway, even bijectivity can hold: an example of that will be Theorem 3.23.

To conclude this subsection we will give a second definition of quasi isometry between spaces that give us the opportunity to define Hausdorff distance and nets, which we will need in the sequel.

Definition 1.6 (Hausdorff distance)**.** Let X be a metric space. Let us denote with $\mathcal{N}_R(A)$ the metric R-neighbourhood of a subset $A \subseteq X$. The *Hausdorff (psuedo)-distance* between two subsets $A, B \subseteq X$ is defined as

$$\operatorname{dist}_{\text{Haus}}(A, B) \stackrel{def}{=} \inf \{R \in \mathbb{R}_+ \text{ s.t. } A \subseteq \mathcal{N}_R(B) \text{ and } B \subseteq \mathcal{N}_R(A)\}.$$

The quantity we have just defined is not a distance on $\mathcal{P}(X)$ since two different sets can be at zero distance from each other; anyway it becomes a metric on the set on compact sets $K(X)$, giving rise to a topology called *Hausdorff-convergence* topology. It is possible to extend this topology to the set of closed subsets $C(X)$ as follows, obtaining the so called *Chabauty topology*, which will be useful several times in the sequel.

Definition 1.7 (Chabauty topology)**.** Let X be a metric space. The *Chabauty topology* on the set of closed subsets $C(X)$ is the topology generated by the following neighbourhoods. Given $\varepsilon > 0$ and a compact set $K \subseteq X$ we can define the neighbourhood $U_{\varepsilon,K}$ of a closed set $C \subseteq X$ as

$$U_{\varepsilon,K} \stackrel{def}{=} \{Z \in C(X) \text{ s.t. } \operatorname{dist}_{\text{Haus}}(Z \cap K, C \cap K) < \varepsilon\}.$$

We will now define nets.

Definition 1.8. Let (X, d) be a metric space. A subset $A \subseteq X$ is said to be *ε-separated* if for any two points $x, y \in X$ it holds $d(x, y) \geq \varepsilon$.

A set $A \subseteq X$ is said to be *R-dense* if its metric neighbourhood $\mathcal{N}_R(A)$ is the whole space X or, equivalently, if $\operatorname{dist}_{\text{Haus}}(A, X) < R$.

Definition 1.9 (Nets)**.** Let (X, d) be a metric space. An *R-net* in X is a R-dense subset $N \subseteq X$. A net is said to be *ε-separated* if it is an ε-separated subset for some ε.

We can now state quasi isometry in terms of nets and bilipschitz maps.

Definition 1.10. Two metric spaces (X, d_X) and (Y, d_y) are said to be *quasi isometric* if there exist separated nets $A \subseteq X$ and $B \subseteq Y$ such that the metric spaces (A, d_X) and (B, d_y) are bilipschitz equivalent, *i.e.* there exists a bilipschitz bijection between them.

The equivalence of this definition and the previous one is an easy argument, although an implication requires the Axiom of Choice. For further details see [D-K, Prop. 5.19, p. 130].

1.1.3 Milnor-Svarc Lemma

A question arising spontaneously from the definition we have given is what we have gained by introducing the metric structure on a group. Since all the information contained in the metric comes from the algebra of the group, and the definition is given in terms of generators it is natural to ask if this path is loopsided, and where we can introduce geometric tools independent from the algebraic structure of the group. The following theorem, known as Milnor-Svarc Lemma, is sometimes called the fundamental observation in geometric group theory, and will give us not only a fundamental tool we will use several times, but will make clear what is the purpose of geometric group theory, and its way to study its objects.

To state the Lemma we will need a couple of definitions generalizing to metric spaces a well known object in Riemannian manifolds: geodesics.

Definition 1.11 (Geodesic)**.** Let X be a metric space. A *geodesic* in X is an isometry $\gamma : [0, T] \longrightarrow X$.

A metric space X is said to be *(uniquely) geodesic* if for each $x, y \in X$ there exists a (unique) geodesic connecting x to y.

We also remark that a metric space is said to be *proper* if every closed ball is compact. We are now able to state Milnor-Svarc Lemma.

Theorem 1.12 (Milnor-Svarc Lemma)**.** *Let (X, d) be a proper, geodesic metric space. Let G be a group acting geometrically (i.e. properly discontinuously and cocompactly) on X. Then G is finitely generated and, for each word-metric d_S on G and any point $x \in X$, the map $G \longrightarrow X$ given by $g \mapsto gx$ is a quasi isometry.*

Proof. We will not prove Milnor-Svarc Lemma, which is relatively straightforward and very well known: a proof can be found in almost any geometric group theory book, see for instance [D-K, Theorem 5.29, p. 134]. □

An easy application of Milnor-Svarc Lemma is to the case of a compact n-dimensional hyperbolic manifold M. The action of $\pi_1(M)$ on the universal cover of M, which is $\mathbb{H}^n$, is geometric and the hypotheses of the theorem are satisfied: it follows that $\pi_1(M)$ is quasi isometric to $\mathbb{H}^n$. This example is a key ingredient of the arguments we will use in the following chapters: when we have to deal with quasi isometries between lattices of hyperbolic isometries (compact hyperbolic manifold fundamental groups will be an example of this more general setting) we will deal instead with quasi isometries between hyperbolic spaces (or particular subsets of them) where we already know the metric structure, and will be able to achieve stronger results by working with a structure which is way more rich and rigid than the word metric on a group we have started with.

1.2 Quasi isometric rigidity

1.2.1 Virtual isomorphisms

It is fairly easy to slightly strengthen Proposition 1.4 obtaining that any two isomorphic groups are quasi isometric. Anyway, since our metric notion in very coarse, and looks like it encodes only a part of the algebraic structure of the group, we actually expect an equivalence relation weaker than isomorphism to imply quasi isometry. This relation is called virtual isomorphism, and will be defined in the following.

Definition 1.13 (Virtual isomorphism)**.** Let G_1, G_2 be two groups. They are said to be *virtually isomorphic* (often shortened in v.i.) if there exist finite-index subgroups $H_i < G_i$ and finite normal subgroups $K_i \lhd H_i$ such that

$$H_1/K_1 \cong H_2/K_2.$$

It is trivial that the virtual isomorphism relation is symmetric and reflexive. Before proving transitivity, and hence that it is indeed an equivalence relation, we want to try to make this really technical definition a little bit more accessible. Virtual isomorphism can equivalently be defined as the relation induced by the operation of replacing a group with a finite-index subgroup or its quotient over a finite normal subgroup and their inverses. This equivalence is clear once one proves transitivity, which is the topic of the next proposition.

Proposition 1.14. *Let G_1, G_2, G_3 be three finitely generated groups such that $G_1 \backsimeq_{v.i.} G_2$ and $G_2 \backsimeq_{v.i.} G_3$. Then $G_1 \backsimeq_{v.i.} G_3$.*

The proof of the proposition relies on the following lemma.

Lemma 1.15. *Let $K_1, K_2 \lhd G$ be finite normal subgroups of G. Then their normal closure K, i.e. the smallest normal subgroup of G containing K_1 and K_2, is still finite.*

Proof. Let us consider the projections

$$\pi_1 : G \longrightarrow G_1 \stackrel{def}{=} G/K_1$$
$$\pi_2 : G_1 \longrightarrow G_1/\pi_1(K_2).$$

Since both maps have finite kernel, so does the composition $\pi_2 \circ \pi_1$, but this kernel is indeed the normal closure required. □

Proof of Proposition 1.14. Let us denote with H_1 and K_1 the groups involved in the virtual isomorphism definition between G_1 and G_2, and with H_1' and K_1' the ones related to that between G_2 and G_3. The group $H_2'' \stackrel{def}{=} H_2 \cap H_2'$ has clearly finite index in G_2. By the above lemma the normal closure of $K_2 \cap H_2''$ and $K_2' \cap H_2''$ in H_2'', which we will denote with K_2'', is finite. Let us consider the quotient maps induced by virtual isomorphisms on H_2''

$$f_i : H_2'' \longrightarrow C_i \stackrel{def}{=} f_i(H_2'') \lhd H_i/K_i$$

for $i = 1, 3$. Those maps have finite kernel and cokernels. Thus the subgroups $E_i \stackrel{def}{=} f_i(K_2'')$, for $i = 1, 3$, are finite and normal in C_i (since the map is surjective). Let now denote with H_i' and K_i', with $i = 1, 3$, the preimages of C_i and E_i under the quotient maps $\pi_i : H_i \longrightarrow H_i/K_i$. The H_i have finite index in the G_i and the K_i are finite. Now, for $i = 1, 3$, it holds:

$$H_i'/K_i' \cong C_i/E_i \cong H_2''/K_2'',$$

giving a virtual isomorphism between G_1 and G_3. □

A relation which is slightly stronger than virtual isomorphism, but sometimes, *e.g.* in the case of nonuniform hyperbolic lattice of Chapter 3, arises more naturally is commensurability.

Definition 1.16 (Commensurability)**.** Two groups G_1, G_2 are said to be *commensurable* if there exist finite-index subgroups $H_i < G_i$ and an isomorphism, called *abstract commensurator*, $\phi : H_1 \longrightarrow H_2$.

Commensurability implying virtual isomorphism is completely trivial. For subgroups another important notion is that of commensurator in the ambient group, which we will use in the sequel.

Definition 1.17 (Commensurator)**.** Let $\Gamma < G$ be a subgroup of an ambient group G. We define the *commensurator* of Γ in G as the subgroup of G defined by

$$\mathrm{Comm}_G(\Gamma) = \left\{ g \in G \text{ s.t. } \Gamma \cap g^{-1}\Gamma g \text{ is of finite index in both } \Gamma \text{ and } g^{-1}\Gamma g \right\}.$$

1.2.2 Rigidity property

Since in both the operations which produce virtual isomorphisms finiteness properties are required, from those it follows that when performing either operation the natural maps are indeed quasi isometries. This leads to the following proposition.

Proposition 1.18. *Let G_1, G_2 be two virtually isomorphic group. Then, given any word-metric of those, they are quasi isometric.*

Proof. It is enough to show the proposition in the case of G_1 being the quotient of G_2 one over a finite normal subgroup K_2, or in the case of G_1 being a finite index subgroup of G_2. In both cases the proof is a straightforward application of Milnor-Svarc Lemma 1.12: the actions are respectively that of G_2 on a Cayley graph for $G_1 = G_2/K_2$, and that of G_1 on a Cayley graph of G_2. For the very few details we omitted see, for instance, [D-K, Cor. 5.31 and Cor. 5.36, p. 135]. □

It is not surprising at all that an algebraic relation implies a geometric relation between groups. In the aim of geometric group theory the interesting part would be the converse, which would let us infer algebraic relations from geometric properties. That implication unfortunately does not hold true: there exist groups which are quasi isometric to each other but not virtually isomorphic.

An example that virtual isomorphism is a stronger relation than quasi isometry can be given in a relatively straightforward manner. Let us define $\mathrm{Hyp}(2,\mathbb{Z})$ as the set of the matrices A in $\mathrm{SL}(2,\mathbb{Z})$ diagonalizable over $\mathbb{R}$, and such that $A^2 \neq \mathrm{Id}$. Let us define an action $\mathbb{Z} \curvearrowright \mathbb{Z}^2$ where the generator is sent to the multiplication by $A \in \mathrm{Hyp}(2,\mathbb{Z})$, and let us define the corresponding semidirect product $G_A \overset{def}{=} \mathbb{Z}^2 \rtimes_A \mathbb{Z}$. The subgroup $\mathbb{Z}^2 \subseteq G_A$ is the maximal normal Abelian subgroup in G_A, if not by computing the product of two elements with nonzero last component and requiring commutativity it would follows that $A^2 = \mathrm{Id}$. By diagonalizing the matrix A over $\mathbb{R}$, the group G_A embeds as a discrete cocompact subgroup in the geometric Lie group

$$\mathrm{Sol}_3 \overset{def}{=} \mathbb{R}^2 \rtimes \mathbb{R}$$

with the action $\mathbb{R} \curvearrowright \mathbb{R}^2$ given by

$$t \mapsto \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Hence, since Sol_3 is torsion-free, so is G_A. Moreover G_A acts geometrically with respect to the metric on Sol_3 induced by the left-invariant Riemannian metric; hence for each matrix $A \in \mathrm{Hyp}(2,\mathbb{Z})$ the group G_A is quasi isometric to Sol_3 via Milnor-Svarc Lemma 1.12, in particular all such groups are quasi isometric to each other. We now want to pick matrices $A, B \in \mathrm{Hyp}(2,\mathbb{Z})$ such that the corresponding groups G_A and G_B are not virtually isomorphic. Let us chose such matrices requiring for each $n, m \in \mathbb{N}$ the powers A^n and B^m not to be conjugate; this request is feasible by choosing matrices whose field generated by eigenvalues over $\mathbb{Q}$ are different, for example we can take

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

Since the corresponding groups are torsion-free it is indeed enough to show that they are not commensurable. Arguing by contradiction, let H_A be a finite-index subgroup in G_A isomorphic to H_B, finite-index subgroup of G_B. H_A intersects the maximal normal Abelian subgroup of G_A, which has rank 2, in another rank 2 Abelian subgroup L_A, since it has finite-index. Moreover the image of H_A under the quotient map $G_A \longrightarrow G_A/\mathbb{Z}^2$ has to be an infinite cyclic group, generate by some $n \in \mathbb{N}$. It follows that we have an isomorphism

$$H_A \cong Z^2 \rtimes_{A^n} \mathbb{Z} = G_{A^n},$$

and, similarly,

$$H_B \cong Z^2 \rtimes_{B^m} \mathbb{Z} = G_{B^m}.$$

An isomorphism between those group would carry $\mathbb{Z}^2$ to $\mathbb{Z}^2$ via a linear map M, since it has to carry L_A to L_B isomorphically. By computing the group operations and requiring them to commute with the isomorphism it turns out that certain power of A and B must be conjugate by such map M, which contradicts our choice of the matrices. Hence the groups G_A and G_B cannot be virtually isomorphic, and the converse to Proposition 1.18 is proved false.

Motivated by the previous example we are particularly interested in those group for which the converse of Proposition 1.18 holds, and this is the reason behind the definition of *quasi isometric rigidity*.

Definition 1.19 (Quasi isometric rigidity)**.** A class of finitely generated groups $\mathcal{C}$ is said to be *quasi isometrically rigid* if each finitely generated group G which is quasi isometric to a group $G' \in \mathcal{C}$ is virtually isomorphic to a (possibly different) $G'' \in \mathcal{C}$.

A finitely generated group G is said to be *quasi isometrically rigid* if every finitely generated group quasi isometric to G is indeed virtually isomorphic to it.

We want to remark that if a class of group is rigid this does not imply that every group is rigid by itself. In what follows we will deal with the proof of quasi isometric rigidity for lattices of isometries of the hyperbolic space of dimension at least 3. Even in this setting we will find an example motivating the previous remark: the cocompact lattices will in fact be rigid as a class, but very far from rigid as single groups, as they are all quasi isometric; however each non cocompact lattice will surprisingly turn out to be rigid by itself.

Chapter 2

Groups q.i. to hyperbolic spaces

In this chapter we will study groups which are quasi isometric to the hyperbolic space $\mathbb{H}^n$ with $n \geq 3$. It turns out, in fact, that these groups are virtually isomorphic to peculiar subgroups of the Lie group $\mathrm{Isom}(\mathbb{H}^n)$. We give the definition of lattices.

Definition 2.1 (Lattice)**.** Let G be a Lie group. A *lattice* Γ is a discrete subgroup of G such that the quotient space G/Γ has finite volume with respect to any left-invariant metric on G. A lattice is said to be *uniform* if the quotient space G/Γ is compact, *nonuniform* otherwise.

In the cases we are interested in, *i.e.* when $G = \mathrm{Isom}\,\mathbb{H}^n$, it turns out that the requirements of finite volume or compactness in the previous proposition can be both verified on the quotient X/G, where X is the symmetric space associated to G, which turns out to be $\mathbb{H}^n$ itself.

The main result we are going to prove in this chapter is the following theorem, due to P. Tukia.

Theorem 2.2 (Tukia)**.** *Let $n \geq 2$, and let G be a finitely generated group which is quasi isometric to $\mathbb{H}^{n+1}$. Then G acts geometrically on $\mathbb{H}^{n+1}$, so it is virtually isomorphic to a uniform lattice in the Lie group* $\mathrm{Isom}(\mathbb{H}^{n+1})$.

Because of the quasi isometry between G and $\mathbb{H}^n$, thanks to the fact that G acts on itself via left translations, we obtain a quasi action of G onto $\mathbb{H}^n$ via Lemma A.5. Our proof of Tukia's Theorem will be based on the fact that this quasi action is conjugate to a geometric action. A big role in the following arguments is played by quasiconformal mappings, which will be introduced in the next section.

2.1 Quasi conformal maps

2.1.1 Quasi symmetry and quasi conformality

In Riemannian geometry conformal maps are the ones which preserves angles; we are interested in introducing a way to measure, and hence control, the deviation of a homeomorphism from being a conformal map. We start dealing with the much simpler case of linear conformal maps. The linear conformal mappings of $\mathbb{R}^n$ are trivially the group $\mathbb{R}_+ \cdot \mathcal{O}(n)$, *i.e.* orthogonal linear maps up to dilation. Linear conformal maps have the property of mapping the unit ball $B \subseteq \mathbb{R}^n$ in another ball (of possibly different radius): for a nonconformal map $M \in \mathrm{GL}(n, \mathbb{R})$ our goal is to measure how much the image $M(B)$, which is an ellipsoid, differs from a round ball. In the two-dimensional case the only way to obtain such a measure is via the eccentricity of the ellipsoid, which is the ratio of the two axes; in higher dimension we can measure eccentricity using different ratios or products of them, so we will give the definition of different distortion quantities. The main tool we will use in order to define those quantities is the *singular values decomposition (SVD)* for matrices.

Theorem 2.3 (Singular values decomposition)**.** *Let $M \in M(n, n, \mathbb{R})$ be a real square matrix. Then it admits a decomposition in the form*

$$M = U\ D\ V$$

where $U, V \in \mathcal{O}(n)$ are orthogonal matrices and D is a diagonal matrix with diagonal elements $\lambda_i \stackrel{def}{=} D_{i,i}$, called singular values*, ordered decreasingly. These values are independent of the decomposition.*

Proof. [G-VL, Theorem 2.4.1, p. 76]. □

Geometrically speaking the singular values of a linear transformation are the half-lengths of the axes of the image of the unit ball, so they are the quantities we are interested in doing ratios with. We are now able to give the definition of the *distortion quantities.*

Definition 2.4. Let $M \in M(n, n, \mathbb{R})$ be a real square matrix with singular values $\lambda_n \geq \cdots \geq \lambda_1$, we define the following quantities:

Linear dilatation $H(M) \stackrel{def}{=} \frac{\lambda_n}{\lambda_1}$. This is the eccentricity of the ellipsoid $M(B)$.

Inner dilatation $H_I(M) \stackrel{def}{=} \frac{\lambda_n \cdots \lambda_1}{\lambda_1^n}$.

Outer dilatation $H_O(M) \stackrel{def}{=} \frac{\lambda_n^n}{\lambda_n \cdots \lambda_1}$.

Maximal dilatation $K(M) \stackrel{def}{=} \max\{H_I(M), H_O(M)\}$.

We can extend the definition of eccentricity from ellipsoids to topological balls: if we choose a point x in the interior of a ball $B \subset \mathbb{R}^n$ the quantity

$$\max_{y,z\in\partial B} \frac{|x-y|}{|x-z|}$$

is exactly the ratio between the radii respectively of the smallest outer round ball and of the biggest inner one. The idea is now to extend the notion of dilatation for a homeomorphism, by measuring eccentricity of topological balls which round balls are mapped into. This leads to the following definition.

Definition 2.5 (Quasi symmetric homeomorphisms)**.** Let U, U' be domains in $\mathbb{R}^n$. A homeomorphism $f : U \longrightarrow U'$ between them is said to be *c-weakly quasi symmetric* if the following inequality

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leq c$$

holds for all $x, y, z \in U$ with $|x-y| = |x-z| \neq 0$. Let us note that we do not request f to preserve orientations.

