TODAY K_2!

Def A central extension of a group G is a surjective group homomorphism $E \xrightarrow{\phi} G$ s.t. $\text{Ker}(\phi) \subseteq Z(E)$

Rmk $x \in \text{Ker}(\phi) \subseteq Z(E)$ $xy = yx$ $\forall y \in E$
 $\Rightarrow \text{Ker}(\phi)$ is abelian group.

Def A central extension $U \xrightarrow{\phi} G$ is called universal if $\forall$ central extension $E \xrightarrow{\phi} G$
 $\exists!$ group homomorphism $U \xrightarrow{g} E$
 s.t. $\psi g = \phi$

Def A group G is called <u>PERFECT</u> if $[G, G] = G$

Ex R ring then $E(R)$ is perfect

 Proof $E(R)$ generated by commutators
 in $E(R)$: $e_{ij}(\lambda) = [e_{ik}(\lambda), e_{kj}(1)]$ $k \neq i, j$

Prop Let G be a perfect group. Then
(a) if $E \xrightarrow{\phi} G$ is a central extension,
 then $[E, E]$ perfect, $\phi[E, E] = G$
(b) G has a unique universal extension
 $U \twoheadrightarrow G$ and U is perfect.
(c) $U \twoheadrightarrow G$ universal central extension
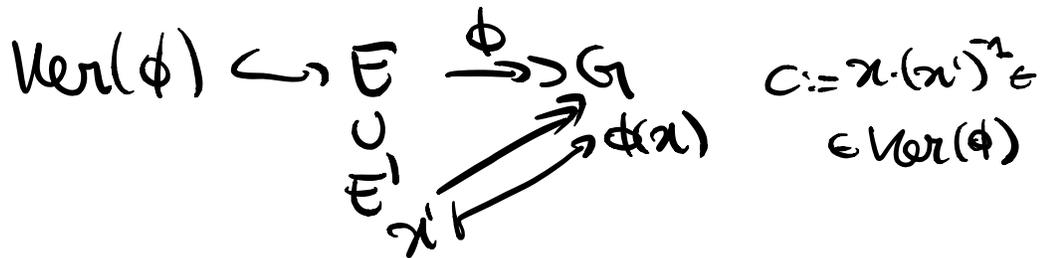 $\iff U$ perfect and every central extension

$E \xrightarrow{\phi} G$ splits $(\exists \psi : G \to E : \phi\psi = 1_G)$

Pf

(a) $\phi[E,E] = G$ $E' \subset G$ is generated by commutators (G perfect)

$E' = [E,E]$ perfect   e.c.: if $x,y \in E$, then $x = c \cdot x'$, $y = d y'$, $c,d \in \ker(\phi)$

$x', y' \in E' = [E,E]$ e.c. $\phi(E') = G$

$$\ker(\phi) \hookrightarrow E \xrightarrow{\phi} G \qquad c := x \cdot (x')^{-1} \in$$

$E' \quad x' \mapsto \phi(x) \qquad \in \ker(\phi)$

E' gen by $[x,y] = [c x', d y'] = [x', y'] \in [E', E']$

    $\uparrow$ $c, d \in Z(E)$

$\Rightarrow [E', E'] \subset E' \subset [E', E']$

(b) uniqueness by definition (universal property)

perfectness by (a)

need to show existence.

choose $F \xrightarrow{} G$ surj. group homomorphism with F a free group with kernel $R \hookrightarrow F$

$\downarrow$

G

$[R,F]$ = subgroup of F gen. by $rfr^{-1}f^{-1}$
$r \in R$, $f \in F$

$\Rightarrow F/_{[R,F]} \longrightarrow\!\!\!\!\!\to G$ well def. with kernel
$R/_{[R,F] \cap R}$

$\Rightarrow F/_{[R,F]} \longrightarrow\!\!\!\!\!\to G$ central extension.
It's commutator subgroup $[F,F]/_{[R,F]} \longrightarrow\!\!\!\!\!\to G$
perfect central extension by (a).

$\underline{\text{claim}}$: $[F,F]/_{[R,F]} \longrightarrow\!\!\!\!\!\to G$ universal
central extension of G.

