Ref: Weibel "The K-Book"

recall: $K_0(R)$ = grp completion of ab. monoid $(P(R), +, 0)$ of iso classes of f.g. proj. R-modules.

$K_0(\mathbb{Z})$, $K_0(\text{field}) = \mathbb{Z}$, $K_0(\text{Art. rings})$, $K_0(\text{0-dim. comm. noeth.})$

today: · K_0(1-dim. comm. rings)

· Def of K_1 and first examples.

R comm. ring

• tensor product of R-modules: M, N f. presented R-modules (e.g. f.g. proj. R-modules)

construct $M \otimes_R N$:

maps into the relations

presentation $R^m \xrightarrow{(a_{ij})} R^n \to N \to 0$

then $M \otimes N$ has presentation:

$$M^m \xrightarrow{(a_{ij})} M^n \to M \otimes N \to 0$$

Defines a map then $M \times N \to M \otimes N$

$M \times N \to M \otimes N$ is R-bilinear + universal property:

$$\begin{tikzcd} M \times N \arrow[r] \arrow[d, "\forall\ \text{R-Bilinear}"'] & M \otimes N \arrow[dl, dashed, "\exists!\ \text{R-lin. map}"] \\ P & \end{tikzcd}$$

Rmk. $M \otimes_R N \cong N \otimes_R M$

- $(M_1 \oplus M_2) \otimes N \cong (M_1 \otimes N) \oplus (M_2 \otimes N)$

- R = Field $\rightsquigarrow$ I know a basis for $M \otimes N$.

Assume from now on R is a 1-dim. comm. domain (i.e. every non-zero prime ideal is maximal).

Let $F = \mathrm{Frac}(R)$ the field of fractions

Def P: f.g. projective R-mod. The <u>RANK</u> of P is $\dim_F (F \otimes_R P) =: \mathrm{rk}(P)$

<u>properties</u>: $\mathrm{rk}(M \oplus N) = \mathrm{rk}(M) + \mathrm{rk}(N)$

$\mathrm{rk}(M \otimes_R N) = \mathrm{rk}(M) \cdot \mathrm{rk}(N)$

$\Rightarrow \mathrm{rk}: P(R) \longrightarrow \mathbb{Z}$ morphism of monoids

$P \longmapsto \mathrm{rk}(P)$

$K_0(R) \xrightarrow{\mathrm{rk}} \mathbb{Z}$

$[P]-[Q] \longmapsto \mathrm{rk}(P) - \mathrm{rk}(Q)$ well def.

$\Rightarrow$ if M, N have rank $=1$ then $M \otimes_R N$ has $\mathrm{rk}=1$

Def $\mathrm{Pic}(R)$ = isom. classes of $\mathrm{rk}=1$ proj. R-modules.

$\rightsquigarrow$ this is an abelian group under $\otimes_R$:

- $[P][Q] := [P \otimes_R Q]$

- $1 = [R]$
- $[P]^{-1} = [P^*]$ where $P^* = \mathrm{Hom}_R(P, R)$

(the dual of a one dim vector space is a 1-dim. vector space)

$$[P]^{-1}[P] = [P^* \otimes_R P] \overset{\circlearrowleft}{=} [R]$$

$$P^* \otimes P \longrightarrow R$$
$$f \otimes x \longmapsto f(x)$$

- First $P^* \otimes_R P \cong \mathrm{End}_R(P)$

$$f \otimes x \longmapsto (y \mapsto f(y) \cdot x)$$

- $$\begin{tikzcd} \mathrm{End}_R(P) \arrow[r, dashed, "\mathrm{Tr}", "\text{defined}"'] \arrow[d] & R \\ \mathrm{End}_R(P \oplus Q) = M_M(R) \arrow[u, "\mathrm{tr}"'] & \end{tikzcd}$$

Q s.t. $P \oplus Q = R^M$

- And $$\begin{tikzcd} R \arrow[r] \arrow[rr, bend right, "rk(P)"'] & \mathrm{End}_R(P) \arrow[r, "\mathrm{Tr}"] & R \end{tikzcd}$$

and if the $rk(P) = 1$ is an iso!

<u>RMK</u>: if R = ring of integers in a field,
$\mathrm{Pic}(R) = \mathcal{C}(R)$ = class group of R.

<u>EX</u> $\mathrm{Pic}(\mathbb{Z}) = 0$ $\qquad$ $\mathrm{Pic}(k[x]) = 0$ k a field

$\mathrm{Pic}(R) = 0$

$\mathrm{Pic}(\mathbb{Z}[\sqrt{-5}]) \cong \mathbb{Z}/2\mathbb{Z}$ $\ni$ $(2, 1+\sqrt{-5}) \subseteq \mathbb{Z}[\sqrt{-5}]$

Non zero in $\mathrm{Pic}(\mathbb{Z}[\sqrt{-5}])$ is not a principal ideal needs more than 1 generator.

nontrivial rk 1

P f.g. projective

$$\Lambda^m_R(P) = P \otimes P \otimes \cdots \otimes P \Big/ \begin{array}{l} R\text{-submodule} \\ \text{gen. by } a_1 \otimes \cdots \otimes a_m \\ \text{with } a_i = a_j \\ \text{for some } i \neq j \end{array}$$

<u>Facts</u>: $\mathrm{rk}\, \Lambda^m_R(P) = \binom{\mathrm{rk}\, P}{m}$

- $\Lambda^m_R(P \oplus Q) \cong \bigoplus_{i+j=m} (\Lambda^i_R P) \otimes_R (\Lambda^j Q)$
- P f.g. projective then so is $\Lambda^m_R P$

  $\Lambda^m_R P = 0$ for $m > \mathrm{rk}\, P$

set $\Lambda^{\max}_R(P) = \Lambda^{\mathrm{rk}\, P}_R(P)$

Hence
$$K_0(R) \longrightarrow \mathrm{Pic}(R)$$
$$[P]-[Q] \longmapsto (\Lambda^{\max} P) \cdot [\Lambda^{\max} Q]^{-1}$$

<u>Prop</u> if R is a 1-dim domain Noeth. then

$$K_0(R) \xrightarrow[\cong]{(\mathrm{rk}, \Lambda^{\max})} \mathbb{Z} \oplus \mathrm{Pic}(R)$$