Let $\eta : [1, +\infty) \longrightarrow [1, +\infty)$ be a continuous divergent increasing function. A homeomorphism $f : U \longrightarrow U'$ is said to be *η-quasi symmetric* if for all $x, y, z \in U$ with $|x-y| \geq |x-z|$ it holds

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leq \eta\left(\frac{|x-y|}{|x-z|}\right).$$

We can now recall the definition of *Moebius transformations.*

Definition 2.6 (Moebius transformation)**.** A *Moebius transformation* of $\mathbb{R}^n$ (or S^n, which we will always think as the 1-point compactification of $\mathbb{R}^n$) is a composition of finitely many inversions in $\mathbb{R}^n$. The group of Moebius transformations will be denoted by $\mathrm{Mob}(\mathbb{R}^n)$ or $\mathrm{Mob}(S^n)$.

The drawback of our definition of quasi symmetric maps is that we cannot apply that definition to Moebius transformations, which may carry a point in $\mathbb{R}^n$ to the infinity, so we give the following definition.

Definition 2.7 (Quasi-Moebius homeomorphisms)**.** A sphere homeomorphism $f : S^n \longrightarrow S^n$ is said to be *quasi-Moebius* if it is the composition of a Moebius transformation with a quasi-symmetric map of $\mathbb{R}^n$ (which extends to a homeomorphism of S^n by fixing the point at infinity).

Moebius transformations are characterized by the property of preserving cross-ratios

$$[x, y, z, w] = \frac{|x-y| \cdot |z-w|}{|x-z| \cdot |y-w|}$$

and it can be proved that similarly a homeomorphism $f : S^n \longrightarrow S^n$ is quasi-Moebius if and only of it exists a homeomorphism $\eta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$[f(x), f(y), f(z), f(w)] \leq \eta([x, y, z, w])$$

for all x, y, z, w for which the two cross-ratios make sense; for further details see [Väi].

We are now ready to introduce quasiconformal mappings, which we will define using the ratio $\frac{|f(x)-f(y)|}{|f(x)-f(z)|}$ with $|x - y| = |x - z| \longrightarrow 0$: the intuitive meaning of the definition we are about to give is that quasiconformal maps are the one which send infinitesimal round balls to infinitesimal ellipsoids of bounded eccentricity.

Definition 2.8 (Quasiconformal homeomorphism)**.** Let U, U' be two domains in $\mathbb{R}^n$ and $f : U \longrightarrow U'$ a homeomorphism between them. For every $x \in U$ we define the quantity

$$H_x(f) = \limsup_{\rho \to 0} \left(\sup_{|x-y|=|x-z|} \frac{|f(x) - f(y)|}{|f(x) - f(z)|} \right).$$

The function $H_x(f)$ is called *dilatation function* of the map f. A homeomorphism $f : U \longrightarrow U'$ is said to be *quasiconformal* if $\sup_{x \in U} H_x(f)$ is finite, and is said to have dilatation less than or equal to H if the essential supremum of $H_x(f)$ is less than or equal to H.

Obviously quasi-symmetric maps are also c-weakly quasi-symmetric and the latter are trivially quasiconformal; it turns out that for $\mathbb{R}^n$ this implications can be reversed and the upon definitions end up to be equivalent; for further details see [Hei, Theorem 11.14, p. 92].

2.1.2 Analytical properties of quasiconformal mappings

We now want to prove some analytical properties of quasiconformal homeomorphisms, essentially because we would like to relate our definition of bounded dilatation with a condition on the Jacobian determinant or, at least, the differential, which a priori can be nowhere defined. Let us start stating a classical result on almost everywhere differentiability we will use.

Theorem 2.9 (Rademacher-Stepanov)**.** *Let $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$. For $c \in \Omega$ let us define*

$$\left|D_x^+(f)\right| \stackrel{def}{=} \limsup_{h \longrightarrow 0} \frac{|f(x+h) - f(x)|}{|h|}$$

and let $E \stackrel{def}{=} \{x \in \Omega \text{ s.t. } |D^+_x(f)| < +\infty\}$. *Then* f *is differentiable almost everywhere (with respect to the Lebesgue measure) in* E.

Proof. See, for instance, [Fed, Theorem 3.1.9, p. 218]. □

We will also need some technical tools arising from measure theory (on $\mathbb{R}^n$), which we will now state.

Definition 2.10 (Derivative of measures)**.** Let μ be a Radon measure on an open set $U \subseteq \mathbb{R}^n$, defined on the σ-algebra of Borel sets, and from now on let λ denote the Lebesgue measure. We define the *derivative* of μ, which will be denoted by μ', as follows

$$\mu'(x) \stackrel{def}{=} \limsup_{x \in B(x',\rho), \rho \to 0} \frac{\mu(B)}{\lambda(B)}.$$

Since this derivative is defined to be integrated over the Lebesgue-measure, in order to avoid any problem concerning our objects being well defined, any two functions which are equal almost everywhere are considered to be the same.

The main result we will use will be the following well-known theorem by Lebesgue, Radon and Nikodym.

Theorem 2.11 (Lebesgue-Radon-Nikodym)**.** *With the notation above the function* μ' *is Lebesgue-measurable and finite almost everywhere in* U. *Furthermore* μ' *is the Radon-Nikodym derivative of the component of* μ *absolutely continuous with respect to the Lebesgue measure.*

Proof. See, for instance, [Fol, Theorem 3.8, p.84; Theorem 3.22, p. 94]. □

We recall that the image of a Borel subset of $\mathbb{R}^n$ via a continuous map $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is not necessarily a Borel set, although it is still Lebesgue-measurable, see [Bog, Section 1.10, p. 35] for further details. This fact let us to give the following definition.

Definition 2.12 (Pull-back measure)**.** Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous map: it is defined a measure μ_f on the Borel σ-algebra of U via the formula

$$\mu_f(E) \stackrel{def}{=} \lambda(f(E)).$$

Lemma 2.13. *Let* $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *be differentiable at* $x \in U$, *then it holds* $\mu'_f(x) = |J_x(f)|$ *where* $J_x(f)$ *denotes the Jacobian determinant.*

Proof. It follows immediately from the change of variables formula. □

Lemma 2.14. *If* $f : U \subseteq \mathbb{R}^n \longrightarrow U' \subseteq \mathbb{R}^n$ *is a homeomorphism then the measure* μ_f *is Radon.*

Proof. We have to check the following facts:

1. $\mu_f(K) < +\infty$ for every compact set $K \subseteq U$.

2. $\mu_f(E) = \inf\{\mu_f(A) \text{ with } E \subseteq A \text{ open set}\}$ for every Borel set E.

3. $\mu_f(E) = \sup\{\mu_f(K) \text{ with } E \supseteq K \text{ compact set}\}$ for every Borel set E.

and all of these follow easily from the fact that the Lebesgue measure is Radon and f is a homeomorphism. □

We are now ready to prove an almost everywhere differentiability theorem for quasiconformal mappings.

Theorem 2.15. *Let $f : U \subseteq \mathbb{R}^n \longrightarrow U' \subseteq \mathbb{R}^n$ be a quasiconformal map. Then it is differentiable almost everywhere in U and, for almost every $x \in U$ the following holds*

$$||D_x f|| \leq H_x(f)\, |J_x(f)|^{\frac{1}{n}}$$

where $J_x(f)$ denotes the Jacobian determinant of f in x.

Proof. Let us define

$$R(\rho) \stackrel{def}{=} \max\{|f(x+h) - f(x)| \text{ with } |h| = \rho\}$$

and

$$r(\rho) \stackrel{def}{=} \min\{|f(x+h) - f(x)| \text{ with } |h| = \rho\}.$$

Using definitions we get

$$\left|D_x^+ f\right| = \limsup_{\rho \to 0} \frac{R(\rho)}{\rho} = H_x(f) \limsup_{\rho \to 0} \frac{r(\rho)}{\rho}.$$

We can now observe that, for $r = r(\rho)$ we have the inclusion

$$B(f(x), r) \subseteq f(B(x, \rho))$$

which implies, denoting with ω_n the volume of the n-dimensional unit ball, that

$$\omega_n r^n = \lambda(B(f(x), r)) \leq \lambda(f(B(x, \rho)))$$

and hence we obtain

$$\frac{\lambda(f(B(x,\rho)))}{\lambda(B(x,\rho))} \geq \frac{r^n}{\rho^n}.$$

Using the definition of the derivative of the pull-back measure, and since it suffices to just consider balls centered in x there, we obtain

$$\mu'_f(x) \stackrel{def}{=} \limsup_{\rho\to 0} \frac{\lambda(f(B(x,\rho)))}{\lambda(B(x,\rho))} \geq \limsup_{\rho\longrightarrow 0} \frac{r^n}{\rho^n} = \left(\frac{|D_x^+(f)|}{H_x(f)}\right)$$

so we get

$$\left|D_x^+(f)\right| \leq H_x(f)(\mu'(x))^{\frac{1}{n}}$$

and, since the right hand side is almost everywhere finite thanks to the Lebesgue-Radon-Nikodym Theorem 2.11 so it is the left hand side, and we can apply the Rademacher-Stepanov Theorem 2.9 obtaining the almost everywhere differentiability of f. Using Lemma 2.13 we can also obtain, almost everywhere in U, that

$$|D_x(f)| = \left|D_x^+(f)\right| \leq H_x(f)(\mu'(x))^{\frac{1}{n}} = H_x(f)\,|J_x(f)|^{\frac{1}{n}}\,. \qquad \square$$

Corollary 2.16. *For $n \geq 2$ every quasiconformal mapping $f : U \longrightarrow \mathbb{R}^n$ has nonvanishing Jacobian on a subset of positive measure.*

Proof. Arguing by contradiction let us assume $J_x(f) = 0$ almost everywhere in U. Because of the inequality

$$|D_x(f)| \leq H_x(f)(\mu'(x))^{\frac{1}{n}}$$

proven in the previous theorem we would have $D_x(f) = 0$ almost everywhere in U. Let now $p \in U$ and $q = p + \lambda e_1 \in U$ be points such that the segment I between them in contained in U. The function f is absolutely continuous, since it is locally Lipschitz indeed, as shown during the proof of Theorem 2.15, hence the Fundamental Theorem of Calculus holds, see [Fol, Theorem 3.36, p. 102], and we get

$$f(p) - f(q) = \int_I \frac{\partial}{\partial X_1} f(x)\mathrm{d}x_1 = 0$$

contradicting the injectivity of f. $\square$

The above corollary can be strengthened by proving that the Jacobian does not vanish almost anywhere on U, but the formulation we have given will fit our purposes. For completeness we are going to give an analytical definition of quasiconformality; we omit the proof of the equivalence with Definition 2.8, which can be found in [Väi, Theorem 34.4, p. 114].

Theorem 2.17 (Analytical definition of quasiconformal mappings)**.** *A homeomorphism $f : U \subseteq \mathbb{R}^n \longrightarrow U' \subseteq \mathbb{R}^n$ is quasiconformal if and only if it belongs to the Sobolev space $W^{1,n}_{loc}(U, V)$, it is almost everywhere differentiable, and*

$$K(f) \stackrel{def}{=} \operatorname*{ess\,sup}_{x\in U} K(D_x(f)) < +\infty$$

where K is the maximal dilatation defined in 2.4.

It is a classical theorem by Liouville that every C^3 conformal map between domains in S^3 is the restriction of a Moebius transformation: many efforts have been required to weaken the differentiability condition, but in the end the result has been proved to hold true just for 1-quasiconformal maps.

Theorem 2.18 (Liouville Theorem). *Let $f : U \subseteq S^n \longrightarrow U' \subseteq S^n$ be a quasiconformal homeomorphism with dilatation $H \leq 1$. Then f is the restriction of a Moebius transformation.*

Proof. See, for instance, [Geh, Theorem 16, p. 389]. □

Liouville's Theorem will play a huge role in what follows: indeed it will be a key part of the proof of quasi rigidity in both the uniform and nonuniform case. The underlying reason this result is so important is that conformal mappings of the boundary sphere of the hyperbolic space arise from isometries of the space, so this will be, in the end, our main tool to distinguish hyperbolic isometries.

Finally we will need a convergenge property for quasiconformal mappings, which will be given by the following result.

Theorem 2.19. *Let $n \geq 3$, and $f_i : U \subseteq S^n \longrightarrow f_i(U) \subseteq S^n$ be a sequence of K-quasiconformal homeomorphisms normalized at three points in U, i.e. such that there exist three points $p_1, p_2, p_3 \in U$ whose limits $f_i(p_j)$ exist and are distinct (after composing with Moebius transformations we can assume those points are fixed). Then there is a subsequence of $(f_i)_{i\in\mathbb{N}}$ which converges to a K-quasiconformal mapping.*

Proof. See, for instance, [Geh, Theorem 13, p. 386] □

2.1.3 Quasi symmetry and hyperbolic geometry

A fundamental result concerning quasiconformal maps we will need is the following, which states that quasi isometries of the hyperbolic space induce quasi symmetric maps of the boundary sphere. In what follows we will be using the half-space model for hyperbolic space.

Theorem 2.20. *Let $f : \mathbb{H}^{n+1} \longrightarrow \mathbb{H}^{n+1}$, with $n \geq 1$ be an (L, A)-quasi isometry and let $f_\infty : S^n \longrightarrow S^n$ be the homeomorphism induced on the boundary (see also Theorem B.9). After composing with an isometry of $\mathbb{H}^{n+1}$ we may assume that $f_\infty(\infty) = \infty$ (recall that we think at the sphere as the Alexandroff compactification of the Euclidean space) then there exists a constant $C = C(L, A)$ depending only on L, A such that for the function*

$$\eta(t) = e^{2C+A}t^L$$

the homeomorphism $f_\infty : S^n \longrightarrow S^n$ *is* η*-quasi-symmetric.*

Proof. With abuse of notation let us denote f_∞ with f. Let us consider an annulus $A \subset \mathbb{R}^n$ given by

$$A = \{x \text{ s.t. } R_1 \leq |x| \leq R_2\}$$

with $0 < R_1 \leq R_2 < +\infty$. We will call the ratio $t = \frac{R_2}{R_1}$ eccentricity of the annulus: it is clear that for y in the outer boundary of A and z in the inner one the ratio $\frac{|y-0|}{|z-0|}$ is exactly the eccentricity of A. It follows that, considering the smallest annulus A' centered in $f(0)$ and which contains $f(A)$ and denoting by t' its eccentricity then

$$\frac{|f(y) - f(0)|}{|f(z) - f(0)|} \leq t'.$$

We now want to prove that $t' \leq \eta(t)$.

After composing f with a translation of $\mathbb{R}^n$ we can assume that $f(0) = 0$. Let $\alpha \subset \mathbb{H}^{n+1}$ be the vertical geodesic connecting 0 to ∞, and let $\pi_\alpha : \mathbb{H}^{n+1} \longrightarrow \alpha$ denote the orthogonal projection to α. This map is obviously the nearest-point projection to α and can be extended to the whole space $\overline{\mathbb{H}^{n+1}} = \mathbb{H}^{n+1} \cup S^n$ by setting $\pi_\alpha(0) = 0$, $\pi_\alpha(\infty) = \infty$ and $\pi_\alpha((v, o)) = (0, \dots, 0, |v|)$ with $v \in \mathbb{R}^n$ (remind that we are using the upper half-space model for $\mathbb{H}^{n+1}$). The image $\pi_\alpha(A)$ of the annulus under this map is the interval σ on α with the first n coordinates zero and last coordinate in $[R_1, R_2]$, whose hyperbolic length is $l = \log(\frac{R_2}{R_1})$. In what follows let δ be the hyperbolicity constant of $\mathbb{H}^{n+1}$. Since we have assumed that f fixes $0, \infty$ the Extended Morse Lemma B.6 gives us that the (L, A)-quasigeodesic $f \circ \alpha$ lies within distance θ from α. This is due to the fact that the geodesic we obtain by the Lemma must be a ray going from 0 to ∞, because it stays at bounded distance from $f(\alpha)$ and so from α, too.

Our goal is now to estimate the diameter of $\pi_\alpha(f(A))$. The ideal boundary of the set $\tilde{A} = \pi_\alpha^{-1}(\sigma)$ is the annulus A, so we can work with the spherical half-shell $\tilde{A}$. Let $p, q \in \tilde{A}$: since quasi isometries and nearest-point projections almost commute by Proposition B.7, there exists a constant C such that

$$\mathrm{d}(f(\pi_\alpha(p), \pi_\alpha(f(p))) \leq C$$

and the same inequality holds for q; since $\mathrm{d}(\pi_\alpha p, \pi_\alpha q) \leq l$ we obtain

$$\mathrm{d}(f(\pi_\alpha(p)), f(\pi_\alpha(q))) \leq Ll + A$$

and by the triangle inequality

$$\mathrm{d}(\pi_\alpha(f(p)), \pi_\alpha(f(q))) \leq 2C + Ll + A$$

so $\operatorname{diam}(\pi_\alpha(f(A))) \leq 2C + Ll + A$.

Due to the estimate above we know that $f(A) \subset \pi_\alpha^{-1}(\sigma')$ where σ' is a geodesic segment of α with length less than or equal to $l' = 2C + Ll + \alpha$. The ideal boundary of such preimage is an annulus of eccentricity bounded by $e^{l'}$, so the eccentricity of $f(A)$ must be bounded by

$$e^{l'} = e^{2C+A} \cdot e^{L\log(t)} = e^{2C+A} t^L$$

which is the statement of the theorem. □

The converse to this statement is also true, and was first proved by Tukia.

Theorem 2.21. *Every η-quasi-symmetric homeomorphism $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ extends to an (A, A)-quasi isometry $F : \mathbb{H}^{n+1} \longrightarrow \mathbb{H}^{n+1}$ where*

$$A = \eta(e) + 2\log(\eta(e)) + \log(\eta(e+1)).$$

Proof. Given a homeomorphism $f \in \operatorname{Homeo}(\mathbb{R}^n)$ we construct the extension $F : \mathbb{H}^{n+1} \longrightarrow \mathbb{H}^{n+1}$ in the following way. For every $p \in \mathbb{H}^{n+1}$ let α be the complete vertical geodesic through p, with endpoints ∞ and $x_p \in \mathbb{R}^n$. Let y_p be a point in $\mathbb{R}^n$ such that $\pi_\alpha(y_p) = p$: this is obviously not unique because all the possible choices form a $(n-1)$-sphere in $\mathbb{R}^n$ centered in x_p. Let us now define α' the complete vertical geodesic with endpoints ∞ and x_p, and we set $F(p) \overset{def}{=} \pi_{\alpha'}(f(y_p))$.

Once it is proven that this map is (L, A)-coarse Lipschitz, which we will do later, it is easy to check that it is in fact a quasi isometry with quasi-inverse $\bar{F}$ given by the extension of f^{-1}. For any point $p \in \mathbb{H}^{n+1}$ we obtain that $\bar{F} \circ F(p)$ will lay on the same vertical geodesic, so we just have to estimate how far the points p and $\bar{F} \circ F(p)$ lay. After applying F the definition of $\bar{F}$ requires the choice of a point with projection $F(p)$ on α': if this point is exactly $f(y_p)$ then we will have $\bar{F} \circ F(p) = p$, for any choice of $y_{F(p)}$ we will have that

$$|f(x_p) - f(y_p)| = \left|f(x_p) - y_{F(p)}\right|.$$

Since the inverse of a quasi symmetric map is also quasi-symmetric (with the same function η) we can now estimate

$$|x_p - y_p| \leq K \left|x_p - f^{-1}(y_{F(p)})\right|$$

so the distance between the two points on the geodesic is

$$\mathrm{d}(\bar{F} \circ F(p), p) \leq \log(K)$$

so F is a quasi isometry.

