Recall: $K_2(R) = \ker(\mathcal{U} \twoheadrightarrow E(R))$ ↳ universal central extension.

$$e_{ij}(\lambda) = \begin{array}{c} \\ \\ \end{array}\begin{pmatrix} 1 & & \lambda \\ & \ddots & \\ & & 1 \end{pmatrix} \leftarrow i \quad \in E(R)$$

j ↓

generate $E(R)$, satisfying:

(1) $e_{ij}(\lambda)e_{ij}(\mu) = e_{ij}(\lambda+\mu) \quad \forall i \neq j$

(2) $[e_{ij}(\lambda), e_{jk}(\mu)] = e_{ik}(\lambda\mu) \quad \forall \begin{matrix} i \neq k \\ i \neq j,\ j \neq k \end{matrix}$

(3) $[e_{ij}(\lambda), e_{kl}(\mu)] = 1 \quad \{i,j\} \cap \{k,l\} = \emptyset$

Def The <u>STEINBERG GROUP</u> $St(R)$ of R is the group generated by $x_{ij}(\lambda)$ $\lambda \in R$ subject to the relations (1), (2), (3) with x_{ij} in place of e_{ij}.

By def. we have a surj. group homomorphism

$$St(R) \twoheadrightarrow E(R)$$
$$x_{ij}(\lambda) \longmapsto e_{ij}(\lambda)$$

THM $St(R) \twoheadrightarrow E(R)$ is the universal extension of $E(R)$ $\quad x_{ij}(\lambda) \longmapsto e_{ij}(\lambda)$

COR $K_2(R) = \ker(St(R) \twoheadrightarrow E(R))$

<u>product</u>: define $K_1(R) \otimes K_1(R) \xrightarrow{\mu} K_2(R)$ for R comm. as follows:

if $A, B \in E(R)$ commute ($[A,B] = ABA^{-1}B^{-1} = 1$)

CHOOSE LIFTS $\tilde{A} \in St(R)$, $\tilde{B} \in St(R)$ $\quad$ note $[\tilde{A}, \tilde{B}] \in \ker(St(R) \twoheadrightarrow E(R)) = K_2(R)$

→ if $\tilde{A}'$, $\tilde{B}'$ are other lifts

$\tilde{A}' = a\tilde{A}$ $\quad a \in K_2(R)$ $\qquad$ central in $St(R)$

$\tilde{B}' = b\tilde{B}$ $\quad b \in K_2(R)$ $\qquad [\tilde{A}', \tilde{B}'] = [a\tilde{A}, b\tilde{B}] = [\tilde{A}, \tilde{B}]$

a, b central

hence $[\tilde{A}, \tilde{B}] \in K_2(R)$ well defined whenever $A, B \in E(R)$, $AB = BA$.

if $A, B \in GL(R)$ commute then

$$A_0 = \begin{pmatrix} A & & \\ & A^{-1} & \\ & & 1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} B & & \\ & 1 & \\ & & B^{-1} \end{pmatrix} \in E(R) \text{ commute}$$

set $A * B := [\tilde{A}_0, \tilde{B}_0] \in K_2(R$

if R commutative, $A \in \underbrace{GL_m(R)}_{Aut(R^m)}$, $B \in GL_m(R)$

then $A \otimes 1_m, 1_m \otimes B \in Aut(R^m \otimes_R R^m) = GL_{mm}(R)$

<u>FACT</u>: R commutative, then

$$K_1(R) \otimes K_1(R) \xrightarrow{\sim} K_2(R)$$
$$[A] \otimes [B] \longmapsto (A \otimes 1) * (1 \otimes B)$$

well defined $A \in GL_n$, $B \in GL_m$

$R^* \longrightarrow K_1(R)$ $\quad$ group homo

$$a \longmapsto \begin{pmatrix} a & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$R^* \otimes_{\mathbb{Z}} R^* \to K_1 R \otimes K_1 R \xrightarrow{\sim} K_2(R)$$
$$(a \otimes b) \longmapsto \{a, b\}$$

<u>FACT</u>: This map factors through the Steinberg relations (R comm.):

$$R^* \otimes_{\mathbb{Z}} R^* \Big/ \underset{a,\,1-a \in R^*}{a \otimes (1-a)} \longrightarrow K_2(R)$$

<u>THM</u> (Matsumoto for fields
Van der Kallen for local rings)

Let F be a field or a local comm. ring with # res. field > 5

then
$$F^* \otimes_{\mathbb{Z}} F^* \Big/ \underset{a,\,1-a \in F^*}{a \otimes (1-a)} \xrightarrow{\cong} K_2(R) \quad \text{is an iso}$$
$$x \otimes y \longmapsto \{x, y\}$$

