Towards the Defn of $K_m(R)$

**Background:**
- Pointed top. spaces
- Definition of higher homotopy groups
- Definition of homotopy groups.

**Hurewicz Thm:**

\[
\pi_m(X) \rightarrow H_m(X) \quad m \geq 1
\]

\[
[	ext{for } S^n \rightarrow X] \rightarrow H_m(f_*)(X)
\]

**Fact (Hurewicz Thm):**

If $X$ space with $\pi_0(X) = 0$ trivial then

\[
\pi_d(X) \rightarrow H_d(X) \text{ factors through } \pi_d(X)^{AB} \rightarrow \pi_d(X)^{AB}
\]

Moreover if $m \geq 2$ odd $\pi_i(X) = 0$ for $i < m$ then $H_i(X) = 0$ (02: $i < m$) odd $\pi_m(X) \rightarrow H_m(X)$.  

**Review:** Quick ref defines $K_m(R)$ as

\[
K_m(R) = \pi_m(B(G_m(R)^+) ) \quad m \geq 2
\]

**Plus Construction**

(All top. spaces be homotopy equiv. to a CW-complex)
THM. Let $X$ be a connected pointed top. space
$P \triangleleft \pi_n(X)$ perfect and maximal subgroup.
Then $\exists$ connected pointed top. space $X^+$
top. with pointed cont. map $X \to X^+$ s.t.

\[ \pi_n(X) \cong \pi_n(X^+) \]

(2) $\pi_*(X) \cong \pi_*(X^+)$

(3) $\exists$ pointed cont. $X \cong Y$ s.t. $\pi_n(Y)(P) = \pi_*(X)^P(Y)$

$\exists$ pointed cont. $X^+ \cong Y$ unique up to
pointed homotopy s.t.

\[
\begin{array}{ccc}
X & \xrightarrow{\gamma} & Y \\
\downarrow & & \downarrow \\
X^+ & \cong & Y
\end{array}
\]

commutes up to
homotopy.

Construction of $X^+$

\textbf{case 1: Assume } $P = \pi_1(X)$ (perfect)

choose a set $I$ of maps $\gamma : (S^2, x_0) \to (X, x_0)$
s.t. \[ \{ \gamma \}_P \subset \pi_2(X) \], $\forall \gamma \in I$, generate $P$
as a group.

Define $X_0$ by

\[ V S^2 \leftarrow \bigcup_{\gamma \in I} V D^2 \]

\[ \xrightarrow{\gamma} \]

\[ \text{this is } S^2 \times X \text{ an component of } \gamma \in I. \]

\[ (1) \]

\[ X \cong X_0 \]
Then Via Van Kampen $\Rightarrow \Pi_4$ reads (*) to a pushout diagram of groups:

\[
\begin{array}{ccc}
\Pi_2(VS^4) & \longrightarrow & \Pi_3(VS^3) \\
\downarrow & & \downarrow \\
\Pi_2X & \longrightarrow & \Pi_2(X_0)
\end{array}
\]

**CONTRACTIBLE**

Free group on $\Gamma$

$\Leftarrow \Pi_2X_0 = \frac{\Pi_2X}{\text{image of } \Pi_2 VS^4_{\partial X}} = \frac{\Pi_2X}{P}$