We are left to prove that the extension F is (A, A)-coarse Lipschitz. Let $p, q \in \mathbb{H}^{n+1}$ and let us assume first that $\mathrm{d}(p, q) \leq 1$; without loss of generality we can assume $p = e_{n+1} \in \mathbb{H}^{n+1}$, the $(n+1)$-th vector from the canonical basis. The first case we deal with is when q belongs to the same vertical geodesic of p: assuming $q = \lambda e_{n+1}$ with $\lambda > 0$ we have that

$$\mathrm{d}(p, q) = \log\left(\frac{|y_q - x_q|}{|y_p - x_p|}\right) \leq 1$$

and, by quasi-symmetry of f, and with the notation $x = x_p = x_q$, we obtain

$$\frac{1}{\eta(e)} \leq \left(\eta\left(\frac{|y_q - x|}{|y_p - x|}\right)\right)^{-1} \leq \frac{|f(y_q) - f(x)|}{|f(y_p) - f(x)|} \leq \eta\left(\frac{|y_q - x|}{|y_p - x|}\right) \leq \eta(e)$$

and, in particular,

$$\mathrm{d}(f(p), f(q)) \leq \log(\eta(e)).$$

The second case is when the two points p, q have the same last coordinate, which equals 1 without loss of generality; we also have $|p - q| \leq e$. The points $F(p), F(q)$ belong to a pair of vertical geodesic lines α'_1, α'_2 with limit points $f(x_p), f(x_q)$: postcomposing with an isometry of $\mathbb{H}^{n+1}$ we can also assume $|f(x_p) - f(x_q)| = 1$. If we denote with y_1, y'_1 points in $\mathbb{R}^n$ such that

$$\pi_{\alpha_1}(p) = y_1$$
$$\pi_{\alpha'_1}(F(p)) = y'_1$$

and the equivalent points y_2, y'_2 for q we now obtain that

$$|y'_1 - f(x_p)| = |F(p) - F(x_p)| = R'_1$$
$$|y'_2 - f(x_q)| = |F(q) - F(x_q)| = R'_2$$

and we can assume $R'_1 \leq R'_2$. Hence we obtain that

$$\mathrm{d}(F(p), F(q)) \leq \frac{1}{R'_1} + \log\left(\frac{R'_2}{R'_1}\right)$$

because we can move from $F(p)$ to $F(q)$ horizontally first, following a path of length $\frac{1}{R'_1}$ and then executing the vertical travel along α'_2, for a path of length $\log\left(\frac{R'_2}{R'_1}\right)$. We now want to give an upper bound of the length of those paths: for the first one let us use the η-quasi-symmetry condition on x_p, y_p, x_q obtaining that

$$\frac{1}{R'_1} = \frac{\left|f(x_p) - y'_p\right|}{|f(x_p) - f(y_p)|} \le \eta\left(\frac{|x_p - x_q|}{|x_p - y_p|}\right) \le \eta(e).$$

Again from the quasi symmetry condition we can now obtain

$$\begin{aligned}\frac{R'_2}{R'_1} &\le \frac{|f(x_p) - y'_2| + |f(x_p) - f(x_q)|}{|f(x_p) - y'_1|} \le \\ &\le \eta\left(\frac{|x_p - y_q|}{|x_p - y_p|}\right) + \frac{1}{\left|f(x_p) - y'_p\right|} \le \eta(e+1) + \eta(e).\end{aligned}$$

Due to the use of these inequalities we can now obtain that

$$\mathrm{d}(F(p), F(q)) \le \eta(e) + \log(\eta(e+1) + \eta(e))$$

ad so, in the general case of $p, q \in \mathbb{H}^{n+1}$ with $\mathrm{d}(p, q) \le 1$ we get

$$\mathrm{d}(F(p), F(q)) \le A.$$

To complete the proof of the theorem we now have to deal with the case of $p, q \in \mathbb{H}^{n+1}$ with $\mathrm{d}(p, q) > 1$, but we can obtain a chain of $\lfloor \mathrm{d}(p, q) \rfloor + 1$ points with distance between them bounded above form 1 and estimate the distance between $F(p), F(q)$ following that path via the triangle inequality obtaining

$$\mathrm{d}(F(p), F(q)) \le (\lfloor \mathrm{d}(p, q) \rfloor + 1)A \le A\mathrm{d}(p, q) + A,$$

so F is a (A, A)-coarse Lipschitz map, and the proof is complete. □

2.2 Tukia's Theorem

2.2.1 Uniformly quasiconformal groups

We can now head towards the proof of Theorem 2.42. From now on let $n \ge 2$ and G be a finitely generated group quasi isometric to $\mathbb{H}^{n+1}$. We already saw that, via conjugation with the quasi isometry, the group G quasi acts on $\mathbb{H}^{n+1}$ via quasi isometries, due to Lemma A.5. We will now state some properties of such quasi-action some of which will guide our proof, while the others will give technical but necessary lemmas.

First, let us note that the quasi action above induces an action by homeomorphisms $\psi_\infty : G \curvearrowright S^n$ via Theorem B.9.

Lemma 2.22. *The action ψ_∞ is composed of uniformly quasiconformal homeomorphisms.*

Proof. The corollary follows immediately from Lemma A.5 and Theorem 2.20. □

The fact that the quasi-action $G \curvearrowright S^n$ is geometric, which follows immediately from Lemma A.5, translates into properties of the natural action $G \curvearrowright \mathrm{Trip}(S^n)$.

Lemma 2.23. *The action* $\phi : G \curvearrowright \mathrm{Trip}(S^n)$ *is properly discontinuous and cocompact.*

Proof. It follows immediately from Theorem B.19. □

One last notable property of our quasi-action is the following.

Lemma 2.24. *Every point of* $S^n = \partial_\infty \mathbb{H}^{n+1}$ *is a conical limit point for the quasi-action* $G \curvearrowright \mathbb{H}^{n+1}$.

Proof. Since the quasi action is geometric the lemma follows immediately from Lemma B.16. □

We can now give the definition of the groups that we will study in the following pages.

Definition 2.25 (Uniformly quasiconformal group)**.** A subgroup of the sphere homeomorphisms $G < \mathrm{Homeo}(S^n)$ consisting of quasiconformal maps is said to be *uniformly quasiconformal* if there exists a constant K such that for every $g \in G$ it holds $K(g) \leq K$.

We saw in Lemma 2.22 that a group G quasi isometric to $\mathbb{H}^{n+1}$ gives us a uniformly quasiconformal (form now on also called uq) group: our goal is to study this group in order to prove that we can conjugate the action $G \curvearrowright \mathbb{H}^{n+1}$ to a geometric action.

The first example of uniformly quasiconformal groups is given by the conjugate group $G^f = f^{-1}Gf$ of a group of Moebius transformations $G < \mathrm{Mob}(S^n)$ by a quasiconformal homeomorphism f. It is in fact trivial that the quasiconformal constant of all the elements of such group are bounded by $K(f)^2$. A uniformly quasiconformal group G is said to be *exotic* if it is not the quasiconformal conjugate of a group of Moebius transformations. We are interested in excluding the possibility of the existence of exotic groups, since from that fact it will follow that our quasi-action is conjugate to an action via Moebius transformations. In dimension 2 it is in fact true that no uniformly quasiconformal group is exotic, and this result was first proved by D. Sullivan: we will give the proof of this fact by the end of the chapter. In higher dimension things fail rather badly: the existence of exotic groups was fist proved by Tukia and, during the years, many other examples of exotic groups followed.

We saw that uniformly quasiconformal actions $G \curvearrowright S^n$ arise from quasi actions $G \curvearrowright \mathbb{H}^{n+1}$: our goal is now to prove the converse.

Theorem 2.26. *Given an uniformly quasiconformal group $G < \mathrm{Homeo}(S^n)$, its natural action on S^n extends to a quasi-action $\phi : G \curvearrowright \mathbb{H}^{n+1}$.*

Proof. Due to Theorem 2.21 every quasiconformal homeomorphism $g \in G$ induces an (A, A)-quasi isometry where A only depends on $K(g)$ and hence is uniformly bounded. It follows that, by sending an element of G to this extension the image of this map consists of (A, A)-quasi isometries.

We now have to prove that such mapping is a quasi action. In order to show that

$$\mathrm{d}(\phi(g_1) \circ \phi(g_2), \phi(g_1 g_2)) \leq C$$

let us note that $f_1 = \phi(g_1) \circ \phi(g_2)$ is an $(A^2, A^2 + A)$ quasi isometry, meanwhile $f_2 = \phi(g_1 g_2)$ is an (A, A)-quasi isometry, so they are both $(L, \tilde{A})$-quasi isometries, but $(f_1)_\infty = (f_2)_\infty$ because on the boundary we have a real action. Hence by Lemma B.13 we get the inequality we needed. The uniformity of the quasi action constants follow from the fact that they depend only on the quasiconformality constant, which is uniformly bounded.

The proof of the fact that

$$\mathrm{d}(\phi(1), \mathrm{Id}) \leq C$$

follows from Lemma B.13 in the exact same way. □

We proved that uniformly quasiconformal groups are equivalent to uniform quasi-actions on the hyperbolic $(n + 1)$-space. In particular we can now define *conical limit points* for a uniformly quasiconformal subgroup $G < \mathrm{Homeo}(S^n)$ as the conical point of the corresponding quasi-action. We are now ready to state the main theorem of this section, which will readily imply Theorem 2.42.

Theorem 2.27. *Let $n \geq 2$ and $G < \mathrm{Homeo}(S^n)$ be a countable uniformly quasiconformal group such that almost every point of S^n is a conical limit point for G. Then G is not exotic, i.e. it is a quasiconformal conjugate of a subgroup of Moebius transformations.*

We will prove this theorem by the end of this chapter but, in order to do so, we will need some rather technical tools we are now going to develop.

2.2.2 Measurable conformal structures

The technical tool we will need are *measurable conformal Riemannian structures* on the sphere, which we are now going to develop.

Definition 2.28 (Measurable conformal structure)**.** A *measurable Riemannian structure* on S^n is a measurable (with respect to the measure induced by the Lebesgue one on $\mathbb{R}^n$) map $\mu : S^n \longrightarrow P(n)$ from the sphere to the space of symmetric positive-definite matrices. Since we are working just with measurable maps, two functions equal almost everywhere will be considered to be the same, and we will always ignore the point ∞.

A *measurable conformal structure* on S^n is a class of measurable Riemannian structures up to multiplication with a positive measurable function. Unless otherwise stated we will assume the metrics normalized in order to have $\det(\mu(p)) = 1$ for all $p \in S^n$.

Definition 2.29 (Pull-back structure)**.** If $f : S^n \longrightarrow S^n$ is a quasiconformal mapping we can define the pull-back of a measurable Riemannian metric μ by the formula

$$f^*\mu(x) \stackrel{def}{=} (D_x(f))\mu(f(x))(D_x(f))^T.$$

If we consider conformal structures (*i.e.* we normalize the metric μ) we can renormalize this pull-back via

$$f^\bullet\mu(x) \stackrel{def}{=} (J_x(f))^{-\frac{1}{2n}}(D_x(f))\mu(f(x))(D_x(f))^T.$$

We will now give some other definitions regarding these metrics.

Definition 2.30 (Bounded metric)**.** A measurable conformal structure μ on S^n is said to be *bounded* if it is represented by a bounded normalized measurable Riemannian metric in $P(n) \cap SL_+(n)$.

Definition 2.31 (Dilatation of Riemannian metric)**.** Let μ be a measurable Riemannian metric on S^n we define its *linear dilatation* to be

$$H(\mu) \stackrel{def}{=} \operatorname*{ess\,sup}_{x \in S^n} \left(\frac{\lambda_n(x)}{\lambda_1(x)}\right)^{\frac{1}{2}}$$

where $\lambda_1(x) \leq \cdots \leq \lambda_n(x)$ are the eigenvalues of $\mu(x)$.

We can now give the fundamental definition of invariant conformal structure and state the technical proposition we will need to prove Tukia's Theorem.

Definition 2.32. A measurable conformal structure μ on $\mathbb{R}^n$ is said to be *invariant* under the action of a quasiconformal group $G < \mathrm{Homeo}(\mathbb{R}^n)$ if, for every $g \in G$ we have

$$g^\bullet\mu = \mu.$$

Proposition 2.33. *Let $G < \mathrm{Homeo}(S^n)$ be a countable uniformly quasiconformal group. There exists a G-invariant bounded measurable conformal structure λ on S^n.*

In order to prove the proposition we will need the existence of the so called *John-Loewner* ellipsoid. We recall that an n-dimensional ellipsoid centered in 0 might be described by

$$E_A = \left\{v \in \mathbb{R}^n \text{ such that } v^T A v \leq 1\right\}$$

where $A \in M_{\mathbb{R}^n}(n, n)$ is a positive-definite symmetric $n \times n$ real matrix. The volume of an ellipsoid, with respect to the Lebesgue measure, is now given by

$$\text{Vol}(E_A) = \omega_n (\det(A))^{-\frac{1}{2}}$$

with ω_n being the volume of the unitary n-ball in $\mathbb{R}^n$. We are now ready to state the proposition we will need.

Proposition 2.34. *Let $X \subset \mathbb{R}^n$ be a compact centrally-symmetric set, i.e. such that $-X = X$, with nonempty interior. There exists an unique ellipsoid $E(X) \supseteq X$ of minimal volume, and it is called the* John-Loewner ellipsoid *of X.*

Proof. Due to the fact that X is compact and so bounded it is clear that an ellipsoid A containing X does exist. Since X has nonempty interior there exists a ball $B(x, r) \subset X$ so, possibly precomposing with an orthogonal matrix, which has norm 1, we can assume that the axes of the ellipsoid are the vector of the canonical basis, so A diagonal. Now it must obviously hold $(|x_i| + r)e_i \in E_A$ for all $i = 1, \ldots, n$, which is equivalent to

$$(|x_i| + r)e_i^T A(|x_i| + r)e_i = (|x_i| + r)^2 \lambda_i \leq 1$$

where λ_i is the i-th eigenvalue: this inequality gives us

$$\lambda_i \leq \frac{1}{(|x_i| + r)^2} \leq \frac{1}{r^2}$$

so $||A|| \leq \frac{n}{r^2}$ so the matrices giving ellipsoids which contain X are a compact set and the determinant attains minimum. We are now left to prove uniqueness.

We want to show that the following function, defined on the set of positive-defined symmetric $n \times n$ real matrices $P(n)$,

$$f(A) = -\frac{1}{2} \log(\det(A))$$

is strictly convex. On a convex subset $V \subseteq P(n)$ so that f_V is bounded from below it attains an unique minimum on V. If we define

$$V_X = \{A \in P(n) \text{ such that } X \subseteq E_A\}$$

this is easily verified to be a convex subset of the convex space $P(n)$, and because X has nonempty interior f_{V_X} is bounded from below and we are done.

Let $A, B \in P(n)$, and let $C_t = tA+(1-t)B$: since A, B can be simultaneously put in diagonal form by a congruence via a matrix M (see [G-VL, Corollary 8.7.2, p. 499]) let us denote $D_t = M^T C_t M$. We obtain

$$\begin{aligned} f(D_t) = f(M^T C_t M) &= -\log(\det(M)) - \frac{1}{2}\log(\det(C_t)) \\ &= -\log(\det(M)) + f(C_t) \end{aligned}$$

and, since D_t is a segment in the set of positive-defined diagonal matrices it suffice to prove the convexity in this case, where it is trivial since

$$f(\mathrm{Diag}(\lambda_1, \dots, \lambda_n)) = -\frac{1}{2}\sum_{i=1}^{n} \log(\lambda_i)$$

which is strictly convex because of the strict concavity of the logarithm. □

Proof of Proposition 2.33. Let μ_0 denote the Euclidean metric on S^n, given by the constantly identical matrix for the metric. Since G is countable then for almost every point $x \in \mathbb{R}^n$ all the pull-backs by elements of G are defined, in particular we have

$$A_{g,x} \stackrel{def}{=} g^\bullet \mu_0(x) = (J_x(g))^{-\frac{1}{2n}} D_x(g)(D_x(g))^T$$

from which it follows that $H(A_{g,x}) = H(D_x(g)) = H_x(g)$. Thus, since we assumed G to be uniformly quasiconformal the family of measurable conformal structures $G^\bullet \mu_0$ is uniformly bounded, and we can define

$$H \stackrel{def}{=} \sup_{g \in G} H(g^\bullet \mu_0) < +\infty.$$

Let $E_{g,x}$ denote the unit metric ball in the tangent space $T_x\mathbb{R}^n$ with respect to the metric $g^\bullet \mu_0$. From the Euclidean viewpoint those are just ellipsoids of volume ω_n, due to the fact that the metric is normalized: the uniform bound of the conformal structures' eccentricity means that the eccentricities of those ellipsoids are bounded, while the volume is fixed. This way we get (in a fashion which is completely analogous to what we have done during the proof of Proposition 2.34) that the diameter of those ellipsoids is uniformly bounded from above and the inradius is bounded from below: *i.e.* there exist constant $0 < r < R < +\infty$ such that the following inclusions hold for every $g \in G$

$$B(0, r) \subseteq E_{g,x} \subseteq B(0, R).$$

Let now

$$U_x \stackrel{def}{=} \bigcup_{g\in G} E_{g,x}$$

denote the union of such ellipsoids for a fixed x: it is trivial that U_x is centrally-symmetric, compact and with nonempty interior: therefore we can apply Proposition 2.34 and denote with E_x the John-Loewner ellipsoid for U_x. Since the set U_x is invariant under the action of G since this action, which is normalized, is actually volume-preserving so $g(E_x)$ must be a least-volume ellipsoid containing U_x, hence it is preserved by G due to its uniqueness. Since ellipsoids E_x have diameters and inradii uniformly bounded respectively from above and below, than their eccentricities must be uniformly bounded from above.

Let now $\mu : \mathbb{R}^n \longrightarrow P(n)$ denote the almost everywhere defined function which sends a point x to the matrix A_x such that the ellipsoid E_x is defined by the matrix A_x. We just have to prove that this function is actually measurable. Since G is countable it is the increasing union of finite subsets $G_i = \{g_j\}_{j\leq i} \subseteq G$: for each $i \in \mathbb{N}$ let us define the sets

$$U_{x,i} \stackrel{def}{=} \bigcup_{g\in G_i} E_{g,x}$$

and let $E_{x,i}$ be the corresponding John-Loewner ellipsoid. We now want to prove that the function $\mu_i : x \mapsto E_{x,i}$ is measurable. In order to do so we use the identity $\mu_i = \nu_i \circ f^i$ where $f^i : \mathbb{R}^n \longrightarrow P(n)^i$ is the function given by

$$x \mapsto (E_{g_1,x}, \dots, E_{g_n,x})$$

which is measurable, and $\mu_i : P(n)^i \longrightarrow P(n)$ sends i ellipsoids to the John-Loewner ellipsoid of the union, which is continuous: hence the composition is measurable (with respect to the σ-algebra of Borel sets).

We can now conclude the proof of the theorem: since the scalar product map $\mathbb{R}^n \times P(n) \longrightarrow \mathbb{R}_+$ given by $(v, M) \mapsto v^T M v$ is open we will think of μ_i as a function $\tilde{\mu}_i : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}_+$

$$(x, v) \mapsto v^T \mu_i(x) v$$

which is measurable if and only if μ_i is so, hence we obtain $\tilde{\mu}_i \geq \tilde{\mu}_{i+1}$ since $E_i \subseteq E_{i+1}$ and, since

$$E_x = \bigcup_{i\in\mathbb{N}} E_{x,i}$$

we obtain that $\lim_i \tilde{\mu}_i = \tilde{\mu}$ and the measurability follows from the Lebesgue monotone convergence Theorem by Beppo Levi. □

2.2.3 Quasiconformality in dimension 2

We are now able to approach the proof of Theorem 2.27. We will start with the case of dimension two, where the situation is simpler and we will not have to use the conical limit points hypothesis. Our proof of Sullivan's Theorem will be built on a version of Riemann Mapping Theorem for conformal maps, which will grant us the equivalence of the bounded conformal structures to the euclidean one: this result, applied to the invariant structure whose existence we have just proved, will provide the non existence of exotic groups in a very straightforward manner.

In this subsection we will work with complex notation, identifying the real plane $\mathbb{R}^2$, whose one point compactification is the way we look at the two dimensional sphere, with the complex line $\mathbb{C}$. From now on we will work on an open domain U. We briefly recall the usual complex differentials $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$ and $\mathrm{d}\bar{z} = \mathrm{d}x - i\mathrm{d}y$, and we will denote with $\partial f \overset{def}{=} \frac{\partial f}{\partial z} = f_z$ and $\bar{\partial} f \overset{def}{=} \frac{\partial f}{\partial \bar{z}} = f_{\bar{z}}$ the holomorphic and antiholomorphic derivatives. A straightforward computation gives the following formula for the Jacobian determinant

$$J_f = |\partial f|^2 - \left|\bar{\partial} f\right|^2$$

and, from now on, we will require our maps to be (almost everywhere) differentiable and orientation-preserving, *i.e.* with positive Jacobian, which means $|\partial f| > \left|\bar{\partial} f\right|$. Given a complex map $w = f(z)$ this induces the complex form $\mathrm{d}w = f_z \mathrm{d}z + f_{\bar{z}} \mathrm{d}\bar{z}$ and it is trivial to check that the following inequality holds

$$(|f_z| - |f_{\bar{z}}|)\,|\mathrm{d}z| \leq |\mathrm{d}w| \leq (|f_z| + |f_{\bar{z}}|)\,|\mathrm{d}z|$$

and both extremal values are indeed attained. It now follows that the eccentricity of the image of tangent circles in a point $z \in \mathbb{C}$ is bounded by

$$D_z(f) = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \geq 1$$

where we have set

$$\mu(z) = \frac{f_{\bar{z}}}{f_z}. \tag{2.1}$$

The previously defined function is called *Beltrami differential*, and the associated PDE is called *Beltrami equation.* We can now define quasiconformality in our context via the Beltrami differential: we have already shown the following.