<u>COR</u> $K_2(\mathbb{F}_q) = 0$

<u>Proof</u> (1) $a, 1-a \in F^* \Rightarrow \{a, 1-a\} = 0 \in K_2(R)$

pf: $-a = \frac{1-a}{1-a^{-1}} \Rightarrow \{a, -a\} = \{a, \frac{1-a}{1-a^{-1}}\} = \underbrace{\cancel{\{a, 1-a\}}}_{=0} \underbrace{- \{a, 1-a^{-1}\}}_{-\underbrace{\cancel{\{a^{-1}, 1-a^{-1}\}}}_{=0}} = 0$

(2) $\mathbb{F}_q^* =$ cyclic group of order $q-1$, generated by some ξ

must to show $\{\xi, \xi\} = 0 \in K_2(F)$

$\{\xi^m, \xi^m\} = m \{\xi, \xi\}$

(a) q even $\Rightarrow \xi = -\xi \Rightarrow \{\xi, \xi\} = \{\xi, -\xi\} \underset{(1)}{=} 0 \Rightarrow K_2(\mathbb{F}_q) = 0$

(b) q odd: $\{\ \}$ skew symmetric, $\{a, b\} = -\{b, a\}$

$\Rightarrow 2\{\zeta,\zeta\}=0$

$\Rightarrow \{\zeta^m,\zeta^n\}=\min\{\zeta,\zeta\}=\{\zeta,\zeta\}$ if m,n odd

$\{\zeta^m \mid m \text{ odd}\}$ non-squares in $\mathbb{F}_q$. Want to find a non square $u\in\mathbb{F}_q$ s.t. $1-u$ is also non-square

$$\# \underbrace{\{u \mid u \text{ non square}\}}_{\frac{q-1}{2}} \cup \underbrace{\{v \mid 1-v \text{ non-square}\}}_{\frac{q-1}{2}} \subseteq$$

$$\subseteq \mathbb{F}_q-\{0,1\} \text{ has } \leq q-2$$

$\Rightarrow \{u \mid u \text{ non-square}\}\cap\{v \mid 1-v \text{ non-square}\}\neq\emptyset$

$K_2(F)$ related to:

- classification of central simple division algebras $/F$.
- classification of non alg. symm. bilinear form $/F$.

<u>Relation w/ division ring</u> (Brauer group)

<u>Def</u> · A division ring is a ring $R\neq 0$ s.t. $R-0=R^*$

· A ring R is called simple if $R\neq 0$ and $0, R \subset R$ ... , 2-sided ideals.

- An F-algebra R is called central if $F = Z(R)$ $\quad (F \to Z(R))$

<u>FACTS</u>: (1) A F-dim. F-alg. A is <u>central simple</u> c.s.

$\Leftrightarrow$ $A \underset{F\text{-Alg.}}{\cong} M_n(D)$, for a(unique) f-dim. central division F-alg.

(2) A, B f. dim. c.s. F-alg. $\Rightarrow A \otimes_F B$ f. dim. c.s. F alg.

(3) A f. dim. c.s. F-alg. $\Rightarrow A \otimes_F A^{op} \xrightarrow{\cong} M_n(F)$

$n = \dim_F(A)$

<u>Def</u> F field, Brauer group of F:

$$Br(F) = \left\{ \begin{matrix} \text{f. dim. central} \\ \text{division } F\text{-alg.} \end{matrix} \right\} = \left\{ \begin{matrix} \text{f. dim. c.s.} \\ F \text{ Alg.} \end{matrix} \right\} \Big/ \begin{matrix} \sim \\ A \sim B \Leftrightarrow \\ M_n(A) = M_m(B) \end{matrix}$$

group under $\otimes_F$ with unit F and inverse $-[A] = [A^{op}]$ by (1)-(3)

<u>Ex</u> $Br(k = \bar{k}) = \{k\}$, $Br(\mathbb{R}) = \{\mathbb{R}, \mathbb{H}\}$

<u>THM</u> (Merkurjev-Suslin) F field with primitive n-th root of unit ζ.

${}_nBr = \ker(Br \xrightarrow{\cdot n} Br)$

$K_2(F)/n \xrightarrow{\cong} {}_nBr(F)$ iso

$\{a, b\} \longmapsto$ cyclic alg.

$A_\zeta(a, b)$ = F alg. gen. by x, y subj. to

C.S. $\sim$ $\qquad x^n = a, \; y^m = b \quad xy = \zeta yx$

F Alg.

<u>EX</u> $F = \mathbb{R}$ $\quad A_{-1}(-1,-1) = \mathbb{H}$.