

# TOWARDS THE DEFN OF $K_n(\mathbb{R})$

## BACKGROUND:

- Pointed top. spaces
- Definition of higher homotopy groups
- Definition of homology groups.

## Hurewicz theorem:

$$\pi_n(X) \longrightarrow H_n(X) \quad n \geq 1$$

$$[f: S^m \rightarrow X] \longmapsto H_m(f)(1)$$

## FACT (HUREWICZ THM)

$X$  space with  $\pi_0(X)$  trivial then

$$\begin{array}{ccc} \pi_2(X) & \longrightarrow & H_2(X) \\ \downarrow & & \uparrow \cong \\ & & \pi_2(X)^{AB} \end{array} \quad \text{factors through} \quad \pi_2(X) \longrightarrow \pi_2(X)^{AB}$$

Moreover if  $n \geq 2$  and  $\pi_i(X) = 0$   $i < n$  then  $H_i(X) = 0$  ( $0 < i < n$ ) and  $\pi_n(X) \xrightarrow{\cong} H_n(X)$ .

Preview: Quillen defines  $K_n(\mathbb{R})$  as

$$K_n(\mathbb{R}) = \pi_n(B(\text{GL}(\mathbb{R})^+)) \quad n \geq 1$$

classifying space. plus construction

## PLUS CONSTRUCTION

(All top. spaces be homotopy equiv. to a CW-cpx)

THM Let  $X$  be a connected pointed top. space  
 $P \trianglelefteq \pi_2(X)$  perfect and normal subgroup.  
 Then  $\exists$  connected pointed top. space  $X^+$   
 top. with pointed cont. map  $X \rightarrow X^+$  s.t.

$$(1) \begin{array}{ccc} \pi_2(X) & \longrightarrow & \pi_2(X^+) \\ P & \longmapsto & 1 \end{array} \quad \text{is } \pi_2(X) \rightarrow \pi_2(X)/P \xrightarrow{\cong} \pi_2(X^+)$$

$$(2) H_*(X) \xrightarrow{\cong} H_*(X^+)$$

(3)  $\forall$  pointed cont.  $X \xrightarrow{f} Y$  s.t.  $\pi_2(f)(P) = \Delta \in \pi_2(Y)$   
 $\exists$  pointed cont.  $X^+ \xrightarrow{F} Y$  unique up to  
 pointed homotopy s.t.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \uparrow \\ & X^+ & \xrightarrow{F} \end{array} \quad \text{commutes up to homotopy.}$$

$\leadsto$  Construction of  $X^+$

cases: Assume  $P = \pi_2(X)$  (perfect)

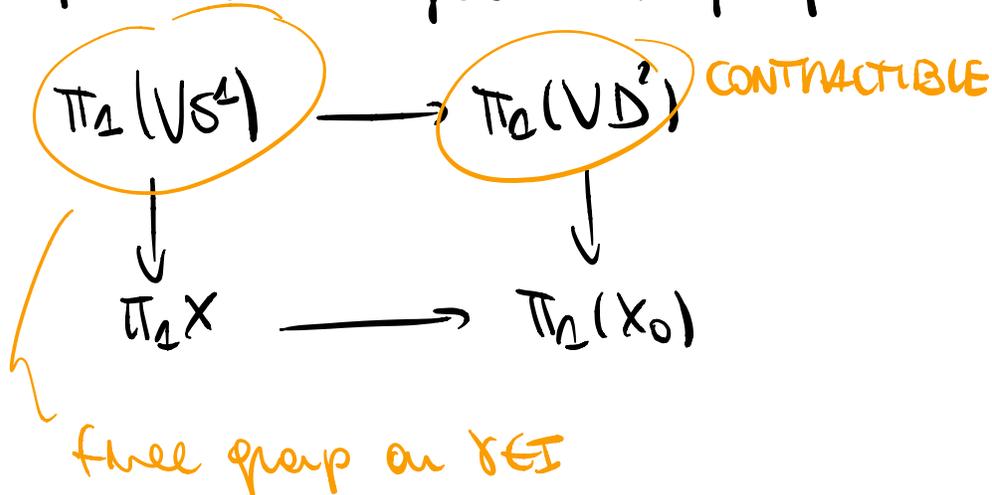
choose a set  $I$  of maps  $\gamma: (S^2, \sigma_0) \rightarrow (X, x_0)$   
 s.t.  $[\gamma] \in P \subset \pi_2(X)$ ,  $\gamma \in I$ , generate  $P$  as  
 a group.

Define  $X_0$  by

$$\begin{array}{ccc} \bigvee_{\gamma \in I} S^2 & \hookrightarrow & \bigvee_{\gamma \in I} D^2 \\ \downarrow \text{(\textcircled{1})} & & \downarrow \text{(*)} \\ X & \hookrightarrow & X_0 \end{array}$$

this is  $S^2 \xrightarrow{\gamma} X$   
 on component of  $\gamma \in I$ .

then Van Kampen  $\Rightarrow \pi_1$  reads (\*) to  
 a pushout diagram of groups:



$$\Rightarrow \pi_1 X_0 = \pi_1 X / \text{image of } \pi_1 \underset{\mathbb{R} \in I}{VS^1} = \pi_1 X / P$$