Proposition 2.35. *For a complex valued orientation-preserving diffeomorphism $f : U \longrightarrow U'$ the following are equivalent:*

1. *The function f is K-quasiconformal, with $K = \frac{1+k}{1-k}$;*

2. If μ is the Beltrami differential of f, then $k = ||\mu||_\infty < 1$.

From this proposition it is trivial to check that 1-quasiconformal maps are the ones whose antiholomorphic derivative is zero almost everywhere, and a theorem of Weyl states that those functions are indeed holomorphic. These facts give the following statement, which resembles Liouville's Theorem in dimension 2.

Proposition 2.36. *A homeomorphism $f : S^2 \longrightarrow S^2$ is 1-quasiconformal if and only if it is a Moebius transformation.*

We will be interested in solving the Beltrami equation when $||\mu||_\infty < 1$, *i.e.* in the quasiconformal case, and this is the following statement.

Theorem 2.37 (Solution of Beltrami equation)**.** *Given any measurable function $\mu : \mathbb{C} \longrightarrow \mathbb{C}$ with $||\mu||_\infty < 1$ there exists a unique quasi conformal mapping f satisfying Equation 2.1 and leaving $0, 1, \infty$ fixed.*

Proof. For a complete proof see, for instance, [Ahl, Chapter 5, Sections A, B, p. 56]. □

We now want to compute, given any bounded quasiconformal structure on $\mathbb{C}$, the Beltrami differential of a transformation which pull that structure back to the Euclidean conformal structure, and then use the solvability of Beltrami equation to state that every bounded measurable conformal structure is equivalent to the Euclidean one. The precise statement we want to prove is the following.

Theorem 2.38 (Measurable Riemann Mapping Theorem)**.** *Given a complex domain U, every bounded measurable conformal structure on U is equivalent to the Euclidean structure on a domain U' via a quasiconformal map $f : U \longrightarrow U'$.*

Proof. We will denote with $w = f(z)$ a quasiconformal mapping $f : U \longrightarrow U'$, and now a very straightforward computation gives

$$|\mathrm{d}w|^2 = |f_z \mathrm{d}z + f_{\bar{z}} \mathrm{d}\bar{z}|^2 = |f_z|^2 \left|\mathrm{d}z + \frac{f_{\bar{z}}}{f_z}\mathrm{d}\bar{z}\right|^2 = |f_z|^2 \left|\mathrm{d}z + \mu \mathrm{d}\bar{z}\right|^2,$$

denoting with μ the Beltrami differential of f. Let now $\mathrm{d}s_\mu^2$ denote the measurable Riemannian metric $|\mathrm{d}z + \mu\mathrm{d}\bar{z}|^2$: if we show that every measurable Riemannian metric can be expressed in such form with $||\mu||_\infty < 1$, our result will follow from Theorem 2.37. Let us now write, for an arbitrary form $\mathrm{d}s^2$,

$$\mathrm{d}s^2 = E\mathrm{d}x^2 + F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2,$$

and it is trivial to check that we can reduce, via a change of variables in the form $z = e^{i\theta}w$, to diagonal forms with $F = 0$. The condition of $\mathrm{d}s^2$ being a multiple of ds^2_μ can be written as

$$\begin{aligned} 1+\mu =& t\sqrt{E}; \\ 1-\mu =& t\sqrt{G}, \end{aligned}$$

for some real $t > 0$, which leads to the solution

$$\mu = \frac{\sqrt{E}-\sqrt{G}}{\sqrt{E}+\sqrt{G}}.$$

It is now trivial that $|\mu| \leq 1$ and, moreover, it holds $\lim_{z\to z_0} |\mu(z)| = 1$ if and only if it holds

$$\lim_{z\to z_0} \frac{E(z)}{G(z)} = 0, +\infty.$$

If we assume our conformal structure to be bounded the previous limit cannot be zero, and so we get $||\mu||_\infty < 1$, and the proof is complete. □

We are now ready to state and prove Sullivan's Theorem on nonexistence of exotic uniformly quasiconformal groups in dimension 2, which will give the first case of Theorem 2.27.

Theorem 2.39 (Sullivan's Theorem)**.** *Every uniformly quasiconformal group $G <$ Homeo(S^2) is conjugate to a group of Moebius transformations.*

Proof. Let μ be the G-invariant bounded measurable conformal structure given by Proposition 2.33: in view of Theorem 2.38 it exists a quasiconformal map $f : S^2 \longrightarrow S^2$ such that $f^\bullet\mu_0 = \mu$, where μ_0 denotes the Euclidean structure. Such map is the conjugating map we are looking for: in fact it is trivial to check that the group $G' = f^{-1}Gf$ preserves μ_o, *i.e.* every element of G' is conformal with respect to the Euclidean metric, so it is 1-quasiconformal, and Proposition 2.36 states that it is a Moebius transformation. □

2.2.4 Non exotic groups in higher dimension

We are now left to prove Theorem 2.27 in the general case $n \geq 3$. In this case the proof will be more cumbersome, since the conical limit point hypothesis, which was unnecessary in dimension two, is really a key technical part of our proof. Before approaching the proof of the theorem, which we will state with slightly more generality, we will need a last definition and a technical convergence property.

Definition 2.40 (Approximate continuity)**.** A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be *approximately continuous* at a point $x \in \mathbb{R}^n$ is, for every $\varepsilon > 0$ it holds:

$$\lim_{r \to 0} \frac{\lambda_n \left(\{y \in B(x,r) \text{ s.t. } ||f(x) - f(y)|| > \varepsilon\}\right)}{\lambda_n(B(x,r))} = 0,$$

where λ_n denotes the Lebesgue measure on $\mathbb{R}^n$.

Proposition 2.41 (Tukia's strong convergence property). *Let $f_i : S^n \longrightarrow S^n$ be a sequence of K-quasiconformal maps converging to a quasiconformal map $f : S^n \longrightarrow S^n$. If there exists a sequence of subsets $E_i \subseteq S^n$ such that*

$$\limsup_{i \to \infty} H(f_{i|S^n \setminus E_i}) = H$$

and it holds

$$\lim_{i \to \infty} \lambda_{S^n}(E_i) = 0,$$

where λ_{S^n} denotes the measure on S^n induced by the Lebesgue measure on S^n, then f is H-quasiconformal.

Proof. See [Tuk86, Lemma B.2, p.322]. □

We are now ready to state again and prove Tukia's Theorem.

Theorem 2.42 (Tukia's Theorem). *Let $n \geq 3$. Let $G < \mathrm{Homeo}(S^n)$ be a uniformly quasiconformal group with bounded measurable invariant conformal structure μ, and assume that such a structure is almost continuous at a conical limit point ξ for G. Then there exists a quasiconformal map $f : S^n \longrightarrow S^n$ such that the corresponding conjugate group of G, called $G' = f^{-1}Gf$, is composed of Moebius transformations.*

Proof. We recall that we will think to S^n as the one-point compactification of the space $\mathbb{R}^n$, which will be identified to the boundary of the upper-half space model we will use for $\mathbb{H}^{n+1}$. Without loss of generality, conjugating G via affine transformations if need be, we can assume the conical limit point ξ to be the origin of the space and that $\mu(0) = \mu_0(0) = \mathrm{Id}_n$ is the Euclidean structure. Let us denote $e = e_{n+1} \in \mathbb{H}^{n+1}$, and let $\phi : G \curvearrowright \mathbb{H}^{n+1}$ be the (L, A)-quasi action extending the action $G \curvearrowright S^n$ via Theorem 2.26.

From the definition of conical limit point for an action it follows that there exist a sequence $(g_i) \in G$ such that

$$\lim_{i \to \infty} \phi(g_i)(e) = 0,$$

a constant $c > 0$ and an infinitesimal sequence of positive real numbers $(t_i) \in \mathbb{R}_+$

such that for each $i \in \mathbb{N}$ it holds

$$d(\phi(g_i)(e), t_i e) \leq c,$$

where d denotes the hyperbolic distance on $\mathbb{H}^{n+1}$.

Let us now define a sequence of hyperbolic isometries γ_i given by the Euclidean dilatations

$$\gamma_i(x) = t_i x,$$

and define a new family of maps by

$$\tilde{\gamma}_i \stackrel{def}{=} g_i^{-1} \circ \gamma_i.$$

Since the γ_i are hyperbolic isometries and the g_i are quasi isometries it follows immediately that the $\tilde{g}_i$ are (L, A)-quasi isometries, which implies trivially that the inequality

$$\mathrm{d}(\tilde{g}_i(e), e) \leq Lc + A$$

holds for all $i \in \mathbb{N}$. Furthermore let us denote with $f_i \stackrel{def}{=} (\tilde{g}_i)_\infty$ the boundary extensions of the $\tilde{g}_i$, which are K-quasiconformal with K independent of i by Theorem 2.20.

Let now T be an ideal triangle with a centroid in e: by Extended Morse Lemma B.6 the quasi geodesic triangles $\tilde{g}_i(T)$ are uniformly close to ideal geodesic triangles T_i with vertices on the ideal boundary and such that, if we denote by $center(T_i)$ the set of their centroids, for all $i \in \mathbb{N}$ it holds

$$\mathrm{d}(center(T_i), e) \leq C,$$

for an appropriate real constant C. The previous inequality follows from the fact that $\tilde{g}_i(e)$ stays at uniformly bounded distance from $center(T_i)$ and all the images $\tilde{g}_i(e)$ stays at uniformly bounded distance from e itself by a previous inequality (mostly by definition of conical limit point). In particular the maps f_i extend to maps $\bar{f}_i : \mathrm{Trip}(S^n) \longrightarrow \mathrm{Trip}(S^n)$ and we can use compactness of S^n to chose a subsequence such that all the three sequences of images under f_i of vertices of T converge. The inequality on the distance of centers from e implies that the limits of such subsequences are different, which means that we get convergence of a subsequence of the f_i to a K-quasiconformal map f via Theorem 2.19.

We can now compute the pullback conformal structure under the maps f_i, and remembering that μ is G-invariant we obtain

$$\mu_i \stackrel{def}{=} f_i^\bullet(\mu) = (\gamma_i)^\bullet(g_i)^\bullet(\mu) = (\gamma_i)^\bullet(\mu),$$

which means that it holds

$$\mu_i(x) = \mu(\gamma_i(x)) = \mu(t_i x).$$

Loosely speaking what we have just done is a "zooming in" procedure, where our structures are obtained simply by zooming into the origin.

We now want to prove that all the elements of $G' \overset{def}{=} f^{-1}Gf$ happen to be indeed 1-quasiconformal and conclude our proof via Liouville's Theorem. It is sufficient to prove this statement locally, so we restrict to a sufficiently small round ball $B(0, R)$ centered at the origin. From the fact that μ is approximately continuous at 0 and the set equality

$$\{x \in B(R) \text{ s.t. } ||\mu_i(x) - \mu(0)|| \geq \varepsilon\} = \{x \in B(t_i R) \text{ s.t. } ||\mu(x) - \mu(0)|| \geq \varepsilon\}$$

it follows that it holds

$$||\mu_i(x) - \mu(0)|| < \varepsilon$$

away from a subset $W_i \subseteq B$ of measure less than or equal to ε_i, with $\varepsilon_i \to 0$. It follows that for $x \notin W_i$ it holds

$$1 - \varepsilon < \lambda_1(x) \leq \cdots \leq \lambda_n(x) < 1 + \varepsilon,$$

where $l_i(x)$ are the eigenvalues of the normalized matrix representing $\mu_i(x)$. Thus, choosing $0 < \varepsilon < \frac{1}{2}$, we get via elementary facts the following inequalities, away from the sets W_i:

$$H_x(\mu_i) < \frac{\sqrt{1+\varepsilon}}{\sqrt{1-\varepsilon}} \leq \sqrt{1+4\varepsilon} \leq 1 + 2\varepsilon.$$

Thus, for every $g \in G$ the conjugate map $g_i' = f_i^{-1} \circ g_i \circ f$ is conformal with respect to the conformal structure μ_i, *i.e.* it is $(1 + 2\varepsilon)$-quasiconformal via the previous calculation. Since the measure of the sets W_i decrease to zero, we can apply Proposition 2.41 obtaining that each limit map $g' = f^{-1} \circ g \circ f$ happens to be $(1 + 2\varepsilon)$-quasiconformal on the ball $B(0, R)$. Since both ε and R were arbitrary we can conclude that the conjugate group $G' = f^{-1}Gf$ is composed of 1-quasiconformal maps, *i.e.* Moebius transformations via Liouville's Theorem 2.18, and the proof is complete. □

Theorem 2.27 follows immediately from the previous statement, as long as the approximate continuity hypothesis on at least a conical limit point is satisfied. The following proposition, which follows easily from Lebesgue differentiation Theorem, assures us that such hypothesis is always satisfied in our case, when the conformal structure is bounded, *i.e.* the function μ is L^∞.

Proposition 2.43. *Every L^∞ function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is approximately continuous at almost every point.*

Proof. See, for instance, [E-G, Theorem 1.37, p.58]. □

2.2.5 Quasi isometric rigidity proof

We are now left to prove Tukia's Theorem 2.2 which states that the class of uniform lattices in $\text{Isom}(\mathbb{H}^{n+1})$, with $n \geq 2$, is quasi isometrically rigid: in fact all uniform lattices are quasi isometric to $\mathbb{H}^{n+1}$ via Milnor-Svarc Lemma, so this formulation is in fact equivalent to the one we have given before.

Proof of Theorem 2.2. The original quasi action $G \curvearrowright \mathbb{H}^{n+1}$ is geometric via Lemma A.5, so we can apply Theorem B.20 obtaining that the topological action $G \curvearrowright S^n$ is a uniform convergence action. We have seen in the various technical lemmas in subsection 2.2.1 that G is a countable (since it is finitely generated) uniformly quasiconformal group of $\text{Homeo}(S^n)$, so we can apply Theorem 2.27 to find a conjugate action by Moebius transformations $G \curvearrowright S^n$, corresponding to an action by isometries $G \curvearrowright \mathbb{H}^{n+1}$. Since being a uniform convergence action is a topological property it is invariant when the actions are conjugated, so even the new action by Moebius transformations $G \curvearrowright S^n$ is a uniform convergence action, and Theorem B.20 now proves that the isometric action $\phi : G \curvearrowright \mathbb{H}^{n+1}$ is geometric. Since the action $\phi : G \curvearrowright \mathbb{H}^{n+1}$ is properly discontinuous it has to have finite kernel: if not, for any $x \in \mathbb{H}^{n+1}$, the set

$$\{g \in G \text{ s.t. } g(\{x\}) \cap \{x\} \neq \emptyset\}$$

would not be finite, hence a contradiction. Similarly if the image of ϕ had an accumulation point $f \in \text{Isom}(\mathbb{H}^{n+1})$ belonging to the image, then for any bounded ball around f the proper discontinuity condition would be violated. We have just proved that the image of $\phi : G \longrightarrow \text{Isom}(\mathbb{H}^{n+1})$ is a discrete subset and, since the kernel is finite, such a map is indeed a virtual isomorphism. The last thing we are left to prove is that the image is a uniform lattice: this follows from the fact that $\mathbb{H}^{n+1}$ is proper, so a cobounded action is in fact cocompact, and since the quotient space is compact *a fortiori* it has to have finite volume. □

Our proof fails in the case of $n = 1$, *i.e.* in the case of isometries of hyperbolic plane, mostly because both Measurable Riemann Mapping Theorem and Liouville's Theorem do not hold anymore in dimension 1. However, it remains true that Theorem B.19 still implies that the group G acts on S^1 as a uniform convergence group. It has been first proven by Tukia, Gabai, Casson and Jungreis between 1988 and 1994 that every uniform convergence group acting on S^1 is in fact a discrete cocompact group of $\mathrm{Isom}(\mathbb{H}^2)$. Another proof of this fact has been given, which however relies on the Thurston Geometrization Theorem, proved by Perelman only in the 2000's: for a sketch of the proof we refer to [D-K, Section 21.8].

The argument we have used to prove Tukia's Theorem is very general and in fact it has been adapted, by both Tukia himself and other authors, to the more general situation of classes of irreducible uniform lattices in semisimple Lie groups, which have been proved to be quasi isometrically rigid.

As last remark, we have seen that the class of uniform lattices in $\mathrm{Isom}(\mathbb{H}^{n+1})$ is quasi isometrically rigid, but we have not proved that each lattice is rigid by itself, and this statement is in fact false. The situation will be very different in the case of nonuniform lattices of $\mathrm{Isom}(\mathbb{H}^{n+1})$ we will deal with in the next chapter, where the structure of the truncated hyperbolic spaces preserved by those groups is more rigid than that of the whole hyperbolic space, and this will allow us to achieve surprisingly stronger results.

Chapter 3

Nonuniform lattices of hyperbolic isometries

In this chapter we will deal with the problem of quasi isometric rigidity for the lattices of hyperbolic isometries we have not covered yet, *i.e.* nonuniform ones. For such lattices a stronger quasi isometric rigidity holds: our goal will be to prove, in a slightly more technical and strong formulation, this theorem, first proved by Richard Schwartz in 1996.

Theorem 3.1. *Let $n \geq 3$. Let $\Gamma < G = \mathrm{Isom}(\mathbb{H}^n)$ be a nonuniform lattice: then Γ is quasi isometrically rigid.*

In the proof of this theorem hyperbolic geometry will play a huge role: in fact the path we will follow relies entirely on the thick-thin decomposition associated to a nonuniform lattice of isometries, and on the rigidity of the behaviour of quasi isometries with respect to such decomposition.

3.1 Geometry of truncated hyperbolic spaces

3.1.1 Thick-thin decomposition

We start our discussion about nonuniform lattices with the theorem that will lead our path throughout the whole proof of rigidity. This is a well-known result from hyperbolic geometry, called thick-thin decomposition, which we state in terms of lattices of isometries.

Theorem 3.2 (Thick-thin decomposition)**.** *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ be a nonuniform lattice. There exists a collection $C \overset{def}{=} \{B_i\}$ of open horoballs with pairwise disjoint closures, whose boundary are called* peripheral horospheres*, such that* *the* truncated hyperbolic space

$$\Omega \overset{def}{=} \mathbb{H}^n \setminus \bigcup_i B_i$$

is Γ*-invariant and the quotient* $M_c \overset{def}{=} \Omega/\Gamma$*, called* thick part*, is compact. The complement in* $M = \Omega/\Gamma$ *of the thick part is called* thin part*. Moreover every parabolic element* $\gamma \in \Gamma$ *preserves exactly one horoball in* C*.*

Proof. See, for instance, [Thu]. □

The first result we will get from this decomposition is a classification of nonuniform lattices which turns out to be surprisingly simple: if fact uniformity is determined just by the absence of parabolic elements. In order to achieve this result we will need a little discussion about virtual algebraic properties of those lattices. The first one is that all hyperbolic lattices are finitely generated: this follows easily from Milnor-Svarc Lemma in both the uniform and nonuniform case with respect to the geometric action on either the entire or truncated hyperbolic space. Since hyperbolic lattices are finitely generate subgroups of $\mathrm{GL}(n+1,\mathbb{R})$, Selberg Lemma (see [D-K, Theorem 3.51, p. 72]) states that they contain torsion-free subgroups of finite-index and are indeed virtually torsion-free. Since a finite index subgroup of a lattice is a lattice itself, and passing to finite-index subgroups does not affect uniformity, we will be able to assume our lattices to be torsion-free if needed. We are now able to characterize nonuniform lattices among torsion-free hyperbolic lattices.

