TODAY K_1!

G group, $[G,G]$ = subg. generated by comm. $[g,h]$

RMK $[G,G] \trianglelefteq G$ normal subgroup (if I conjugate it I end in the same subgroup.)

$$G^{ab} := G/[G,G]$$ Abelianization of G (Abelian + universal)

$\leadsto$ R ring $GL_n(R)$ = group of inv. matrices

$GL_n(R) \hookrightarrow GL_{n+1}(R)$ inclusion of groups

$A \longmapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}$

Def The infinite general linear group of R is the group $GL(R) = \bigcup_{n \geq 1} GL_n(R)$

So elements of $GL(R)$ are of the form

$$\begin{pmatrix} A & & \\ & 1 & \\ & & 1 \ddots \end{pmatrix}$$ $A \in GL_n(R)$ for some n.

Def $K_1(R) = GL(R)^{Ab} = GL(R)/[GL(R), GL(R)]$

$\leadsto$ elements in $[GL(R), GL(R)]$?

$$e_{ij}(\lambda) = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \\ \begin{pmatrix} 1 & & \overset{j}{\lambda} \\ & \ddots & \\ & & 1 \end{pmatrix} \end{array} \begin{array}{l} i \\ \\ \\ \end{array}$$ $\lambda \in R$, $i \neq j$

$e_{ij}(\alpha)e_{ij}(\beta)=e_{ij}(\alpha+\beta)$

$e_{ij}(0)=1 \qquad e_{ij}(\alpha)e_{ij}(-\alpha)=1$

<u>lemma</u>: (1) $[e_{ij}(\alpha), e_{jk}(\beta)] = e_{ik}(\alpha\beta)$

in particular $e_{ik}(\alpha)\in[GL(R), GL(R)]$ $\forall\alpha\in R$ $i\neq 1$ (take $\beta=1$)

<u>Def</u> The infinite elementary linear group of R is the subgroup $E(R)$ of $GL(R)$ gen. by $e_{ij}(\lambda)$ $i\neq j$ $\lambda\in R$

note that (1) $\to E(R)\subseteq[GL(R), GL(R)]$

<u>lemma</u> (2) (Whitehead) $\forall A\in GL_n(R)$ COMPUTATION

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1}\end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -A^{-1} & 1\end{pmatrix}\begin{pmatrix} 1 & A \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\in E(R)$$

<u>proof</u>

$$\left(\begin{array}{c|c} 1 & A \\ \hline 0 & 1\end{array}\right) = \prod_{1\leq i,j\leq n} e_{i,j+n}(\alpha_{ij}) \in E(R)$$

$A=(\alpha_{ij})\in GL_n(R)$

similarly $\begin{pmatrix} 1 & 0 \\ A^{-1} & 1\end{pmatrix}\in E(R)$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}\in E(R)$$

lemma (3) in $K_1(R)$ we have $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & \\ & 0 \end{pmatrix}$
$\forall A, B \in GL_n(R)$

proof $\begin{pmatrix} A & \\ & B \end{pmatrix} \underbrace{\begin{pmatrix} B & \\ & B^{-1} \end{pmatrix}}_{\in E(R)} = \begin{pmatrix} AB & \\ & 1 \end{pmatrix}$

lemma (4) $E(R) = [GL(R), GL(R)]$

proof "$\subseteq$" ✓
"$\supseteq$" $A, B \in GL_n(R)$

$$[A, B] = \begin{pmatrix} ABA^{-1}B^{-1} & \\ & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} A & \\ & A^{-1} \end{pmatrix}}_{\in E(R)} \underbrace{\begin{pmatrix} B & \\ & B^{-1} \end{pmatrix}}_{\in E(R)} \underbrace{\begin{pmatrix} (BA)^{-1} & \\ & BA \end{pmatrix}}_{\in E(R)} \in E(R)$$

EXERCISE in $K_1(R)$ we have
$\begin{pmatrix} A & \\ & 1_n \end{pmatrix} = \begin{pmatrix} 1_n & \\ & A \end{pmatrix}$ $A \in GL_n(R)$
and also $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

§ COMPUTATIONS FOR SOME RINGS

$A \in M_n(R)$ $\quad \det(A) = \sum_{\sigma \in \Sigma_n} \operatorname{sign}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$
$(A = (a_{ij}))$

if R is commutative then
$\det(A \cdot B) = \det(A)\det(B)$,

$\det(I_n) = 1$

$\det\begin{pmatrix} A & \\ & B \end{pmatrix} = \det(A)\det(B)$

Hence if R comm then $K_1(R) \xrightarrow{\det} R^*$, $A \longmapsto \det(A)$ well defined group homomorphism!

EX $R = F$ field claim: $K_1(F) \xrightarrow[\det]{\cong ?} F^*$

recall left (Right) multiplication with $e_{ij}(\lambda)$ is an elementary row (column) operation

up to elementary row and column operations any $A \in GL_n(F)$ is of the form

$$\sigma_1 \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \sigma_2 = \quad \sigma_1, \sigma_2 \in \Sigma_n$$

$$\overset{=}{\scriptstyle K_1(F)} \begin{pmatrix} \operatorname{sign}(\sigma_1) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} a_1 \cdots a_n & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix} \begin{pmatrix} \operatorname{sign}(\sigma_2) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$\Rightarrow \begin{tikzcd} F^* \arrow[r, two heads, "\text{surj}"] \arrow[rr, bend right, "1"', "\text{(exercise)}"] & K_1(R) \arrow[r, "\det"] & F^* \end{tikzcd}$$

$\rightsquigarrow F^* \twoheadrightarrow K_1(F)$ inj. hence iso

$\Rightarrow \det : K_1(F) \xrightarrow{\cong} F^*$

EX R local comm. $\Rightarrow K_1(R) \xrightarrow[\cong]{\det} R^*$ (same proof: exercise!)

EX $K_1(\mathbb{Z}) \xrightarrow[\cong]{\det} \mathbb{Z}^* = \{\pm 1\} = \mathbb{Z}/2\mathbb{Z}$

Proof

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & & \end{pmatrix} \in GL_n(\mathbb{Z}) \Rightarrow a_{11}\mathbb{Z} + \cdots +$

wlog $|a_{11}|$ minimal among $a_{11}, \ldots a_{nn}$

$a_{1i} = c_i a_{11} + r_{1i}$

use elem. op^s to replace a_{1i} w/ r_{1i}

up $\rightsquigarrow A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \Rightarrow \mathbb{Z}^* \twoheadrightarrow K_1(\mathbb{Z}) \to \mathbb{Z}^*$ iso

Ex R euclidean domain

$K_1(R) \xrightarrow{\cong} R^*$ (same proof)

$\exists R$ PID where $K_1(R) \underset{\det}{\overset{\cong}{\to}} R^*$

Thm (Milnor Bass Serre)

R ring of integers in a field

then $K_1(R) \xrightarrow{\cong} R^*$