

ALGEBRAIC K-THEORY

Q// What is algebraic K-theory?

- in a broad sense: use of homotopy theory (homolog. algebra) to solve problems in algebra / alg. geom.
- in a strict sense: cohomology theory for rings (schemes, ...)

RINGS

AB. GROUPS

$$R \longmapsto K_i(R) \quad i \in \mathbb{Z}$$

today: $K_0(R)$ based on projective modules

Def An R -module P is projective if

\exists surj. R -mod homomorphism

$$M \twoheadrightarrow N$$

R -mod homomorph. $P \xrightarrow{f} N$

$\exists R$ -mod $P \rightarrow M$

$$\begin{array}{ccc} & \exists \bar{f} \nearrow & M \\ & & \downarrow g \neq \\ P & \xrightarrow{f} & N \end{array}$$

EX $M=0$ projective

Def M is called f.g. if $\exists m \in M, \alpha_1, \dots, \alpha_n \in R$

$$\text{s.t. } M = R\alpha_1 + \dots + R\alpha_n$$

EX R^m is projective.

Exercise: M R -mod M f.g. is projective \Leftrightarrow

$\exists Q$ R -mod s.t. $P \oplus Q = R^m$ for some $m \in \mathbb{N}$

PROPERTIES: P_1, P_2 f.g. noj. $\Rightarrow P_1 \oplus P_2$ f.g. noj.
 $P_1 \oplus P_2 \cong P_2 \oplus P_1$

Def $P(R)$ = set of isomorphism classes of f.g. noj. R -modules " $=$ " {f.g. noj. R -modules} / \cong

$P(R)$ is a unital abelian monoid.

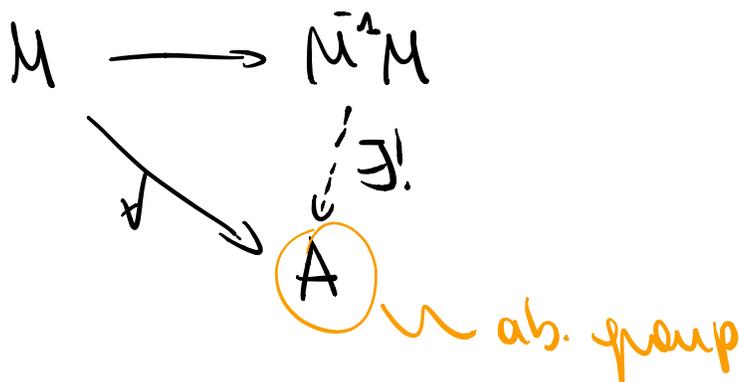
$[P]$ is class.

Def A unital abelian monoid is a set M top with $M \times M \rightarrow M$ associative and unital for $0 \in M$.
 $x, y \mapsto x+y$

EX $(\mathbb{N}, +, 0)$
 $(\mathbb{N}, \cdot, 1)$

$\rightsquigarrow P(\mathbb{N}), +, 0 \quad [P] + [Q] = [P \oplus Q]$
 $0 = [0]$

Def The GROUP COMPLETION of a unital abelian monoid M is an ab. group $M^{-1}M$ + homomorphism $M \rightarrow M^{-1}M$ of unital abelian monoids. s.t.



Lemma Let M be a unital ab. monoid, "the" group completion $N^{\times}M$ of M exists and is unique (up to unique iso).

pt unique: since given by universal property.

existence: $N^{\times}M = M \times M / \sim$

$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow \exists z \in M \text{ s.t.}$

$$x_1 + y_2 + z = y_1 + x_2 + z$$

\therefore notation: $(x, y) =: x - y$

Def $K_0(R) =$ group completion $(P(R), +, 0)$

Ex $(\mathbb{N}, +, 0) \rightsquigarrow (\mathbb{Z}, \dots)$

Ex. $K_0(\mathbb{C}) = \mathbb{Z}$ $(P(\mathbb{C}), +, 0) \xrightarrow{\cong} (\mathbb{N}, +, 0)$

given by dimension.

• same argument for every field

Ex $K_0(\mathbb{Z}) = ?$

every f.g. proj. \mathbb{Z} -mod. $\subseteq \mathbb{Z}^m$

and is itself free

no option $K_0(\mathbb{Z}) = \mathbb{Z}$

\rightsquigarrow same argument applies for every other PID (e.g. $F[x]$, $\mathbb{Z}[i]$, ...)

because every submodule of R^m is free if R is a PID.

Ex R semi-simple ring \rightsquigarrow has the property that every R -module is direct sum of simple R -module. Every simple R -module is projective

$$\Rightarrow K_0(R) \cong \mathbb{Z}^m \quad m = \# \text{ simple } R\text{-modules}$$

FACT $I \in R$ nilpotent ideal ($I^m = 0$ for some m)

$$\text{then } K_0(R) \xrightarrow{\cong} K_0(R/I)$$

$$P \longmapsto P/IP$$

Cor if R is artinian ring, $J \in R$ jacobson radical ($J = \bigcap_{\substack{\text{all maximal} \\ \in R}} \text{ ideals}$)

$$R \text{ artinian} \Rightarrow J \text{ nilpotent on } R/J \text{ semi-simple} \\ \Rightarrow K_0(R) = K_0(R/J) = \mathbb{Z}^m$$

$$\rightsquigarrow K_0(0\text{-dim. ring}) = \mathbb{Z}^m.$$

Q What about $K_0(R)$ R 1-dim noetherian domain (a domain in which every nonzero prime ideal is maximal) (ex: $R = \mathbb{Z}$, $F[x]$ rings of integers in a number field)