Proposition 3.3. *A torsion-free hyperbolic lattice* $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ *is non uniform if and only if it contains a parabolic element* $\gamma \in \Gamma$*.*

Proof. See, for instance, [D-K, Corollary 22.5, p. 517]. □

The previous proposition is a first application of the thick-thin decomposition, which however will play a fundamental role in the proof of Schwartz's Theorem. Via Milnor-Svarc Lemma it follows that if two nonuniform lattices are quasi isometric then we have an induced quasi isometry between their respective truncated hyperbolic spaces: our goal is to extend such quasi isometry to the whole hyperbolic space, in a way such that the boundary extension is Moebius, and then use this property of preservation of truncated hyperbolic space to conclude that those lattices are indeed virtually isomorphic. The case of generic groups will follow from that.

3.1.2 Coarse topology of truncated spaces

The first things we are going to study are some coarse topological and metric properties of truncated hyperbolic spaces. Given such a space Ω, with respect to a lattice Γ, we can put two different natural metrics of it: the first one, which we will denote with d, is the distance function induced by the restriction of the Riemannian metric of $\mathbb{H}^n$ to Ω. The second metric, which we will denote with dist, is simply the restriction of the hyperbolic distance on $\mathbb{H}^n$ to Ω. It is clear that Γ acts geometrically on both those metric spaces. In order to state this in a more precise manner we need a definition and a technical lemma.

Definition 3.4 (Uniformly proper map)**.** Let (X, d_X) and (Y, d_Y) be proper metric spaces. A coarse-Lipschitz map $f : X \longrightarrow Y$ is called *uniformly proper* if there exists a function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, called *upper distortion function*, such that, for all $y \in Y$ and $r \in \mathbb{R}_+$ it holds:

$$\mathrm{diam}(f^{-1}(B(y, r))) \leq \psi(r).$$

Equivalently a proper continuous function $\eta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, called *lower distortion function*, exists such that for all $x, x' \in X$ it holds:

$$d_Y(f(x), f(x')) \geq \eta(d_X(x, x')).$$

Lemma 3.5. *Let X be a proper geodesic metric spaces, and Y a metric space. Let $G \curvearrowright X$ be a geometric action, $G \curvearrowright Y$ a properly discontinuous action by isometries and $f : X \longrightarrow Y$ a G-equivariant coarsely Lipschitz map. Then f is uniformly proper.*

Proof. See, for instance, [D-K, Lemma 5.34, p. 136]. □

Lemma 3.6. *The identity map* Id : $(\Omega, d) \longrightarrow (\Omega, \mathrm{dist})$ *is* 1*-Lipschitz and uniformly proper.*

Proof. It is trivial to check that, for all $x, y \in \Omega$ it holds $\mathrm{dist}(x, y) \leq d(x, y)$ since dist is an infimum taken over a larger class of paths connecting x, y. Uniform properness follows from Lemma 3.5, since actions are geometric and the identity is obviously Γ-equivariant. □

We will now state and prove a couple of technical lemmas, and then the coarse topology property which distinguishes the case of hyperbolic plane, where the quasi isometric rigidity theorem fail, from the higher dimensional case.

Lemma 3.7. *The restriction of d to each peripheral horosphere is a flat metric.*

Proof. We will use the upper half-space model for $\mathbb{H}^n$ and, without loss of generality, we can assume our peripheral horosphere Σ to have footpoint at infinity, and be $\Sigma = \{x \in \mathbb{H}^n \text{ s.t. } x_n = 1\}$, so we have that it holds the inclusion $\Omega \subseteq \{x \in \mathbb{H}^n \text{ s.t. } x_n \leq 1\}$. The restriction of the hyperbolic metric to Σ is a flat metric, so we just have to prove that the distance induced by such metric is the same of the restriction of the distance d on Ω. In order to prove that let $\pi : \Omega \longrightarrow \Sigma$ be the projection given by

$$\pi(x_1, \ldots, x_{n-1}, x_n) \stackrel{def}{=} (x_1, \ldots, x_{n-1}, 1),$$

then its differential is easily seen to be

$$\mathrm{d}\pi = \begin{bmatrix} \mathrm{Id}_{n-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

A straightforward computation now gives us, for all paths $\gamma : [0,T] \longrightarrow \Omega$ and all times,

$$\begin{aligned} ||(\pi \circ \gamma)'(t)||_{\pi(\gamma(t))} = ||\mathrm{d}\pi(\gamma'(t))||_{\pi(\gamma(t))} = \sum_{i=1}^{n-1} \gamma_i'(t)^2 \leq \sum_{i=1}^{n} \gamma_i'(t)^2 \\ \leq \frac{\sum_{i=1}^{n} \gamma_i'(t)^2}{\gamma_n(t)^2} = ||\gamma'(t)||_{\gamma(t)}, \end{aligned}$$

hence for each path p in Ω between two points of Σ it holds that its lengths is greater than or equal to the length of the path $\pi \circ p$ which is contained in Σ, so the lemma is proved. □

Lemma 3.8. *For every horoball $B \subseteq \mathbb{H}^n$ the R-neighbourhood $\mathcal{N}_R(B)$ is also a horoball.*

Proof. We will use again the upper half-space model, placing the horoball at infinity with the assumptions of the previous lemma, and let π denote the same projection. It is trivial to check, writing the integral representing the lengths of a path between a point $p \in \omega$ and $x \in \Sigma$ horosphere associated to B, that π in in fact the nearest point-projection on Σ, *i.e.* it holds

$$\mathrm{dist}(p, \Sigma) = \mathrm{dist}(p, \pi(p)) = -\log(p_n),$$

so $\mathcal{N}_R(B)$ is indeed the horoball $\left\{x \in \mathbb{H}^n \text{ s.t. } x_n \geq e^{-R}\right\}$. □

We can now state the coarse topological difference between the two dimensional and the higher dimensional case, which comes form the notion of coarse separation given in Appendix C.

Lemma 3.9. *Let $n \geq 3$. Then each peripheral horosphere $\Sigma \subseteq \Omega$ does not coarsely separate (Ω, d).*

Proof. Let Σ be the horoball at the boundary of a horosphere B. Let $R < +\infty$ and $\mathcal{N}_R^{\text{dist}}(\Sigma)$ be a metric neighbourhood of Σ in $\mathbb{H}^n$ (*i.e.* with respect to the previously defined metric dist in Ω). From Lemma 3.8 it follows immediately that $B \cup \mathcal{N}_R(\Sigma)$ is again a horoball, say B' in $\mathbb{H}^n$. We want to prove that such a horoball does not separate (in the usual sense) Ω.

First we observe that B' does not separate $\mathbb{H}^n$. Let now $x, y \in \Omega \setminus B'$ be distinct points. There exists a piecewise-geodesic path connecting those points in $\mathbb{H}^n \setminus B'$ and such a path has to be either entirely contained in Ω or can be divided in subpaths which alternately are in Ω or lie inside one of the horoballs connecting pairs of points on the boundary. In the first case we are done since $\Omega \setminus B'$ is still connected.

For the second case let us note that the intersection between a horoball and a horosphere is isometric to an Euclidean ball of dimension $n-1$. We also use here the assumption on the dimension which tells us that such a ball cannot separate $\mathbb{E}^{n-1}$ since $n-1 > 1$. From these two remarks it follow that B' cannot disconnect Ω since the subpaths inside horoballs can be replaced with paths on the horospheres, even when they intersect B'.

We can now conclude that Σ does not coarsely separate (Ω, d). Let us suppose on the contrary that there exists $R < +\infty$ such that $Y \overset{def}{=} \Omega \setminus \mathcal{N}_R^d(B)$ contains two deep components C_1, C_2. For each pair of points $x_i \in C_i$ it is not hard to see, from the definition of deep component, that there exist two continuous paths $\alpha_i : \mathbb{R}_+ \longrightarrow C_i$ with $\alpha_i(0) = x_i$ and such that

$$\lim_{t \to \infty} d(\alpha(t), \Sigma) = \infty.$$

Gives $s \in \Sigma$ it follows from Lemma 3.6 that

$$\text{dist}(\alpha(t), s) \geq \eta(d(\alpha(t), s))$$

for a proper map $\eta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$: it follows that it has to hold

$$\lim_{t \to \infty} \text{dist}(\alpha(t), \Sigma) = \infty,$$

too. It follows that there exists $T \in \mathbb{R}_+$ such that $\alpha_i(T) \notin B'$ for each i. Since $\mathcal{N}_R^d(\Sigma) \subseteq \mathcal{N}_R^{\text{dist}}(\Sigma)$ it follows that $Y \subseteq \Omega \setminus \mathcal{N}_R^d(B)$. From the fact that $\Omega \setminus \mathcal{N}_R^d(B)$ is connected, as previously shown, we reach the conclusion that x, y might be path-connected, which is absurd. The lemma is proved. □

It is clear that the previous lemma fails in the case of $n = 2$, and the proof suggest how a counterexample can be constructed. From now on we will assume $n \geq 3$.

Let now $\Gamma, \Gamma' < \mathrm{Isom}(\mathbb{H}^n)$ be quasi isometric nonuniform lattices with associated truncated hyperbolic spaces Ω, Ω', and let $f : \Omega \longrightarrow \Omega'$ be the corresponding (L, A)-quasi isometry. Our final goal is to extend such map to the whole hyperbolic space, and in order to do so we start by showing that peripheral horospheres are mapped at uniformly bounded distance from peripheral horospheres.

Proposition 3.10. *In the previous hypotheses let Σ be a peripheral horosphere for Γ. There exists a unique peripheral horosphere $\Sigma \subseteq \partial\Omega'$ within finite Hausdorff distance from $f(\Sigma)$.*

Proof. The uniqueness part follows easily from the triangle inequality for Hausdorff distance, since two horospheres cannot lay at finite distance from each other.

In order to prove existence let us first note the since $\Omega'/_{\Gamma'}$ is compact (in $\mathbb{H}^n/_{\Gamma'}$) it has finite diameter and so there exists $D < +\infty$ such that, for each $x \in \Omega'$ it holds

$$\mathrm{dist}(x, \partial\Omega') \leq D.$$

Since the horosphere Σ is isometric to $\mathbb{E}^{n-1}$ via Lemma 3.7 it has bounded geometry and is uniformly contractible: therefore it follows from Theorem C.5 that $f(\Sigma)$ coarsely separates $\mathbb{H}^n$. We now want to prove that $f(\Sigma)$ cannot coarsely separate Ω'. Otherwise let $N' \in \mathbb{R}_+$ such that $\mathcal{N}_{N'}^{\Omega'}(f(\Sigma))$ separates Ω' in two deep components C_1, C_2: let us take $\bar{x}_i \in Ci \cap f(\Omega)$, which exist since the image of a quasi isometry is a net, and chose it so that $\bar{x}_i = f(x_i)$. Since Σ does not coarsely separate Ω by Lemma 3.9, then if the points $\bar{x}_i$ were far enough from $f(\Sigma)$ then they would belong to the same deep component of $\Omega \setminus \mathcal{N}_N^{\Omega}(\Sigma)$. Moreover we can choose N big enough to ensure $\frac{N}{L} - A > N'$ where (L, A) are the quasi isometry constants for f. Let now α be a path connecting x_1 to x_2, parametrized by $[0, 1]$. We choose a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ with $d(\alpha(t_i), \alpha(t_{i+1})) < K$ with K small enough to obtain that $H = LK + A < N'$ and the each H-ball centered at $f(\alpha(t_i))$ is a geodesic space with respect to the induced metric. We can now join $\bar{x}_1$ and $\bar{x}_2$ via a path which between each pair $f(\alpha(t_i))$ to $f(\alpha(t_{i+1}))$ is a geodesic, contradicting the fact that $f(\Sigma)$ coarsely separates Ω'.

Let now $R < +\infty$ be such that $\mathcal{N}_R(f(\Sigma))$ separates $\mathbb{H}^n$ in two deep components, say C_1, C_2. Let us suppose, arguing by contradiction, that for each horoball B'_i in the thick-thin decomposition with respect to Γ' the following containment

$$\mathcal{N}_R^{\complement}(B'_i) \overset{def}{=} B'_i \setminus \mathcal{N}_R(\Sigma'_i) \subseteq C_1$$

holds. Then, for every $x \in \Omega'$ it holds

$$\text{dist}(x, C_1) \leq R + D$$

and it follows that indeed for each $x \in \mathbb{H}^n$ it holds

$$\text{dist}_{\mathbb{H}^n}(x, C_1) \leq 2R + D,$$

and the component C_2 cannot be deep, giving a contradiction.

We are left to consider the case where there exist distinct horoballs B_1, B_2 for Ω' such that

$$\mathcal{N}_R^{\complement}(B_i') \subseteq C_i.$$

Let $\Sigma_i' = \partial B_i'$. Since $f(\Sigma)$ does not coarsely separate Ω' without loss of generality we can suppose that there exists $R' < +\infty$ such that, denoting Σ_1' by Σ', it holds

$$\Sigma \subseteq \mathcal{N}_{R'}(f(\Sigma)).$$

We are left to check that there exists $R'' < +\infty$ such that

$$f(\Sigma) \subseteq \mathcal{N}_{R''}(\Sigma').$$

The nearest-point projection $\pi : \Sigma' \longrightarrow f(\Sigma)$ gives rise to a quasi isometric embedding $h : \Sigma' \longrightarrow \Sigma$. If this is indeed a quasi isometry our assertion in proved. This fact follows from the next technical result. □

Lemma 3.11. *Let $X = \mathbb{E}^n$, i.e. $\mathbb{R}^n$ endowed with the Euclidean metric. Let $f : X \longrightarrow X$ be a (L, A) quasi isometric embedding. Then f is indeed a quasi isometry, and furthermore there exists $C = C(L, A)$ depending only on the coarse Lipschitz constant, such that $\mathcal{N}_C(f(X)) = X$.*

Proof. See, for instance, [D-K, Lemma 7.71, p.193] □

As in the case of Morse Lemma, whose analogous this proposition can be seen as, we want the distance involved in the previous lemma to be uniformly bounded, independently of Σ. Indeed this is the content of the following lemma.

Lemma 3.12. *In the above proposition there exists a constant $R(L, A)$ depending only on the quasi isometry constants such that*

$$\text{dist}_{\text{Haus}}(f(\Sigma), \Sigma') \leq R.$$

Proof. We just have to check that all the choices we have done in the proof of Proposition 3.10 depend only on the quasi isometry constants. We retain the notation of the previous proof and state that the first constant R just depends on the uniform geometry and contractibility bounds on $f(\Sigma) \cong \mathbb{E}^{n-1}$ and $\mathbb{H}^n$ which are constant or depend on the quasi isometric constants of f (see [D-K, Section 6.6] for further details). Expanding the estimates used to prove that $f(\Sigma)$ does not coarsely separate Ω' it can be seen that the inradii of shallow components of $\Omega' \setminus f(\Sigma)$ just depends on (L, A), so the choice of R' can be uniform. Finally the constant given from Lemma 3.11 depends only on the quasi isometry constant of the map h. □

3.1.3 Hyperbolic extension

We are now able, thanks to the coarse geometry results from the previous subsection, to state and prove the extension theorem for quasi isometries between truncated hyperbolic spaces we were looking for.

Theorem 3.13 (Horoball q.i. extension theorem)**.** *Let $\Gamma, \Gamma' < \mathrm{Isom}(\mathbb{H}^n)$ be nonuniform lattices associated with truncated hyperbolic spaces Ω, Ω', and let $f : \Omega \longrightarrow \Omega'$ be a quasi isometry. Such map extends to a quasi isometry $\tilde{f} : \mathbb{H}^n \longrightarrow \mathbb{H}^n$. Moreover if f is ρ-equivariant for an isomorphism $\rho : \Gamma \longrightarrow \Gamma'$ then the extension $\tilde{f}$ is ρ-equivariant, too.*

Proof. Let us denote with Σ' the only horosphere of Ω' within finite distance (depending only on quasi isometry constants) from a horosphere $\Sigma \subseteq \Omega$ arising from Proposition 3.10 and the related Lemma 3.12. By uniqueness of such a horosphere and the action of the lattice on the set of horospheres of its thick-thin decomposition C the map $\theta : C \longrightarrow C'$ mapping $\Sigma \mapsto \Sigma'$ is ρ-equivariant if so f is.

Postcomposing f_Σ with the nearest-point projection to Σ' we can alter f on $\partial\Omega$ in a ρ-equivariant way thus obtaining a new quasi isometry, which we will still denote with f, which is at finite distance from the first one, because of Lemma 3.12. This new f has the nice property of mapping horospheres to horopheres: we are now left to extend such a map to a quasi isometry on each horoball.

From now on we will use the half-space model for $\mathbb{H}^{n+1}$ and, composing with hyperbolic isometries if need be, we can suppose both horoballs B, B', bounded by horospheres Σ, Σ', to be

$$\left\{(x_1, \ldots, x_{n-1}, 1) \text{ with } (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\right\}.$$

For each vertical unit-speed geodesic ray $\rho(t)$ in B over $\rho(0)$ we define the extension $\tilde{f}$ to be

$$\tilde{f}(\rho(t)) = \rho'(t) \text{ for all } t \in \mathbb{R}_+$$

where ρ' is the vertical unit-speed geodesic ray over $f(\rho(0))$.

We want now to check that such an extension is coarse Lipschitz. Let us pick $x, y \in B$ such that $\mathrm{dist}(x,y) = 1$: it is clearly sufficient to check the coarse Lipschitz condition just for those pairs of points. If x, y belong to the same vertical ray then f acts as an isometry on them: hence, by triangle inequality it sufficient to check the condition for pairs having the same last coordinate, say h, *i.e.* they both belong to the horosphere given by:

$$\Sigma_h \stackrel{def}{=} \left\{(x_1, \ldots, x_{n-1}, h) \text{ with } (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\right\}.$$

Indeed, points belonging to the interior of the truncated hyperbolic space are mapped exactly in the same points, although the metric changes: the fact that the map is still a quasi isometry follows immediately from the fact that the two metrics are locally the same, since being a coarse Lipschitz map is a local property. Denoting $d = |x-y|_{\mathbb{R}^n}$ and computing the distance $\mathrm{dist}(x,y)$ in the hyperbolic space we get

$$1 = \mathrm{dist}_{\mathbb{H}^n}(x,y) = \operatorname{arccos}\left(1 + \frac{d^2}{2h^2}\right) = \ln\left(1 + \frac{d^2}{2h^2} + \sqrt{\frac{d^4}{4h^4} + \frac{d^2}{h^2}}\right).$$

Thus, it follows that

$$e - 1 = \frac{d^2}{2h^2}\left(1 + \sqrt{1 + a\frac{h^2}{d^2}}\right) \geq \frac{d^2}{2h^2}$$

hence we get

$$d \leq h\sqrt{2(e-1)}.$$

It is now easy to check that, denoting with x_0, y_0 the projection of x and y to Σ it holds

$$\mathrm{dist}_\Sigma(x_0, y_0) \leq h\sqrt{2(e-1)}$$

and so we get

$$\mathrm{dist}(f(x_o), f(y_0) \leq Lh\sqrt{2(e-1)} + A$$

where (L, A) are the coarse Lipschitz constants for f: it follows immediately that

$$\mathrm{dist}(\tilde{f}(x), \tilde{f}(y)) \leq \mathrm{dist}_{\Sigma_h}(\tilde{f}(x), \tilde{f}(y)) \leq \frac{Lh\sqrt{2(e-1)} + A}{h} \leq L\sqrt{2(e-1)} + A$$

and we are done. Called f' the quasi-inverse of f then $\tilde{f}'$ is the quasi inverse for $\tilde{f}$, so $\tilde{f}$ is a quasi isometry. The equivariance property follows from the equivariance of f. □

The map $\tilde{f}$ we have just constructed admits a quasiconformal extension $h : \partial_\infty \mathbb{H}^n \longrightarrow \partial_\infty \mathbb{H}^n$ by Theorem 2.20. From continuity of boundary extension it follows that, if f is ρ-equivariant so is h. Moreover, if Λ, Λ' denote the set of footpoints of peripheral horospheres, continuity of h easily implies that $h(\Lambda) = \Lambda'$.

Our main goal is now to show that such an extension h is Moebius, *i.e.* h is the extension at infinity of a global hyperbolic isometry. The proof of this fact is the core of our argument, and will be done by contradiction in the next chapters. Let us begin with the statement we are going to contradict in the case of h hyperbolic extension as above.

Proposition 3.14. *Let assume that the previously defined h is not Moebius. Then there exists a quasi isometry $F : \Omega \longrightarrow \Omega'$ with extension to the sphere at infinity which is a linear map that is not a similarity, i.e. it does not belong to the group $\mathbb{R} \cdot O(n-1)$.*

In order to prove the theorem above we will need a compactness result, which is a coarse version of the Ascoli-Arzelà Theorem. Before stating it, we need a definition.

