OVERVIEW THH e TC SEMINAR

Recall A a ring (Associative, unital, comm. not req.)

$$HH_*(A) = H_*\left(A \xleftarrow{d} A\otimes_{\mathbb{Z}} A \leftarrow A\otimes_{\mathbb{Z}} A\otimes_{\mathbb{Z}} A \leftarrow \dots\right)$$

$$a|b \mapsto ab - ba \longleftarrow a\otimes b$$

$$a|b\otimes c + \dots \longleftarrow a\otimes b\otimes c$$

$$- a\otimes bc + ca\otimes b$$

THH replaces $\otimes_{\mathbb{Z}}$ with $\underset{\mathbb{S}}{\otimes} = \wedge$

ALGEBRA	TOPOLOGY
Set	Spaces
$F \downarrow \uparrow U$	$\Sigma^\infty_+ \downarrow \uparrow \Omega^\infty$
Ab	Spectra
$\otimes$ on Ab	$\wedge$ on Spectra
$\mathbb{Z}$ is the $\otimes$-unit	$\mathbb{S}$ the $\wedge$-unit
Rings = Monoids $(Ab, \otimes)$	Ring Spectra = Monoids(Spectra, $\wedge$)

$$\begin{tikzcd} & Ch \arrow[dr] & \\ Ab \arrow[ur] \arrow[rr, "H"] & & Spectra \end{tikzcd}$$

Eilenberg Mac Lane functor

(AX-SYM LONGUN)

$$\begin{tikzcd} \text{Rings} \arrow[rr, "\mathbb{H}"] \arrow[dr] & & \text{Ring spectra} \\ & \text{DGA} \arrow[ur] & \end{tikzcd}$$

REALIZATION

$$THH(A) = \left| HA \leftleftarrows HA \wedge HA \leftleftarrows HA \wedge HA \wedge HA \ldots \right|$$

$= $ a spectrum!

$THH_*(A) = \pi_*(THH(A))$ Homotopy groups of this spectrum

<u>Idea</u>: $S \longrightarrow H\mathbb{Z}$ $\wedge = \otimes_S$ "kills off fewer relations than $\otimes_{\mathbb{Z}}$"

<u>EX</u>

$$\begin{array}{ccccc} HH_*(\mathbb{F}_p) & \longleftarrow & HH^{ch}_*(\mathbb{F}_p) & \longleftarrow & THH_*(\mathbb{F}_p) \\ \| & & \| & & \| \\ \mathbb{F}_p[0] & & \mathbb{F}_p\langle x \rangle & & \mathbb{F}_p[x] \\ & & & & |x|=2 \end{array}$$

BÖKSTEDT

$S^1 \circlearrowright THH(A)$ $\quad \pi_* S^1 \otimes THH_*(A) \longrightarrow THH_*(A)$

$TC^-(A) = THH(A)^{hS^1}$

$THH(A) \longrightarrow THH(A)^{tC_p} \rightsquigarrow TC(A)$

<u>Consequences</u>:

p-typical Witt vectors

- k perfect field of char p, A f.g. $W(k)$-Algebra

$$K(A)^\wedge_p \xrightarrow[\pi_{*\geq 0}\text{-iso}]{} TC(A)^\wedge_p$$

$$K\left(k[x]/(x^m)\right)^\wedge_p \cong W_{+m-1}(k)/V^m W_{+1}(k)$$

$$K_{even}(\quad)^\wedge_p = 0 \qquad \text{[Hesselholt–Madsen]}$$

- A commutative smooth k-algebra

$$\pi_*(THH(A)^{C_{p^\cdot-1}}) \cong W_\cdot \Omega^*_A$$

DE RHAM WITT COMPLEX

PRO-ABELIAN GROUPS

[Hesselholt]

- <u>BMS</u> [BHATT–MANEW–SCHOLZE]

$$R\Gamma_{crys}(A/W(k)) \cong R_{syn}\Gamma(A; \pi_0 TC^-(-;\mathbb{Z}_p^\wedge))$$

TALKS

① BACKgrnd on ∞-categories	OCT 16
② Ring Spectra	OCT 23
③ Def of THH(A) MARCO	OCT 30
④ THH ($\mathbb{F}_p$) JOHN	NOV 6
⑤ CYCLOTOMIC spectra and TC	NOV 13
⑥ TC($\mathbb{F}_p$)	NOV 20
⑦ and ⑧ Applications	