ALGEBRAIC K-THEORY

Q// What is algebraic K-Theory?

- in a broad sense: use of homotopy theory (homolog. algebra) to prove results in algebra / alg. geom.
- in a strict sense: cohomology theory for rings (schemes, ...)

RINGS		AB. GROUPS
R	$\longmapsto$	$K_i(R)$ $\quad i \in \mathbb{Z}$

today: $K_0(R)$ based on projective modules

<u>Def</u> An R-module P is projective if $\forall$ surj. R-mod homomorphism $M \overset{g}{\twoheadrightarrow} N$

R-mod homomorph. $P \xrightarrow{f} N$

$\exists$ R-mod $P \to M$

$$\begin{tikzcd} & M \arrow[d, two heads, "g\,\forall"] \\ P \arrow[ur, dashed, "\exists \bar{f}"] \arrow[r, "f", "\forall"'] & N \end{tikzcd}$$

<u>EX</u> $M=0$ projective

<u>Def</u> M is called f.g. if $\exists m \in \mathbb{N}$, $x_1, \dots, x_m \in M$ s.t. $M = Rx_1 + \dots + Rx_m$

<u>EX</u> R^m is projective.

<u>Exercise</u>: M R-mod M f.g. is projective $\Leftrightarrow$ $\exists$ Q R-mod s.t. $P \oplus Q = R^m$ for some $m \in \mathbb{N}$

<u>PROPERTIES</u>: P_1, P_2 f.g. proj. $\Rightarrow$ $P_1 \oplus P_2$ f.g. proj.

$P_1 \oplus P_2 \cong P_2 \oplus P_1$

<u>Def</u> $P(R)$ = set of isomorphism classes of f.g. proj. R-modules "=" $\{\text{f.g. proj. R-modules}\}/\cong$

$P(R)$ is a unital abelian monoid.

$[P]$ iso class.

<u>Def</u> A unital abelian monoid is a set M tog with $M \times M \to M$, $x, y \mapsto x+y$ associative and unital for $0 \in M$.

<u>EX</u> $(\mathbb{N}, +, 0)$

$(\mathbb{N}, \cdot, 1)$

$\rightsquigarrow$ $P(R), +, 0$ $\quad [P]+[Q] = [P \oplus Q]$

$0 = [0]$

<u>Def</u> The GROUP COMPLETION of a unital abelian monoid M is an ab. group $M^{-1}M$ + homomorphism $M \to M^{-1}M$ of unital abelian monoids. s.t.

$$\begin{tikzcd} M \arrow[r] \arrow[dr, "\forall"'] & M^{-1}M \arrow[d, dashed, "\exists!"] \\ & A \end{tikzcd}$$

$\rightsquigarrow$ ab. group

Lemma Let M be a unital ab. monoid, "the" group completion $M^{-1}M$ of M exists and is unique (up to unique iso).

pf unique: since given by universal property.

existence: $M^{-1}M = M \times M/\sim$

$(x_1, y_1) \sim (x_2, y_2) \iff \exists z \in M$ s.t.

$x_1 + y_2 + z = y_1 + x_2 + z$

notation: $\boxed{(x,y) =: x - y}$

Def $K_0(R)$ = group completion $(P(R), +, 0)$

Ex $(\mathbb{N}, +, 0) \rightsquigarrow (\mathbb{Z}, \ldots$

Ex. $K_0(\mathbb{C}) = \mathbb{Z}$ $\qquad (P(\mathbb{C}), +, 0) \xrightarrow{\cong} (\mathbb{N}, +, 0)$

given by dimension.

- same argument for every field

Ex $K_0(\mathbb{Z}) = ?$ $\qquad$ every f.g. proj. $\mathbb{Z}$-mod. $\subseteq \mathbb{Z}^n$ and is itself free

no again $K_0(\mathbb{Z}) = \mathbb{Z}$

$\rightsquigarrow$ same argument applies for every other PID (e.g. $F[x]$, $\mathbb{Z}[i]$, ...)

because every submodule of R^n is free if R is a PID.

Ex R semi-simple ring $\rightsquigarrow$ has the property that every R-module is direct sum of simple R-module. Every simple R-module is projective

$\Rightarrow K_0(R) \cong \mathbb{Z}^m$ $\quad m = \#$ simple R-modules

FACT $I \subseteq R$ nilpotent ideal ($I^n = 0$ for some n)

then $K_0(R) \xrightarrow{\cong} K_0(R/I)$

$$P \longmapsto P/IP$$

Cor if R is artinian ring, $J \subseteq R$ jacobson radical ($J = \bigcap m$ all maximal left ideals $\subseteq R$)

R artinian $\Rightarrow J$ nilpotent an R/J semi-simple

$\Rightarrow K_0(R) = K_0(R/J) = \mathbb{Z}^m$

$\rightsquigarrow K_0(0\text{-dim. ring}) = \mathbb{Z}^m$.

Q What about $K_0(R)$ R 1-dim noetherian domain (a domain in which every nonzero prime ideal is maximal) (ex: $R = \mathbb{Z}$, $F[x]$, rings of integers in a number field)