Definition 3.15. Let X be a proper metric space, Y a metric space. A sequence of maps $(f_i) : X \longrightarrow Y$ is said to *coarsely uniformly converge on compacts* to a map $f : X \longrightarrow Y$ if there exists a constant $R < +\infty$ such that for every compact subset $K \subseteq X$ there exists an index i_K so that for all $i > i_K$ the inequality

$$d(f_i(x), f(x)) \leq R$$

holds for all $x \in K$.

We can now state the coarse Ascoli-Arzelà Theorem, and then proceed to the proof of the proposition.

Theorem 3.16. *Let D be a fixed real positive number. Let X, Y be proper metric space such that X admits a separated net. Let $(f_i) : X \longrightarrow Y$ be a sequence of coarse Lipschitz maps such that there exist two points x_0, y_0 with the property*

$$d(f_i(x_0), y_0) \leq D.$$

Then there exists a subsequence of (f_i) coarsely converging uniformly on compacts to a coarse Lipschitz map (with possibly different constants) $f : X \longrightarrow Y$. Furthermore if all the f_i are (L, A) quasi isometries then the limit map is still a quasi isometry (possibly with different constants).

Proof. See, for instance, [D-K, Prop. 5.26, p. 133]. $\square$

Proof of Proposition 3.14. Since a quasiconformal map is differentiable almost everywhere by Theorem 2.15, there exists a point $\xi \in S^{n-1} \setminus \Lambda$ in the ideal boundary of $\mathbb{H}^n$ which is not the footpoint of a horoball in the thick-thin decomposition, and such that h is differentiable in ξ and $Dh(\xi) \in \mathrm{GL}(n-1)$ is invertible but not a similarity. Indeed if Dh was a similarity almost everywhere then h would be 1-quasiconformal and indeed Moebius, by Liouville's Theorem 2.18. The points where Dh is not a similarity form a set of positive measure, and from Corollary 2.16 it follows that the Jacobian does not vanish on a positive-measure subset: this provides us the existence of a point ξ with the properties we were looking for. Composing f with isometries of $\mathbb{H}^n$ if need be, we can assume that $\xi = h(\xi) = 0 \in \mathbb{R}^{n-1}$.

We now use once again a "zooming in" argument. Let L be the vertical geodesic having the point ξ at infinity. Since ξ is not the footpoint of a horoball, there exists a sequence $(z_i) \in \Omega \cap L$ converging to ξ. Let us now define again, for $t > 0$, the dilatation maps $\gamma_t \in \mathrm{Isom}(\mathbb{H}^n)$ given by Euclidean dilatations $\gamma_t(x) = tx$ in the upper half-space model. Let (t_i) be such that $z_i = \gamma_{t_i}(z_1)$. Let us set

$$\tilde{f}_i = \gamma_{t_i}^{-1} \circ \tilde{f} \circ \gamma_{t_i}$$

and let us remark that the boundary extensions of those function are given by

$$h_i(z) = \frac{h(t_i z)}{t_i}.$$

From the definition of differentiability it immediately follows that

$$\lim_{i\to\infty} h_i = \mathrm{D}\, h(0) \overset{def}{=} A \in \mathrm{GL}(n-1)$$

where the convergence is intended to be uniform on compact sets in $\mathbb{R}^{n-1}$.

We now want to prove that there exists a subsequence of $(\tilde{f}_i)$ converging to a quasi isometry of $\mathbb{H}^n$. In order to do so we will make use of the coarse Ascoli-Arzelà Theorem 3.16. We want to show that a subsequence of $(\tilde{f}_i(z_1))$ is bounded in $\mathbb{H}^n$. Let L_1, L_2 be a couple of distinct geodesics in $\mathbb{H}^n$ through z_1 with distinct endpoints different from ∞. Then we can apply Morse Lemma B.5 obtaining that the quasi geodesics $\tilde{f}_i(L_j)$ are within uniformly bounded distance from geodesics denoted with $L^*_{1,i}, L^*_{2,i}$. Those geodesics subconverge to geodesics L^*_1, L^*_2: indeed their endpoints must subconverge in S^{n-1} due to compactness, and the fact that A, which is the limit of the maps on the boundary, is injective and fixes ∞ implies that the four limits of the endpoints must remain distinct from each other and we obtain distinct geodesics from the process. If the sequence $(\tilde{f}_i(z_1))$ were unbounded then the limit geodesics L^*_1, L^*_2 would have a common endpoint at infinity, which is a contradiction.

Having verified the hypotheses of the coarse Ascoli-Arzelà Theorem we can now assume, up to passing to subsequences, that $(\tilde{f}_i)$ coarsely converge to a quasi isometry $f_\infty : \mathbb{H}^n \longrightarrow \mathbb{H}^n$; anyway nothing implies that f_∞ has to send Ω to Ω'. Since the quotient spaces Ω/Γ and Ω'/Γ' are compact there exist sequences $(g_i) \in \Gamma$ and $(g_i') \in \Gamma'$ such that both sequences $(g_i(z_i))$ and $(g_i'(\tilde{f}(z_i)))$ belong to a compact subset of $\mathbb{H}^n$. It follows that the sequences given by

$$\beta_i \stackrel{def}{=} \gamma_{t_i}^{-1} \circ g_i^{-1}$$

and

$$\beta_i' \stackrel{def}{=} \gamma_{t_i}^{-1} \circ g_i'^{-1}$$

are precompact in $\mathrm{Isom}(\mathbb{H}^n)$ and so they subconverge to isometries

$$\beta_\infty, \beta_\infty' \in \mathrm{Isom}(\mathbb{H}^n).$$

Defining the sets

$$\Omega_i \stackrel{def}{=} \gamma_{t_i}^{-1}\Omega = \beta_i \Omega$$

and

$$\Omega_i' \stackrel{def}{=} \gamma_{t_i}^{-1}\Omega' = \beta_i'\Omega'$$

it follows that we can define the restriction $\tilde{f}_i : \Omega_i \longrightarrow \Omega_i'$.

Let us remark that the just defined spaces Ω_i and Ω_i' are isometric copies of Ω and Ω', respectively. Those sets subconverge in the Chabauty topology to subsets $\beta_\infty\Omega$ and $\beta_\infty'\Omega'$ respectively, and those spaces are still isometric copies of Ω and Ω'. The map $\tilde{f}_\infty$ is now a quasi isometry between $\beta_\infty\Omega$ and $\beta_\infty'\Omega'$, and our assertion follows. □

In order to show that a linear map different from a similarity cannot be the extension at infinity of a quasi isometry of a truncated hyperbolic space we will show that such a map admits, for each peripheral horosphere Σ with footpoint different from ∞, a sequence of peripheral horospheres Σ_k at uniformly bounded distance from Σ but such that

$$\lim_i \mathrm{dist}(\Sigma', \Sigma_i') = +\infty,$$

where Σ' is the horosphere $f(\Sigma)$ is at bounded distance from. Indeed this easily proves that f cannot even be coarse Lipschitz.

The proof of the contradiction we have just outlined will take the next two subsections, and will require the introduction of the machinery of Inverted linear mappings, and a very technical statement known as Scattering Lemma. In the end the result which will follow is Theorem 3.23.

3.1.4 Inverted linear mappings

We will now introduce a tool which lets us distinguish similarity maps. This will prove itself useful in finding the contradiction we have outlined at the end of the previous subsection. From now on let $A \in \mathrm{GL}(n-1, \mathbb{R})$ and J denote the unit sphere inversion

$$J(x) = \frac{x}{||x||^2},$$

which we recall to have order two.

Definition 3.17 (Inverted linear mappings)**.** An *inverted linear mapping* is the conjugation of a linear automorphism with the inversion J, extended by mapping the origin to itself

$$h \stackrel{def}{=} J \circ A \circ J.$$

It holds the equivalent formula

$$h(x) = \frac{||x||^2}{||Ax||^2} Ax.$$

We will now prove a couple of lemmas about such functions, which will be useful later.

Lemma 3.18. *The function $\phi(x) = \frac{||x||^2}{||Ax||^2}$ is asymptotically constant,* i.e. *it holds*

$$||\nabla \phi(x)|| = O(||x||^{-1}),$$

and

$$||\mathrm{Hess}(\phi(x))|| = O(||x||^{-2})$$

as $||x|| \to +\infty$.

Proof. The function ϕ is a nonconstant rational function of degree 0 in the vector entries: it follows that either it is constant or its gradient has to be a rational function with degree -1 and each component of the Hessian matrix has to be rational with degree -2. □

Despite the previous lemma we still have to observe that the function ϕ is not constant, unless A is a similarity, which means that an inverted linear mapping is linear if and only if it is the conjugate of a similarity. Nonetheless from the previous lemma we get a way to construct a sequence of functions, built from inverted linear mappings, converging to an affine map.

Corollary 3.19. *Let $R > 0$ and h be an inverted linear mapping. Let us set $(v_i)_{i \in \mathbb{N}} \in \mathbb{R}^{n-1}$ to be be a diverging sequence in $\mathbb{R}^{n-1}$. Then the sequence of* *maps*

$$h_i(x) \overset{def}{=} h(x+v_i) - h(v_i)$$

subconverges, uniformly on $B(0,R)$, to an affine map.

Proof. We have the following equalities:

$$\begin{aligned} h(x+v_i) - h(v_i) &= \phi(x+v_i)A(x+v_i) - \phi(v_i)A(v_i) = \\ &= \phi(x+v_i)A(x) - (\phi(x+v_i) - \phi(v_i))A(v_i). \end{aligned}$$

Since the function ϕ is asymptotically constant via the previous lemma it follows that

$$\lim_{i\to+\infty} \phi(x+v_i)A(x) = cA(x)$$

for some constant $c \in \mathbb{R}$, and this convergence is uniform on each ball $B(0,R)$.

Since it holds

$$\phi(x+v_i) - \phi(v_i) = O(|v_i|^{-1}) \text{ for } i \to \infty$$

it follows that the sequence

$$(\phi(x+v_i) - \phi(v_i))A(v_i)$$

in uniformly bounded in each ball $B(0,R)$. Moreover for every partial derivative of the second order it holds

$$\begin{aligned} \frac{\partial^2}{\partial x_j \partial x_k}(\phi(x+v_i) - \phi(v_i))A(v_i) &= \frac{\partial^2}{\partial x_j \partial x_k}\phi(x+v_i)A(v_i) \\ &= O(|v_i|^{-2})A(v_i) = O(|v_i|^{-1}), \end{aligned}$$

hence the Hessians of $h_{i|B(0,R)}$ converge uniformly to zero as $i \to \infty$. □

The last lemma we are going to prove is a stronger fact about the function ϕ not being constant when A is not a similarity.

Lemma 3.20. *Let h be an inverted linear mapping not deriving from a similarity. Let G be a group of Euclidean isometries acting geometrically on $\mathbb{E}^{n-1}$. Let $v \in \mathbb{E}^{n-1}$. There exist $R > 0$ and a diverging sequence of points $(v_i) \in Gv$ such that the restrictions of the function ϕ on $B(v_i,R) \cap Gv$ are not constant for all $i \in \mathbb{N}$.*

Proof. Let $R < +\infty$ be such that

$$\bigcup_{g\in G} B\left(gv, \frac{R}{2}\right) = \mathbb{E}^{n-1},$$

which exists by coboundedness of the action. Let us suppose that the required sequence v_i does not exists: then there exists a constant $r < +\infty$ such that the restriction of ϕ to the set $B(v_i, R) \cap Gv$ is constant for each choice of points $v_i \in Y \stackrel{def}{=} Gv \setminus B(v, r)$. Since we are assuming $n \geq 3$, the dimension of the Euclidean space $\mathbb{E}^{n-1}$ is at least two, so the union of balls

$$\bigcup_{v_i \in Y_i} B(v_i, R)$$

is connected. It follows that the function ϕ is constant on $Gv \setminus B(v, r)$ since we can connect any two points there via a sequence of ball each containing at least two points of the set.

In order to conclude let us note that the set

$$\left\{ \frac{y}{|y|} \text{ with } y \in Gx \setminus B(v, r) \right\}$$

is dense in the unit sphere of $\mathbb{E}^{n-1}$. Since it holds that $\phi(\frac{y}{|y|}) = \phi(y)$ it follows that ϕ is constant on the unit sphere and is a constant function overall, *i.e.* h derives from a similarity, hence a contradiction. The proof of the lemma is now complete. □

3.1.5 Scattering Lemma

Before moving to the statement and the proof of the Scattering Lemma, which will be the key ingredient for proving the contradiction we are looking for, we will have to introduce a big amount of notation, which we will use shortly after. Retaining the previous notation we are dealing with an invertible affine $A : \mathbb{E}^{n-1} \longrightarrow \mathbb{E}^{n-1}$, which we can assume to fix the origin, *i.e.* to be an invertible linear mapping, such that $A(\Lambda) = \Lambda'$, where $\Lambda, \Lambda' \in \mathbb{E}^{n-1}$ where the sets of footpoints of peripheral horopheres in the thick-thin decompositions Ω, Ω'. After composing with translations, if need be, we can also assume that $0 \in \Lambda$ and $0 \in \Lambda'$.

With abuse of notation let us still denote with $J \in \text{Isom}(\mathbb{H}^n)$ the isometry inducing the inversion of $\mathbb{E} \cup \{\infty\}$ with respect to the unit sphere. We can now conjugate quasi isometry $\tilde{f} : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ obtaining that the map

$$J \circ \tilde{f} \circ J$$

sends $J(\Omega)$ to $J(\Omega')$. Since $\{\infty\} \in J(\Lambda) \cap J(\Lambda')$ we still have two horoballs with footpoint at infinity in the complements of $J(\Omega), J(\Omega')$ in $\mathbb{H}^n$: we will denote these horoballs with B_∞, B'_∞.

In order to simplify our notation we will now set $\Gamma = J\Gamma J$ and $\Gamma' = J\Gamma' J$. Moreover we set $\Omega = J(\Omega)$ and $\Omega' = J(\Omega')$. Furthermore we will abuse notation setting $\Lambda = J(\Lambda)$ and $\Lambda' = J(\Lambda')$. For last we will use h to identify the inverted linear map $J \circ A \circ J$.

Let Γ_∞ and Γ'_∞ be the stabilizers of the footpoint ∞ in Γ and Γ' respectively. Then Γ_∞ and Γ'_∞ act geometrically on $\mathbb{E}^{n-1}$: given $x \in \mathbb{E}^{n-1}$ let us define $h_*(x) \stackrel{def}{=} h(\Gamma_\infty x)$.

We are now ready to state and prove the Scattering Lemma.

Lemma 3.21 (Scattering Lemma)**.** *Let h be the inverted linear mapping associated to a linear automorphism A which is not a similarity. Then for each $x \in \mathbb{E}^{n-1}$ the set $h_*(x)$ is not contained in the union of finitely many Γ'_∞-orbits.*

Proof. We will argue by contradiction. Let us suppose that $h_*(x)$ is not contained in the union of finitely many Γ'_∞-orbits. Since Γ'_∞ is a subgroup of a lattice it is indeed discrete, hence for each metric ball $B = B(x, R) \subseteq \mathbb{E}^{n-1}$ the intersection

$$(\Gamma'_\infty h_*(x)) \cap B$$

has to be finite. We will show that this cannot happen.

Given a sequence $(x_i) = (\gamma_i x) \in \Gamma_\infty x$, let us denote with $R < \infty$ the constant given by Lemma 3.20, with $G = \Gamma_\infty$. Since the action $\Gamma'_\infty \curvearrowright \mathbb{E}^{n-1}$ is geometric it follows that there exists a sequence $(g'_i) \in \Gamma'_\infty$ such that the set

$$\{\gamma'_i h(x_i) \text{ with } i \in \mathbb{N}\} \subseteq \mathbb{E}^{n-1}$$

is relatively compact. We can now apply Lemma 3.20 to prove that the map h restricted to $B(x_i, R) \cap \Gamma_\infty x$ is not linear, for every index $i \in \mathbb{N}$. It follows that the maps

$$h_i \stackrel{def}{=} \gamma'_i \circ h \circ \gamma_k$$

cannot be affine maps on $B(x, R) \cap \Gamma_\infty x$. On the other hand it follows from Corollary 3.19 that a subsequence of (h_i) converges to an affine map uniformly on $B(x, R)$. This fact implies that the sequence of images of these maps restricted to $B(x, R) \cap \Gamma_\infty x$ cannot be finite, and indeed the union

$$\bigcup_{i \in \mathbb{N}} h_i(\Gamma_\infty x \cap B(0, R)) \subseteq (\Gamma'_\infty h_*(x)) \cap B$$

is an infinite set, giving us the contradiction we were looking for, and proving the lemma. □

We are now ready to prove that the extensions of quasi isometries of truncated hyperbolic spaces are indeed similarities.

Theorem 3.22. *Let h be an inverted linear mapping which does not arise from a similarity. Then h admits no quasi isometric extension to a map from a truncated hyperbolic space to another.*

Proof. Let $B \neq B_\infty$ be a horoball in the thick-thin decomposition of Ω, with footpoint x. It follows from the Scattering Lemma 3.21 that h_* cannot be contained in the union of finitely many Γ'_∞-orbits. Hence there exists a sequence $(\gamma_i) \in \Gamma_\infty$ such that the Γ'_∞-orbits of the $h(\gamma_i x)$ are all distinct from each other. Since the action $\Gamma'_\infty \curvearrowright \mathbb{E}^{n-1}$ is geometric it follows that there exists a sequence (γ'_i) such that all the points $x_i \stackrel{def}{=} \gamma'_i h(\gamma_i x)$ belong to a compact subset $K \subseteq \mathbb{E}^{n-1}$: we remark that these points are all distinct from each other for the assumption on their Γ'_∞-orbits.

For each $i \in \mathbb{N}$ let us denote with B'_i the complementary horoball in the thick-thin decomposition of Ω' whose footpoint is x_i. If there exists a subsequence of B'_i with Euclidean diameter bounded from below then, since footpoints were contained in a compact set of $\mathbb{E}^{n-1}$, those distinct horoballs will have to eventually intersect, which is absurd. It follows that the Euclidean diameter of those horoball must converge to zero, as $i \to \infty$.

Let now B_i be the complementary horoball of Ω with footpoint at $\gamma_i x$. Since $\gamma_i \in \Gamma_\infty$ it follows that

$$\operatorname{dist}(B_i, B_\infty) = \operatorname{dist}(B_1, B_\infty) \stackrel{def}{=} D$$

while it also holds that

$$\operatorname{dist}(B'_i, B'_\infty) = C - \log(\operatorname{diam}_{Eucl}(B'_i)) \to +\infty.$$

If $f : \Omega \longrightarrow \Omega'$ were a (L, A) quasi isometry between truncated hyperbolic spaces whose quasiconformal extension is the map h in our hypothesis, then it would follow from Lemma 3.12 that

$$\operatorname{dist}(B'_i, B'_\infty) \leq R(L, A) + LD + A,$$

giving a contradiction and proving the theorem. □

The previous theorem directly contradicts Proposition 3.14 and in the end we have obtained the following result we will need to prove rigidity theorems.

Theorem 3.23. *Let $n \geq 3$. Let $f : \Omega \longrightarrow \Omega'$ be a quasi isometry between truncated hyperbolic spaces. Then f admits a unique extension to $\partial_\infty \mathbb{H}^n$, which is Moebius. In particular such an extension is within finite distance from a unique hyperbolic isometry.*

Proof. We are left to prove uniqueness: this follows from the fact that f determines the homeomorphism at infinity on the dense subset $S^{n-1} \setminus \Lambda$, alongside Lemma B.13. □

3.2 Rigidity theorems

3.2.1 Isometries almost preserving truncated spaces

We are approaching the precise statement and the proof of Schwartz's Theorem.

It is trivial to see that commensurability implies virtual isomorphism. The converse does not hold true in general, but it does in particular cases, one of those being that of nonuniform hyperbolic lattices. From our statement of Schwartz's Theorem it will be clear that the relation between two quasi isometric nonuniform lattices of hyperbolic isometries is indeed commensurability, so we obtain a little bit stronger property than virtual isomorphism. This introduction should explain why, in the following statements, we will look for commensurability results, which are *a priori* stronger than virtual isomorphism ones.

