∞-CATEGORY OF RING SPECTRA

- $x, y \in C_0$ $\Delta^1 = N(0 \to 1)$

$$\begin{tikzcd}
\mathrm{Map}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Map}(\Delta^1, C) \arrow[d, "(ev_0, ev_1)"] \\
* \arrow[r, "(x,y)"] & C \times C
\end{tikzcd}$$

- x is initial if $\mathrm{Map}(x,y) \simeq * \quad \forall y \in C_0$

- $F : C \to D$ consider diagrams

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, hook] & D \\
\Delta \star C \arrow[ur, dashed, "\hat{F}"'] &
\end{tikzcd}$$

these are the objects of an ∞-category

$C_{/F} : [n] \longmapsto \mathrm{Hom}_p(\Delta^n \star C, D)$

<u>Def</u> A limit of F is a final object of $C_{/F}$

- $Sp = \underbrace{\lim}_{Cat_\infty} \left(Top_* \xleftarrow{\Omega} Top_* \xleftarrow{\Omega} \cdots \right)$

objects : $X_0, X_1, \ldots$

$X_0 \simeq \Omega X_1$

$X_1 \simeq \Omega X_2$

⋮

Ring spectra = "Monoids in Sp"

§1 SYM. MONOIDAL ∞-CAT

$(\mathcal{C},\otimes)$ symm. monoidal 1-category (classical)

⇝ we package as follows:

$$\begin{tikzcd}
\mathrm{Fin}_* \arrow[r] & \mathrm{Cat} \\
\langle n\rangle \arrow[r, mapsto] \arrow[u, phantom, "\in" rotate=90] & \mathcal{C}^{\times n}
\end{tikzcd}$$

$=\{1,\ldots,n\}_+$

↕ Groth. CONSTRUCTION

$$\mathcal{C}^{\otimes} = \begin{cases} \text{objects: } \{\langle n\rangle, c_1,\ldots,c_n\} \\ \underline{\text{maps}}: (c_1,\ldots,c_n) \to (d_1,\ldots,d_k) \\ \quad \ldots\ldots \end{cases}$$

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \arrow[d] \\
\mathrm{Fin}_*
\end{tikzcd}$$

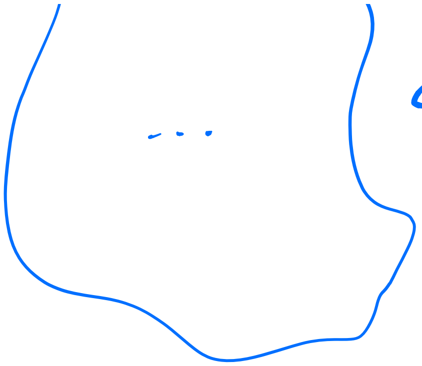
Maps of simplicial sets.

$\underline{\underline{\text{THM}}}$ [LURIE] S an ∞-category. Then

$\mathrm{Fun}(S,\mathrm{Cat}_\infty) \simeq \{ X \xrightarrow{p} S$ co-cartesian fibrations $\}$

"$s \longmapsto X_s = p^{-1}(s)$" ⟻

- . p an inner fibration
- . for every $f: s\to t$ in S and every $x \in p^{-1}(s)$, there exists



$\hookleftarrow$ $\hat{f}: x' \to y$ s.t. $p(\hat{f}) = f$ & $\hat{f}$ "a cocartesian edge"

hom condition but just for inner homs.

Def A symmetric monoidal ∞-category is a cocartesian fibration

JUST NOTATION

$$C^{\otimes} \xrightarrow{p} NFin_* \quad \text{s.t.}$$

$$C^{\otimes}_{\langle m\rangle} := p^{-1}(\langle m\rangle) \xrightarrow{\sim} (C^{\otimes}_{\langle 1\rangle})^{\times m}$$

$$\int_j : \langle m\rangle \longrightarrow \langle 1\rangle$$
$$j \longmapsto 1$$
$$\text{else} \longmapsto +$$

$C := C^{\otimes}_{\langle 1\rangle}$ is the underlying category

$$\otimes : C \times C \xleftarrow{\sim} C^{\otimes}_{\langle 2\rangle} \longrightarrow C^{\otimes}_{\langle 1\rangle} = C$$

$\rightsquigarrow$ $\langle 2\rangle \to \langle 1\rangle$

$\begin{matrix} 1 \\ 2 \end{matrix} \longmapsto 1$

$$\begin{tikzcd} \langle 3\rangle \arrow[r] \arrow[d] & \langle 2\rangle \arrow[d] \\ \langle 2\rangle \arrow[r] & \langle 1\rangle \end{tikzcd}$$

$\rightsquigarrow$ ASSOCIATIVITY

$\langle 2\rangle \xrightarrow{\cong} \langle 2\rangle$ $\rightsquigarrow$ SYMMETRY.

Ex. • $\mathcal{C}$ has (co)products

$\exists$ a symm. monoidal category

$\mathcal{C}^{\times}$ s.t. underlying category $= \mathcal{C}$ $\quad \otimes = \times$.

• $\exists$ a symmetric monoidal ∞-category

$Sp^{\otimes}$, $\otimes = \wedge$

$Sp \simeq Fun^{ex}(Top_*^{fin}, Top_*)$

$$\begin{tikzcd}
Top_*^{fin} \times Top_*^{fin} \arrow[r, "F \times G"] \arrow[d, "\wedge"] & Top_* \times Top_* \arrow[d, "\wedge"] \\
Top_*^{fin} \arrow[r, dashed, "Lan"] & Top_*
\end{tikzcd}$$

§2 ALGEBRAS IN SYMM. MONOIDAL CATEGORIES

Def A (colored) operad is an inner fibration $\mathcal{O}^{\otimes} \longrightarrow NFin_*$ s.t.