$\underline{\text{proof}}$ Let $E \xrightarrow{\psi}\!\!\!\!\to G$ be a central
extension of G.
F free group $\Rightarrow \exists$ morphism $F \xrightarrow{\psi} E$
s.t.
$$F \xrightarrow{\phi}\!\!\!\!\to G$$
$$\psi \downarrow \quad \circlearrowleft \quad \nearrow \varphi$$
$$E$$
commutes

$rfr^{-1}f^{-1}$ $[R,F] \subset F$

$$\downarrow \qquad \downarrow \psi$$
$$E$$

....
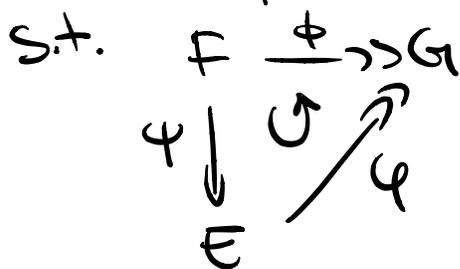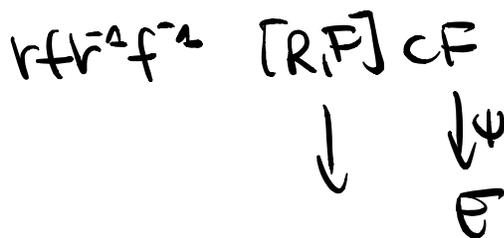
the map g is unique by
$\underline{\text{lemma}}$ G perfect, $E_1, E_2 \longrightarrow\!\!\!\!\!\to G$ central $\begin{pmatrix} \text{with} \\ E_1 \\ \text{PERFECT} \end{pmatrix}$
extension. Then $\exists$ at most one
morphism $E_1 \to E_2$ s.t.

$$E_1 \longrightarrow E_2 \quad \text{commutes.}$$

with a diagonal arrow $E_1 \to G$ and $E_2 \to G$.

proof Assume $E_1 \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} E_2$ commutes

$$\phi_1 \searrow \quad G \quad \swarrow \phi_2$$
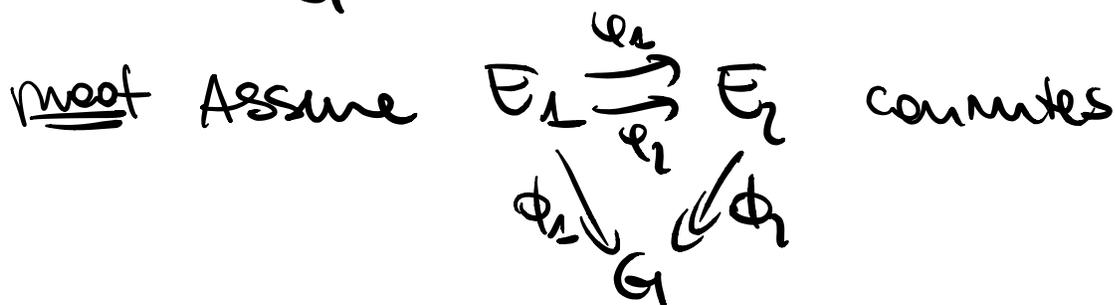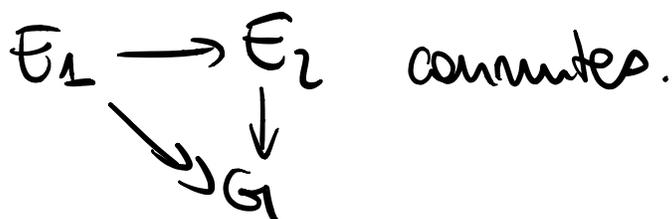
$x \in E_1$, $\varphi_1(x)\varphi_2(x)^{-1} \in \ker \phi_2 \subseteq Z(E_2)$

E_1 perfect $\Rightarrow x = c_1 \ldots c_m$ $\quad c_i$ commutators

Show $\varphi_1(c_1 \ldots c_m) = \varphi_2(c_1 \ldots c_m)$ by induction

on c_i commutators.

$$\varphi_1(x)\varphi_2(x)^{-1} = \varphi_1(c)\varphi_2[g,h]\left(\varphi_2[g,h]\right)^{-1}\varphi_2(c)$$

$\ldots$ A lot of computation.

$$\Rightarrow \varphi_1 = \varphi_2 \quad \square$$

$\underline{\underline{\text{Def}}}\ K_2(R) = \ker\left(\mathfrak{u} \twoheadrightarrow E(R)\right)$

↳ universal central extension of $E(R)$