Definition 3.24. Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ be a nonuniform lattice with associated truncated hyperbolic space Ω. A subset $H \subset \mathrm{Isom}(\mathbb{H}^n)$ is said to *almost preserve* Ω is there exists a positive constant $C < +\infty$ such that, for all $h \in H$, it holds:

$$\mathrm{dist}_{\mathrm{Haus}}(\Omega, h\Omega) \leq C.$$

Let us start by proving the following lemma about sequences of isometries almost preserving a truncated space.

Lemma 3.25. *Let $(g_i) \in \mathrm{Isom}(\mathbb{H}^n)$ be a converging sequence almost preserving a truncated hyperbolic space Ω. Then such a sequence has to be definitively constant.*

Proof. We will argue by contradiction. Without loss of generality we can assume the limit of the sequence to be the identity. Let us suppose on the contrary that the sequence (g_i) consists of infinitely many distinct elements. It follows that, passing to a subsequence if need be, there exists a footpoint ξ of a peripheral horosphere Σ such that the sequence $(\xi_i) \stackrel{def}{=} g_i(\xi)$ consists of elements distinct from each other. The sequence of horospheres $\Sigma_i \stackrel{def}{=} g_i(\Sigma)$ converges in the Chabauty topology to Σ. Since the sequence g_i is almost preserving Ω there exists a sequence of complementary horospheres for Ω, denoted by $\tilde{\Sigma}_i$, and a constant $C < +\infty$ such that

$$\mathrm{dist}_{\mathrm{Haus}}(\Sigma_i, \tilde{\Sigma}_i) \leq C$$

for all $i \in \mathbb{N}$. Hence the horospheres $\tilde{\Sigma}_i$ will definitely have to intersect Σ: on the other hand the footpoints of those horospheres, which we remark to be the same of the Σ_i, are all distinct from each other, and so the horospheres Σ and $\tilde{\Sigma}_i$ of the thick-thin decomposition of Ω are different from each other but have to intersect, which is a contradiction. □

We will now use the previous lemma in order to prove the following proposition, which will be the main tool used in the proof of Schwartz's Theorem.

Proposition 3.26. *Let* $\Gamma, \Gamma' < \mathrm{Isom}(\mathbb{H}^n)$ *be nonuniform lattices such that* Γ' *almost preserves the truncated hyperbolic space* Ω *associated to* Γ*. Then* Γ *and* Γ' *are commensurable.*

Proof. We will prove the proposition by contradiction. Let us suppose that Γ and Γ' are not commensurable: then the projection of Γ' to $G/_{\Gamma}$ is infinite. Hence we can pick an infinite sequence $(\psi_i) \in \Gamma'$ with distinct projections to $G/_{\Gamma}$. We will continue to denote by Λ the set of footpoints of peripheral horospheres in the thick-thin decomposition. The set $\Lambda/_{\Gamma}$ is finite since $\Omega/_{\Gamma}$ is compact and so can have only finitely many boundary components so, after passing to a subsequence if need be, we can assume that for some horosphere Σ the footpoints of all the horospheres $\psi_i(\Sigma)$ lie in the same Γ-orbit in Λ. Hence there exist elements $\gamma_i \in \Gamma$ such that every isometry $\alpha_i \overset{def}{=} \gamma_i\psi_i$ fixes the footpoint ξ of Σ. Since elements γ_i lie in Γ they actually preserve Ω, hence the set $A \overset{def}{=} \{\alpha_i\}_{i\in\mathbb{N}}$ still almost preserve Ω, and still projects injectively to $G/_{\Gamma}$.

Working in the upper half-space model, without loss of generality we can assume ξ to be ∞: therefore the elements of A are Euclidean similarities of $\mathbb{E}^{n-1}$. Since the previously defined stabilizer Γ_∞ acts cocompactly on $\mathbb{E}^{n-1}$ there exist a constant K and a sequence $(\tau_i) \in \Gamma_\infty$ such that, after setting $\beta_i \overset{def}{=} \tau_i\alpha_i$, the inequality

$$|\beta_i(0)|_{\mathbb{E}} \leq K$$

holds for all $i \in \mathbb{N}$. Set $B \overset{def}{=} \{\beta_i\}_{i\in\mathbb{N}}$.

The set B is infinite and still almost preserves Ω and projects injectively to $G/_{\Gamma}$. The almost preservation definition gives us a constant C such that for every $\beta \in B$ it holds

$$\mathrm{dist}_{\mathrm{Haus}}(\Sigma, \beta(\Sigma)) \leq C.$$

Thus the set B is contained in the set of similarities given by

$$\left\{g(x) = \lambda U x + v \text{ with } e^{-C} \leq |\lambda| \leq e^{C} \text{ and } U \in O(n-1) \text{ and } |v| \leq K\right\}$$

which is compact. Hence the closure of B is compact and since the set was indeed infinite, we obtain a converging sequence contradicting Lemma 3.25: the proposition is now proved. □

3.2.2 Schwartz Rigidity Theorem

We are approaching Schwartz Rigidity Theorem: from the results we have already achieved regarding the geometry of truncated hyperbolic spaces and quasi isometries between them the theorem will follow in a pretty straightforward manner. We will need just another very easy construction to be able to state the theorem.

Elements of the commensurator of Γ can be seen as quasi isometries of the group on itself. Indeed, given $g \in \mathrm{Comm}_G(\Gamma)$, the Hausdorff distance between Γ or $g^{-1}\Gamma g$ and $H = \Gamma \cap g^{-1}\Gamma g$ is finite: in fact if H has index n in Γ then there exists $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that

$$\bigcup_{i=1}^{n} \gamma_i H = \Gamma.$$

It follows that in Γ it holds

$$\mathrm{dist}_{\mathrm{Haus}}(\Gamma, H) \leq \max_i d(e, \gamma_i) < +\infty,$$

and an identical statement holds true for $g^{-1}\Gamma g$. The quasi isometry induced by an element $\gamma \in \mathrm{Comm}_G(\Gamma)$ is given by the conjugation via γ postcomposed with the nearest-point projection on H.

We are now ready to state and prove the rigidity theorem we have worked for throughout this whole chapter.

Theorem 3.27 (Schwartz Rigidity Theorem)**.** *Let $n \geq 3$. From now on let $\Gamma < G = \mathrm{Isom}(\mathbb{H}^n)$ be a nonuniform lattice. Then the following statements hold:*

1. *Each quasi isometry $f : \Gamma \longrightarrow \Gamma$ is within finite distance from a quasi isometry f induced by an element $\gamma \in \mathrm{Comm}_G \Gamma$. Such distance depends only on Γ and the quasi isometry constants of f;*
2. *Let Γ' be a nonuniform lattice quasi isometric to Γ. Then there exists $g \in G$ such that Γ' is commensurable to $g^{-1}\Gamma g$;*
3. *Let Γ' be a finitely generated group quasi isometric to Γ. Then Γ is virtually isomorphic to Γ'.*

Proof. Let us prove part 1. From Theorem 3.23 the quasi isometry f extends to a quasi isometry of the entire hyperbolic space within finite distance from an isometry $\alpha \in \mathrm{Isom}(\mathbb{H}^n)$. Such distance only depends on the lattice Γ and the quasi isometry constants of f via Lemma B.13. We are left to prove that $\alpha \in \mathrm{Comm}_g(\Gamma)$: in order to prove this it is enough to observe that the lattice

$$\Gamma' \stackrel{def}{=} \alpha\Gamma\alpha^{-1}$$

almost preserves Ω, and then apply Proposition 3.26.

The proof of part 2 is analogous to the previous one. The quasi isometry $f : \Omega \longrightarrow \Omega'$ arising via Milnor-Svarc Lemma from the quasi isometry between Γ and Γ' lies within finite distance from an isometry $\alpha \in \mathrm{Isom}(\mathbb{H}^n)$ because of Theorem 3.23. The lattice

$$\Gamma'' \stackrel{def}{=} \alpha\Gamma'\alpha^{-1}$$

almost preserves Ω and hence is commensurable to Γ via Proposition 3.26.

Let us now prove part 3. Let $f : \Gamma' \longrightarrow \Gamma$ be a quasi isometry with quasi inverse $f' : \Gamma \longrightarrow \Gamma'$. We can conjugate the left multiplication action of Γ' on itself via Lemma A.5 obtaining a quasi action $\rho : \Gamma' \curvearrowright \Omega$. According to part 1 each quasi isometry $\rho(\gamma) \stackrel{def}{=} g$ is induced by an element $g^* \in \mathrm{Comm}_G(\Gamma)$. We claim that the map $\psi : \Gamma' \longrightarrow \mathrm{Comm}_G(\Gamma)$ given by

$$\gamma \mapsto \rho(\gamma)^* \in \mathrm{Comm}_G(\Gamma)$$

is a group homomorphism with finite kernel. From Lemma A.5 it follows that the quasi action $\rho : \Gamma' \curvearrowright \Omega$ is geometric; hence the kernel K of the induced action at infinity $\psi_\infty : \Gamma' \longrightarrow \mathrm{Mob}(S^{n-1}) \cong \mathrm{Isom}(H^n)$ is finite due to Lemma B.13. The conclusion of the proof resembles those of the previous statements: the group

$$\psi(\Gamma') = \psi(\Gamma'/K) \simeq_{v.i.} \Gamma'/K \simeq_{v.i.} \Gamma'$$

almost preserves Ω, and we conclude via Proposition 3.26 just as before. □

Schwartz's Theorem proves not only a strong rigidity property for nonuniform hyperbolic lattices, but also that among those groups the notions of commensurability and virtual isomorphism coincide, which is in general very far from being true.

3.2.3 Mostow Rigidity Theorem

One of the most celebrated results of hyperbolic geometric is Mostow Rigidity Theorem, which states that for complete finite volume hyperbolic manifolds of dimension at least 3 the metric is determined just by the topology, in particular by the fundamental group. This result, which is another proof of how rigid the hyperbolic structure on a manifold is, has been proven for the first time by

Mostow in 1968 (at least for compact manifolds) in a way that has nothing to do with geometric group theory. Nowadays several different proofs of the theorem, relying on various techniques, have been given: we will state and prove it by using the tools developed in the previous chapters, and this can be seen as a witty application of the geometric properties we have used in this thesis.

Theorem 3.28 (Mostow Rigidity Theorem)**.** *Let $n \geq 3$. Let the two groups $\Gamma, \Gamma' < G = \mathrm{Isom}(\mathbb{H}^n)$ be lattices and $\rho : \Gamma \longrightarrow \Gamma'$ be an isomorphism. Then ρ is induced by a hyperbolic isometry, i.e. there exists $h \in \mathrm{Isom}(\mathbb{H}^n)$ such that*

$$h \circ \gamma = \rho(\gamma) \circ h$$

for all $\gamma \in \Gamma$.

Before approaching the proof of the theorem let us state a couple of technical lemmas we will use throughout the proof.

Lemma 3.29. *There exists a ρ-equivariant quasi isometry $f : \mathbb{H}^n \longrightarrow \mathbb{H}^n$.*

Proof. Let Ω and Ω' the truncated hyperbolic spaces associated with the lattices Γ and Γ' respectively, and if a lattice is uniform we associate the whole hyperbolic space to it. We can observe that a uniform lattice cannot be isomorphic to a nonuniform one. Indeed a uniform lattice is hyperbolic and hence cannot contain a non cyclic free Abelian group, whereas in a nonuniform lattice the stabilizers of peripheral horospheres in the thick-thin decomposition contain free Abelian subgroups of rank $n-1 \geq 2$. Our lemma now follows directly from Lemma A.6 combined, in the nonuniform case, with Theorem 3.13. □

Let us remark that the previous lemma does not hold true anymore if we remove our hypothesis on the dimension being greater than or equal to 3, at least in the non uniform case, and in fact the whole Rigidity Theorem does not hold anymore.

We can now state the second technical lemma.

Lemma 3.30. *Let $g \in \mathrm{Isom}(\mathbb{H}^n)$ be such that $g(\infty) \neq 0, \infty$. Moreover let $A \in \mathrm{GL}(n-1, \mathbb{R})$ be such that $A^{-1}gA \in \mathrm{Isom}(\mathbb{H}^n)$. Then A is an Euclidean similarity, i.e. $A \in \mathbb{R}_+ \cdot O(n-1)$.*

Proof. We will use the upper half-space model for $\mathbb{H}^n$. Let us suppose, arguing by contradiction, that A is not a similarity. Let P be a hyperplane in $\mathbb{R}^{n-1}$ containing the origin but not containing $A\gamma^{-1}(\infty)$. It follows that the subset $\gamma \circ A^{-1}(P)$, since it is the image of a hyperplane under a Moebius transformation, can only be either a hyperplane or a round sphere: since it does not contain ∞ it has to be a round sphere, say Σ. Since A is not a similarity the image $A(\Sigma)$ is an ellipsoid which is not a round sphere, hence the map $A\gamma A^{-1}$ does not send planes to round spheres and hence cannot be a Moebius transformation, which is absurd. □

We are now ready to prove the Mostow Rigidity Theorem.

Proof of Theorem 3.28. Let f be the ρ-equivariant quasi isometry given by Lemma 3.29, and let us denote with $h : S^{n-1} \longrightarrow S^{n-1}$ its quasiconformal boundary extension granted by Theorem 2.20. If we show that h is indeed a Moebius transformation then h arises from an isometry $\tilde{f} : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ and we are left to prove that this isometry has to be ρ-equivariant. Indeed, since it is equivariant on the boundary sphere the two maps $\rho \circ \tilde{f}$ and $\tilde{f} \circ \rho$ lie within finite distance due to Lemma B.13. For both whole hyperbolic spaces and truncated ones the map $\mathrm{Isom}(X) \longrightarrow QI(X)$ is injective: indeed for hyperbolic spaces this is a consequence of the fact that quasi isometries correspond to quasi conformal mappings while isometries are one to one with Moebius transformations, for truncated hyperbolic spaces this is Theorem 3.23. It now follows immediately that $\rho \circ \tilde{f}$ and $\tilde{f} \circ \rho$ must coincide, hence $\tilde{f}$ is ρ-equivariant.

For nonuniform lattices Theorem 3.23 already ends the proof, anyway we will deal with nonuniform lattices alongside uniform ones for the remaining part of this proof. The argument we will use to prove that h is Moebius will resemble closely that of Proposition 3.14. It will be be once again crucial the almost-everywhere differentiability of h, and moreover we will need the hypothesis on the dimension in order to be able to apply Corollary 2.16. We will once again use the upper half-space model for the hyperbolic space.

Let $z \in S^{n-1}$ be a point where h is differentiable and its Jacobian is not vanishing, which exists by Theorem 2.15 and Corollary 2.16 combined as in the proof of Proposition 3.14. Since fixed points of parabolic elements are at most countable we can assume z not to be one of them, and by composing with a Moebius transformation we can assume $z = h(z) = 0$ and $h(\infty) = \infty$.

We will once again use a "zooming in" argument. Let $L \subseteq \mathbb{H}^n$ be the vertical geodesic with endpoint 0, and let $y_0 \in L$. Since 0 is a conical limit point for the action of Γ by Lemma B.16 there exist a sequence of elements $(\gamma_i) \in \Gamma$ such that

$$\lim_{i \to +\infty} \gamma_i(y_0) = z,$$

and a constant $C < +\infty$ such that the inequality

$$\mathrm{dist}(\gamma_i(y_0), L) \leq C$$

holds for each $i \in \mathbb{N}$. Let us denote with y_i the nearest point projection of $\gamma_i(y_0)$ on L: let λ_i be the infinitesimal sequence such that $y_i = \lambda_i y_0$. Let $g_i \in \mathrm{Isom}(\mathbb{H}^n)$

be the hyperbolic isometries given by $g_i(x) = \lambda_i x$. The sequence of $k_i \stackrel{def}{=} \gamma_i^{-1} \circ g_i$ is relatively compact in $\mathrm{Isom}(\mathbb{H}^n)$, so after passing to a subsequence, if needed, we can assume the k_i to converge to a certain $k \in \mathrm{Isom}(\mathbb{H}^n)$.

Let us now define the sequence of quasiconformal mappings of $\mathbb{R}^{n-1}$ given by

$$h_i(x) \stackrel{def}{=} \lambda_i^{-1} h(\lambda_i x) = (g_i)_\infty{}^{-1} h(g_i)_\infty(x).$$

From the differentiability of h in 0, along with the fact that the Jacobian is not zero, it follows that there exists a linear transformation $A \in \mathrm{GL}(n-1, \mathbb{R})$ such that

$$\lim_{i \to +\infty} h_i(x) = Ax$$

for all $x \in \mathbb{R}^n$, hence from the fact that $h(\infty) = \infty$ it follows that the h_i pointwise converge to A on S^{n-1}.

It follows from our construction that the maps h_i conjugate Γ to groups $\Gamma_i \stackrel{def}{=} g_i^{-1} \Gamma g_i \in \mathrm{Isom}(\mathbb{H}^n)$. It holds

$$\Gamma_i = g_i^{-1} \Gamma g_i = (k_i^{-1} \gamma_i) \Gamma (k_i^{-1} \gamma_i)^{-1} = k_i^{-1} \Gamma k_i.$$

Therefore the sequence of sets Γ_i converges to $\Gamma_\infty \stackrel{def}{=} k^{-1} \Gamma k$ in the Chabauty topology on $\mathrm{Isom}(\mathbb{H}^n)$. Given a sequence $(\beta_i) \in \Gamma_i$ which converges to some $\beta \in \mathrm{Isom}(\mathbb{H}^n)$ it holds

$$\lim_{i \to +\infty} h_i \beta_i h_i^{-1} = A \beta A^{-1}.$$

It follows that the right hand side $A\beta A^{-1}$ is sill an isometry for each $\beta \in \Gamma_\infty$, hence

$$A \Gamma_\infty A^{-1} \subseteq \mathrm{Isom}(\mathbb{H}^n).$$

The orbit of ∞ under the action of Γ_∞ is infinite, hence Γ_∞ must contain an element γ such that $\gamma(\infty) \notin \{0, \infty\}$. We are now able to apply Lemma 3.30 to prove that h is a conformal mapping, and conclude the proof. $\square$

Appendix A

Quasi actions

Group actions play a huge role throughout the whole thesis; in this appendix we will remind some definitions and properties of actions and of their mild generalization, quasi actions.

For starters we will give the fundamental definition of geometric action, which is used several times both in the previous chapters and in the following appendix.

Definition A.1 (Geometric action)**.** Let X be a metric space and $\phi : G \curvearrowright X$ an action. This action is said to be *properly discontinuous* if for every bounded set $B \subseteq X$ the set

$$\{g \in G \text{ s.t. } \phi(g)(B) \cap B \neq \emptyset\}$$

is finite.

The action ϕ is said to be *cobounded* if there exist a point $x \in X$ a constant $R < +\infty$ such that

$$\bigcup_{g \in G} \phi(g)(B(x, R)) = X.$$

A *geometric action* $G \curvearrowright X$ is a cobounded properly discontinuous action by isometries.

When the metric space X is proper, *i.e.* its closed balls are compact, the set in the definition of properly discontinuous action can be replaced with a compact set. Moreover a cobounded action is indeed cocompact, *i.e.* the space $X/_G$ is compact.

We now want to extend our definition to include groups "acting" by quasi isometries on a metric space. We are interested in associating to each element in the group G a quasi isometry of the space x whose constants are uniformly bounded. Moreover, since quasi isometries do not have inverse maps but just quasi inverse, we cannot request our association to exactly commute with the group law, and will ask only for commutation up to finite error. These are the technical reasons we are giving the following definition.

Definition A.2 (Quasi action)**.** Let G be a group and (X, d) a metric space. An (L, A)-*quasi action* is a map $\phi : G \longrightarrow X^X$ such that:

1. For each $g \in G$ the map $\phi(g)$ is an (L, A)-quasi isometry of X;
2. It holds $d(\phi(e), \mathrm{Id}_X) \leq A$;
3. For all $g, g' \in G$ it holds $d(\phi(gg'), \phi(g)\phi(g')) \leq A$.

By abusing notation we will still denote a quasi action via $\phi : G \curvearrowright X$.

The definition of quasi action is given in a way such that it induces a true group homomorphism $G \longrightarrow QI(X)$, since maps at finite distance from each other are identified in that group, hence we have the usual commutation rule.

The notion of geometric action extends naturally to the setting of quasi actions, and can be stated this way.

Definition A.3 (Geometric quasi action)**.** Let G be a group and (X, d) a metric space. A quasi action $\phi : G \curvearrowright X$ is said to be *properly discontinuous* if for every $x \in X$ and $R \in \mathbb{R}_+$ the set

$$\{g \in G \text{ s.t. } d(x, \phi(g)(x)) \leq R\}$$

is finite.

