TOWARDS THE DEFN OF $K_n(R)$

BACKGROUND:

- Pointed top. spaces
- Definition of higher homotopy groups
- Definition of homology groups.

Hurewicz thm:

$$\pi_n(X) \longrightarrow H_n(X) \qquad n \geq 1$$
$$[f: S^n \to X] \longmapsto H_n(f)(1)$$

FACT (HUREWICZ THM)

X space with $\pi_0(X)$ trivial then

$$\begin{tikzcd} \pi_1(X) \arrow[rr] \arrow[dr, two heads] & & H_1(X) \\ & \pi_1(X)^{AB} \arrow[ur, "\cong"'] & \end{tikzcd}$$

factors through $\pi_1(X) \twoheadrightarrow \pi_1(X)^{AB}$.

Moreover if $n \geq 2$ and $\pi_i(X) = 0$ $i < n$ then $H_i(X) = 0$ $(0 < i < n)$ and $\pi_n(X) \xrightarrow{\cong} H_n(X)$.

PREVIEW: Quillen defines $K_n(R)$ as

$$K_n(R) = \pi_n(\underbrace{B}_{\text{classifying space.}}(GL(R)^{\underbrace{+}_{\text{plus construction}}})) \qquad n \geq 1$$

PLUS CONSTRUCTION

(All top. spaces be homotopy equiv. to a CW-cpx)

THM Let X be a connected pointed top. space
$P \trianglelefteq \pi_1(X)$ perfect and normal subgroup.
Then $\exists$ connected pointed top. space X^+
top. with pointed cont. map $X \to X^+$ s.t.

(1) $\pi_1(X) \longrightarrow \pi_1(X^+)$ is $\pi_1(X) \twoheadrightarrow \pi_1(X)/P \xrightarrow{\cong} \pi_1(X^+)$
$\quad P \longmapsto 1$

(2) $H_*(X) \xrightarrow{\cong} H_*(X^+)$

(3) $\forall$ pointed cont. $X \xrightarrow{f} Y$ s.t. $\pi_1(f)(P) = 1 \in \pi_1(Y)$
$\exists$ pointed cont. $X^+ \xrightarrow{\bar{f}} Y$ unique up to pointed homotopy s.t.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & X^+ \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

commutes up to homotopy.

$\rightsquigarrow$ <u>Construction of X^+</u>

<u>case 1</u>: Assume $P = \pi_1(X)$ (perfect)
choose a set I of maps $\gamma : (S^1, s_0) \to (X, x_0)$
s.t. $[\gamma] \in P \subset \pi_1(X)$, $\gamma \in I$, generate P as a group.

Define X_0 by

$$\begin{tikzcd} \bigvee_{\gamma \in I} S^1 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \bigvee_{\gamma \in I} D^2 \arrow[d] \\ X \arrow[r, hook] & X_0 \end{tikzcd}$$

(*)

this is $S^1 \xrightarrow{\gamma} X$ on component of $\gamma \in I$.

then Van Kampen $\Rightarrow$ π_1 sends (*) to a pushout diagram of groups:

$$\begin{tikzcd} \pi_1(\bigvee S^1) \arrow[r] \arrow[d] & \pi_1(\bigvee D^2) \arrow[d] \\ \pi_1 X \arrow[r] & \pi_1(X_0) \end{tikzcd}$$

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free group on $\gamma \in I$

$$\Rightarrow \pi_1 X_0 = \pi_1 X \Big/ \text{image of } \pi_1 \bigvee_{\gamma \in I} S^1 = \pi_1 X / P$$