1) Every inert morphism $\langle m \rangle \xrightarrow{f} \langle n \rangle$ has a cocartesian lift $|f^{-1}(j)| = 1 \ \forall j \neq +$

2) ....

3) $\mathcal{O}^{\otimes}_{\langle m \rangle} \xrightarrow{\sim} \left(\mathcal{O}^{\otimes}_{\langle 1 \rangle}\right)^{\times m}$

Ex. comm. operad:

$Comm^{\otimes} = NFin_* \xrightarrow{Id} NFin_*$ (even cocartesian fibration.

- $\mathrm{Ass}^{\otimes} = N \begin{cases} \mathrm{ob} = \mathrm{Fin}_* \\ \mathrm{Hom}(\langle n \rangle, \langle h \rangle) = \{ f : \langle n \rangle \to \langle h \rangle \\ \quad + \text{linear orderings on each } f^{-1}(j),\ j \in \{1, \dots, h\} \} \end{cases}$

- Every symm. monoidal ∞-category

Def An $\mathcal{O}^{\otimes}$-algebra in a symm. mon. ∞-category $\mathcal{C}^{\otimes}$ is a map

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, dashed, "A"] \arrow[dr] & & \mathcal{C}^{\otimes} \arrow[dl] \\ & N\mathrm{Fin}_* & \end{tikzcd}$$

s.t. A(cocartesian lift of an inert morphism) is cocartesian.

These form an ∞-category

$$\mathrm{Alg}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}) \subseteq \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$$

Def $\mathrm{Mon}(\mathcal{C}^{\otimes}) := \mathrm{Alg}_{\mathrm{Ass}^{\otimes}}(\mathcal{C}^{\otimes})$

$E_\infty\text{-}\mathrm{Mon}(\mathcal{C}^{\otimes}) := \mathrm{Alg}_{\mathrm{Comm}^{\otimes}}(\mathcal{C}^{\otimes})$

Ex let A be a monoid in this sense:

$$\begin{tikzcd}
A: \mathrm{Ass}^{\otimes} \arrow[rr] \arrow[dr] & & C^{\otimes} \arrow[dl] \\
& N\mathrm{Fin}_{*} &
\end{tikzcd}$$

$a := A(\langle 1 \rangle)$

$A(\langle m \rangle) \in C^{\otimes}_{\langle m \rangle} \simeq C^{\times m}$

$\longmapsto (a_1, \dots, a_m)$ $\boxed{a_i \simeq a}$ uses COCARTESIAN EDGES IN $C^{\otimes}$

$\left(\begin{array}{l} \langle m \rangle \longmapsto \langle 1 \rangle \\ i \longmapsto 1 \\ + \longmapsto + \end{array} , \text{ order on } \{1, \dots, m\} \right)$

$\rightsquigarrow \boxed{\mu_m} = A(\quad) : a^{\otimes m} \longrightarrow a$

multiplication map.

Aside: Nice exercise: make sense of the homotopy coherent nerve:

C a top. category. $N^h C$ is a sSet

$$N^h C = \begin{cases} 0 \longmapsto \mathrm{ob}(C) \\ 1 \longmapsto \mathrm{Hom}(C) \\ 2 \longmapsto \\ \vdots \end{cases}$$

$$\begin{tikzcd}
& \cdot \arrow[dr, "g"] & \\
\cdot \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[r, phantom, "\iota \Downarrow"] & & \cdot
\end{tikzcd}$$

ι a homotopy

Trying to understand a functor

$N\mathbb{Z}/2\mathbb{Z} \longrightarrow N^h Top$

$\bullet \qquad\qquad X$

$\bullet \xrightarrow{\tau} \bullet \qquad\qquad X \circlearrowleft \tau$

$$\begin{tikzcd}
 & & & & & X \arrow[dr, "\tau"] & \\
\bullet \arrow[r, "\tau"] & \bullet \arrow[r, "\tau"] & \bullet & & X \arrow[ur, "\tau"] \arrow[rr, "\tau^2=Id"'] & & X
\end{tikzcd}$$

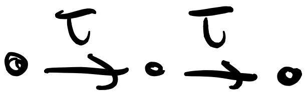
$\vdots \qquad\qquad \vdots$

just the nondegenerate simplices

0

X

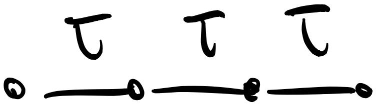
$X \circlearrowleft \tau$

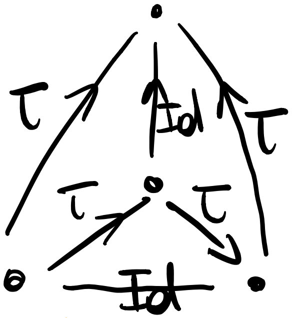


$$\begin{tikzcd}
 & X \arrow[dr, "\tau"] \arrow[d, phantom, "\Downarrow"] & \\
X \arrow[ur, "\tau"] \arrow[rr, "Id"'] & & X
\end{tikzcd}$$

⇝ homotopy comm.

⇒ $\tau^2 \simeq Id$





⇒ $\tau^3 \simeq \tau$

with homotopy commutative maps