A quasi action $\phi : G \curvearrowright X$ is said to be *cobounded* if there exist a point $x \in X$ and a constant $R < +\infty$ such that

$$\bigcup_{g \in G} B(\phi(g)(x), R) = X.$$

A quasi action is said to be *geometric* if it is both properly discontinuous and cobounded.

Given an action by isometries $G \curvearrowright X$, conjugation with an isometric mapping $X \longrightarrow Y$ let us to define a new action $G \curvearrowright Y$ which is still by isometries. It is not hard to conceive that a similar behaviour must occur while conjugating via a quasi isometry, except from the fact that we expect to obtain a quasi action on the space. This does happen indeed, and while in the isometric setting conjugation does not add any meaningful information, in the setting of quasi isometries it is a very powerful tool, other that the first and more natural way quasi actions arise. We now formalize the idea we have just given.

Definition A.4 (Conjugate quasi action)**.** Let X, Y be metric spaces, and let $f : X \longrightarrow Y$ be a quasi isometry between them with quasi inverse $\bar{f}$, and let

$\psi : G \curvearrowright Y$ a quasi action. We can define the *conjugate quasi action* $f^*(\psi)$ by the pullback

$$f^*(\psi)(g) = \bar{f} \circ \psi(g) \circ f.$$

We can now state the following lemma, proving that the map just defined is indeed a quasi action. Moreover this conjugation procedure preserves geometric quasi actions: this is not obvious *a priori*, and is a fact we have used repeatedly throughout the thesis.

Lemma A.5. *Under the above notation* $f^*(\psi) : G \curvearrowright X$ *is a quasi action. Moreover if* $G \curvearrowright Y$ *is geometric so is* $f^*(\psi)$.

Proof. See, for instance, [D-K, Lemma 5.60, p. 150]. □

We conclude this appendix with another lemma giving a nice equivariance property of geometric actions, which is used during the proof of the Mostow Rigidity Theorem.

Lemma A.6. *Let* X, X' *be proper geodesic metric spaces. Let* G, G' *be groups acting geometrically on* X *and* X' *respectively. Given a group isomorphism* $\rho : G \longrightarrow G'$ *there exists a* ρ*-equivariant quasi isometry* $f : X \longrightarrow X'$.

Proof. See, for instance, [D-K, Lemma 5.35, p. 136]. □

Appendix B

Hyperbolic metric spaces

In this section we will define the fundamental notion of a hyperbolic metric space, and state some proposition used throughout the main matter of the thesis. For references and proofs, along with further details, we will refer mostly to [D-K].

Most of the properties which characterize classical hyperbolic spaces, in particular the extension of isometries on the ideal boundary, can be generalized in the context of metric spaces. It turns out that the crucial property does not depend on the Riemannian structure, but just on the metric. We will define hyperbolicity according to Rips, which is defined for geodesic spaces; this notion can be generalized to any metric space by introducing Gromov hyperbolicity, but we will not need that.

Without further ado let us define hyperbolic spaces according to Rips.

Definition B.1. Let (X, d) be a geodesic metric space, and T a geodesic triangle with edges τ_1, τ_2, τ_3 (with subscripts considered in $\mathbb{Z}/3\mathbb{Z}$). The *thinness radius* of such a triangle is the quantity

$$\delta(T) \overset{def}{=} \max_i \left(\sup_{p \in \tau_i} d(p, \tau_{i+1} \cup \tau_{i+2}) \right).$$

A triangle T is said to be δ*-thin* if $\delta(T) \leq \delta$.

Definition B.2 (Rips' hyperbolicity)**.** Let X be a geodesic metric space. It is called δ*-hyperbolic* (in the sense of Rips) if each geodesic triangle in X is δ-thin. A space which is δ-hyperbolic for some $\delta < +\infty$ is simply said to be *(Rips) hyperbolic.*

The first thing to show is that this definition in not empty, which is the content of our next proposition.

Proposition B.3. *Let $n \geq 2$. The hyperbolic space $\mathbb{H}^n$ is Rips-hyperbolic with hyperbolicity constant* $\operatorname{arccos}(\sqrt{2})$*, uniformly in n.*

Proof. See, for instance, [D-K, Prop. 8.43, p. 212]. □

The property of being of Rips hyperbolicity is preserved under quasi isometry between geodesic spaces, so it is natural to define a group to be *hyperbolic* if so is a Cayley graph for it. From Milnor-Svarc Lemma 1.12 it follows immediately that uniform hyperbolic lattices are hyperbolic. An example of nonhyperbolic groups is easily given by free Abelian groups of rank strictly greater than one.

Once we have defined geodesics such as in Definition 1.11 it is not hard at all to loosen the definition, obtaining quasi geodesics.

Definition B.4 (Quasi geodesics)**.** Let X be a metric space. An (L, A)-*quasi geodesic* in X is an (L, A) quasi isometric embedding $\gamma : [0, T] \longrightarrow X$.

One of the main properties of hyperbolic spaces is the possibility to approximate quasi geodesics with a unique geodesic up to finite error. This property is the so called Morse Lemma, and is crucial for many arguments, including the extension at infinity that we will discuss in a moment. The proof of the Lemma is rather technical and uses the machinery of ultralimits, hence that of ultrafilters, and ultimately requires the Axiom of Choice to work.

Theorem B.5 (Morse Lemma)**.** *Let X be a δ-hyperbolic geodesic space. Then for every (L, A)-quasigeodesic α the Hausdorff distance between the image of α and a geodesic segment between the two endpoints is at most θ, where θ is a constant depending only on L, A, δ.*

Proof. See, for instance, [D-K, Theorem 9.38, p. 223]. □

An extended version of the previous theorem can be given also for geodesic rays, and complete geodesics, and is called Extended Morse Lemma.

Lemma B.6 (Extended Morse Lemma)**.** *Let X be a proper δ-hyperbolic geodesic space, and let ρ be an (L, A)-quasigeodesic ray or a complete quasigeodesic. Then there exists either a geodesic ray or a complete geodesic ρ' in X such that the Hausdorff distance between* $\operatorname{Im}(\rho)$ *and* $\operatorname{Im}(\rho')$ *is less than or equal to $\theta(L, A, \delta)$, where θ is the function given by the Morse Lemma B.5.*

Moreover there are two functions $s(t), s'(t)$ such that the following inequalities hold

$$\frac{1}{L}t - B \leq s(t) \leq Lt + B$$
$$\frac{1}{L}(t - B) \leq s'(t) \leq L(t + b)$$

with $B = A + \theta$, and such that for every t

$$\mathrm{d}(\rho(t), \rho'(s)) \leq \theta$$
$$\mathrm{d}(\rho'(t), \rho(s')) \leq \theta.$$

Proof. See, for instance, [D-K, Lemma 9.80, p. 238]. □

One of the main consequences of the Extended Morse Lemma is the property of quasi isometries to commute with nearest-point projections up to finite error.

Proposition B.7. *Let X be a δ-hyperbolic geodesic metric space and let the map $f : X \longrightarrow X$ be an (L, A)-quasi isometry. Let α be a geodesic in X and β a geodesic which is closer than $\theta(L, A, \delta)$ to α, with respect to the Hausdorff distance, then f almost commutes with the the nearest-point projections π_α, π_β, i.e.*

$$\mathrm{d}(f(\pi_\alpha(x)), \pi_\beta(f(x))) \leq C$$

for all $x \in X$, where $C = C(L, A, \delta)$ depends only on L, A, δ.

Proof. See, for instance, [D-K, Prop. 9.38, p. 239]. □

We are now ready to explain one of the main features of hyperbolic spaces, which is the extension of quasi isometries to the ideal boundary. Let us define such a boundary, in a way that is general but should remind the classical construction used for hyperbolic spaces $\mathbb{H}^n$.

Definition B.8 (Ideal boundary)**.** Let X be a geodesic metric space. Two geodesic rays ρ_1, ρ_2 are said to be *asymptotic* if the lie at finite Hausdorff distance, *i.e.* the function $d(\rho_1(t), \rho_2(t))$ is bounded.

The *ideal boundary* of X is the set of asymptotic classes of geodesic rays. We will denote it by $\partial_\infty X$. The space of geodesic rays is endowed with the natural compact-open topology: we endow the ideal boundary with the quotient topology induced by it.

It is possible to endow the entire set $X \cup \partial_\infty X$ with a topology respecting the previous ones. This is the quotient topology induced on the quotient of the compact-open topology on the set of geodesic segments and rays with initial point $p \in X$. This topology is actually independent of the choice of the basepoint, and defines the topology we are looking for; for more details see [D-K, Exercise 2.74, p. 49].

It is a well known fact regarding isometries of hyperbolic spaces $\mathbb{H}^n$, and the core of their classification, that they extend to the sphere at infinity, which is indeed the ideal boundary of hyperbolic spaces. This fact continues to hold true in a way more general framework, that of hyperbolic metric spaces. Moreover, above all it is crucial the fact that not only isometries but also quasi isometries can be extended at infinity, as the following theorem states.

Theorem B.9. *Let X, Y be two Rips-hyperbolic proper metric spaces, and let $f : X \longrightarrow Y$ be a quasi isometry. Then f admits a homeomorphic extension $f_\infty : \partial_\infty X \longrightarrow \partial_\infty X$ such that the extended function is continuous at every boundary point with respect to the topology of $\bar{X}$.*

Moreover, this extension satisfies the following functoriality properties:

1. *For every pair of quasi isometries $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ it holds*

$$(g \circ f)_\infty = g_\infty \circ f_\infty.$$

2. *For every two quasi isometries $f, g : X \longrightarrow Y$ such that* $\mathrm{dist}(f, g) < \infty$ *it holds $f_\infty = g_\infty$.*

Proof. The idea of the proof is very straightforward. Given a point $\xi \in \partial_\infty X$ this is represented by the class of a geodesic ray ρ: if we consider the quasi geodesic ray $F \circ \rho$, than the geodesic ray it stay within bounded distance, given by Extended Morse Lemma B.6, represents the image of ξ. This way we can define the map we were looking for; for all the details we refer to [D-K, Theorem 9.83, p. 239]. □

While working with ideal boundaries, it becomes natural to consider triangles in the space $X \cup \partial_\infty X$, called ideal triangles, which are defined as follows.

Definition B.10 (Ideal triangles)**.** Let X be a δ-hyperbolic geodesic metric space. A *generalized triangle* is a concatenation of three geodesics (which can be finite, half-infinite or infinite) connecting three distinct points

$$p, q, r \in \bar{X} = X \cup \partial_\infty X.$$

An *ideal triangle* is a generalized triangle with all the vertices lying on the ideal boundary $\partial_\infty X$.

It is still possible to define a notion which resembles that of center for ideal triangles, thanks to the following lemma.

Lemma B.11. *Let X be a δ-hyperbolic geodesic metric space. Then every generalized triangle in X is 5δ-thin.*

Proof. See, for instance, [D-K, Exercise 9.71, p. 236]. □

Definition B.12 (Centroids)**.** Let T be a generalized triangle, with edges τ_1, τ_2, τ_3, in a δ-hyperbolic geodesic metric space (X, d). An *R-centroid* for T is a point $p \in X$ such that, for $i = 1, 2, 3$ it holds:

$$d(p, \tau_i) \leq R.$$

A 5δ-centroid (which exists by Lemma B.11) is simply called a *centroid*.

Centroids let us to state the following result, which is an inverse of the second functoriality property of Theorem B.9. It holds for most spaces, in particular for hyperbolic spaces $\mathbb{H}^n$.

Lemma B.13. *Let X, Y be proper geodesic δ-hyperbolic metric spaces, and assume that the centroids of ideal triangles in X form a R-net. Given two maps $f, f' : X \longrightarrow Y$ which are (L, A)-quasi isometries with $f_\infty = f'_\infty$ there exists a constant $D = D(L, A, R, \delta)$ such that* $\operatorname{dist}(f, f') \leq D$.

Proof. See, for instance, [D-K, Lemma 9.86, p. 241]. □

While the topology of the ideal boundary arising from the compact-open topology is usually the one the boundary is endowed with, while considering some action at infinity it is indeed useful to define a new topology, called conical topology.

Definition B.14 (Conical topology)**.** Let X be a proper geodesic hyperbolic metric space. We say that a sequence $(x_n)_{n \in \mathbb{N}}$ converges to $p = \rho(\infty) \in \partial_\infty X$ in the *cone topology* if there exists a constant C such that the points x_n belong to a C-neighbourhood of ρ and the geodesic segments $[x_1, x_n]$ converge to a geodesic ray asymptotic to p.

The above definition leads naturally to that of conical limit points.

Definition B.15. Let X be a Rips-hyperbolic space, and $\phi : G \curvearrowright X$ a quasi-action. A point $p \in \partial_\infty X$ is called *conical limit point* for ϕ if there exists a sequence of elements $g_i \in G$ such that $\phi(g_i)(x)$ converges to p in the conical topology for all $x \in X$. In other words we are requiring that for every geodesic ray $\gamma \subset X$ which is asymptotic to p and every point $x \in X$ there is a constant $R < \infty$ such that $\lim_i \phi(g_i)(x) = p$ and $d(\phi(g_i)(x), \gamma) \leq R$ for all $i \in \mathbb{N}$.

The following proposition granting the existence of conical limit points is key to the proof of Theorem 2.2.

Lemma B.16. *Let $\phi : G \curvearrowright X$ be a cobounded quasi-action. Then every point of the ideal boundary of X is a conical limit point for ϕ.*

Proof. See, for instance, [D-K, Corollary 9.92, p. 242]. □

The last notion we will introduce in this appendix in that of convergence group, which is used during the proof of Tukia's Theorem. Let us start with some notation and then with the definition we are interested in.

Definition B.17. Given a topological space X we define $\mathrm{Trip}(X)$ the space of triplets of distinct elements of X, endowed with the subspace topology in $X \times X \times X$.

Definition B.18 (Convergence group)**.** Let X be a compact space. A subgroup $G < \mathrm{Homeo}(X)$ is said to be a *convergence group* if the naturally induced action $G \curvearrowright \mathrm{Trip}(X)$ is properly discontinuous. A convergence group is called *uniform* if the quotient space $\mathrm{Trip}(X)/_G$ is compact.

The following two technical results of convergence actions, are key ingredients to the proof of Theorem 2.2, too.

Theorem B.19. *Let X be a δ-hyperbolic proper geodesic metric space such that the centroids of ideal triangles in X form a R-net. Let $\phi : G \curvearrowright X$ be a geometric quasi-action. Then the extension $\phi_\infty : G \longrightarrow \mathrm{Homeo}(\partial_\infty X)$ is a uniform convergent action.*

Proof. See, for instance, [D-K, Theorem 9.107, p. 246]. □

Theorem B.20. *Let X be a proper δ-hyperbolic geodesic metric space such that its ideal boundary Z consists of at least three points. Let $G \curvearrowright X$ be an isometric action and $G \curvearrowright Z$ be the corresponding action on the boundary. Then the action $G \curvearrowright X$ is geometric if and only of the corresponding topological action $G \curvearrowright Z$ is a uniform convergence action.*

Proof. See, for instance, [D-K, Theorem 9.104, p. 245]. □

Appendix C

Coarse separation

Topology plays a very little role in geometric group theory. In fact quasi isometries, which are the maps we are working with, are in general very far from being continuous, and usual topological notion are not preserved by them. Sometimes some topological properties can however be useful, even though they are quite different from the ones we are used to. The results we state in this appendix regard coarse separation, which is a coarse notion which is used exclusively during the proof of Proposition 3.10. This notion of coarsely separated space replaces that of non connected topological space, but it's weaker and metric-related, since we have to "inflate" separating spaces, and will not consider components which in some sense are "small" (*i.e.* with finite inradius).

During this appendix our setting will be that of triangulated manifolds. We first give the definition of bounded geometry.

Definition C.1 (Bounded geometry)**.** A metric simplicial complex X is said to have *bounded geometry* if it is connected and there exist constant $L \geq 1$ and $N < +\infty$ such that every vertex of X is incident to at most N edges, and the length of each edge is uniformly bounded, *i.e.* it belongs to $[L^{-1}, L]$.

From now on X will be a connected simplicial complex of bounded geometry which is the triangulation on a (possibly noncompact) n-dimensional closed manifold. We will denote by W a subcomplex triangulating a submanifold, possibly with boundary. Let C_R be the complementary set of the open R-neighbourhood $\mathcal{N}_R(W)$ of W in X.

Definition C.2. For a connected component $C \subseteq C_R$ we define the *inradius*, denoted by $\text{inrad}(C)$ as the supremum of the radii of metric balls contained in C. A component is said to be *shallow* if it has finite inradius, *deep* otherwise.

We are now able to define the coarse separation property.

Definition C.3 (Coarse separation)**.** A subcomplex $W \subseteq X$ is said to *coarsely separate* X if there exists $R \in \mathbb{R}$ such that C_R has at least two deep components.

Another topological definition will allow us to state the main theorem of this appendix.

Definition C.4 (Uniform contractibility)**.** A metric cell complex X is said to be *uniformly contractible* if there exists a continuous function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that for every $x \in X^{(0)}$, denoting with $\mathcal{B}(x, r)$ the set of cells with center at distance from x less than or equal to r, the map

$$\mathcal{B}(x, r) \longrightarrow \mathcal{B}(x, \psi(r))$$

is null-homotopic.

The following theorem, which gives a sufficient condition to have a coarsely separating set, is indeed an algebraic topology statement, and the main tool used for its proof is Poincaré Duality.

Theorem C.5. *Let X, Y be two uniformly contractible simplicial complexes of bounded geometry, respectively homeomorphic to $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$. Then for each uniformly proper cellular map $f : X \longrightarrow Y$ the set $f(X)$ coarsely separates Y and, moreover, for sufficiently large $R \in \mathbb{R}_+$ the set $\mathcal{N}_R(f(X))^{\complement}$ has exactly two deep components.*

Proof. See, for instance, [D-K, Theorem 6.55, p.172]. □

Bibliography

[Ahl] L. V. Ahlfors, *Lectures on quasiconformal mappings*, second edition, University Lecture Series, vol. 38, American Mathematical Society, Providence, 2006.

[B-P] R. Benedetti, C. Petronio, *Lectures on hyperbolic geometry*, Springer-Verlag, Berlin Heidelberg, 1992.

[Bre] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer-Verlag, New York, 2010.

[Bog] V. Bogachev, *Measure theory*, Springer, 2007.

[D-K] Cornelia Druţu, Michael Kapovich, *Lectures on Geometric Group Theory*, book preview available at Professor Kapovich's web page.

[E-G] L. C. Evans, R. F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, 2015.

[Fed] H. Federer, *Geometric measure theory*, Springer, 1996.

[Fol] G.Folland, *Real analysis: modern techniques and their application*, Pure and applied mathematics, Wiley-Interscience, 1999.

[Fri] R. Frigerio, *Quasi isometric rigidity of piecewise geometric manifolds*, survey available at Arxiv.

[F-S] R.Frigerio, A. Sisto, *Characterizing hyperbolic spaces and real trees*, Geom. Dedicata **142** (2009), 139-149.

[G-R] H. Garland and M. S. Raghunathan, *Fundamental domains for lattices in ($\mathbb{R}$-)rank 1 semisimple Lie Groups*, Annals of Mathematics Second Series, Vol. 92, **2** (1970), 279-326.

[Geh] F. W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. 103 (1962), 353-393.

[G-VL] G. H. Golub, C. F. Van Loan, *Matrix computations*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, 2012.

[Hei] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001.

[Mar] B. Martelli, *An introduction to geometric topology*, Università di Pisa, 2016. Online version available at Professor's Martelli web page.

[Rat] J.G. Ratcliffe, *Foundations of hyperbolic manifolds*, Springer, 1994.

[Sch] R. Schwartz, *The quasi-isometry classification of rank one lattices*, Pubb. Math. IHES **82** (1996), 133-168.

[Thu] W. Thurston, *Three-dimensional geometry and topology*, First Edition, Princeton Mathematical Series, vol. 35, Princeton University Press, 1997.

[Tuk86] P. Tukia, *On quasiconformal groups*, J. Analyse Math. **46** (1986), 318-346.

[Tuk94] P.Tukia, *Convergence groups and Gromov's metric hyperbolic spaces*, New Zealand J. Math. **23** (1994), no.2, 457-187.

[Väi] J. Väisälä. *Quasi-Moebius maps*, J. Analyse Math. 44 (1984/85), 218-234.