

# 7. PLUS CONSTRUCTION

15/08/19

THM (Plus construction) Let  $X$  be a connected, pointed CW-complex,  $P \trianglelefteq \pi_2(X)$  a normal subgroup. Then  $\exists$  a pointed connected CW-complex  $X^+$  and a pointed cont. map  $X \rightarrow X^+$  such that:

(1)  $\pi_2 X \rightarrow \pi_2 X^+$  is  $\pi_2 X \rightarrow \pi_2 X/P$

(2)  $H_n X \xrightarrow{\cong} H_n X^+ \quad \forall n$

(3)  $\forall$  pointed map  $X \xrightarrow{f} Y$  s.t.  $(\pi_2 f)(P) = 0 \in \pi_2 Y$

$\exists$  pointed cont. map  $X^+ \xrightarrow{\hat{f}} Y$  unique up to homotopy s.t.

up to homotopy.  $X \rightarrow X^+$  commutes up to homotopy.

$$\begin{array}{ccc} X & \xrightarrow{\quad} & X^+ \\ f \downarrow & & \uparrow \hat{f} \\ Y & & Y \end{array}$$

## CONSTRUCTION OF $X^+$ :

**Case 1**  $P = \pi_2 X$  perfect

choose  $I = \{ \gamma : (S^1, s_0) \rightarrow (X, x_0) \}$  s.t.

$[\gamma] \in P = \pi_2 X$  generates  $P$  as a group

define  $X_0$  as a pushout:

best fit:

$\pi_2 X \rightarrow \pi_2 X_0$  is  
 $\pi_2 X \rightarrow \pi_2 X/P, \pi_2 X_0 = 0$

$$\begin{array}{ccc} X & \hookrightarrow & X_0 \\ \uparrow \Gamma = \bigvee_{\gamma \in I} \gamma & & \uparrow (*) \\ \bigvee_{\gamma \in I} S^1 & \hookrightarrow & \bigvee_{\gamma \in I} D^2 \end{array}$$



Define  $X^+$  as the pushout:

$$\begin{array}{ccc}
 X_0 & \hookrightarrow & X^+ \\
 \uparrow & & \uparrow \\
 VS^2 & \longrightarrow & VD^3 \\
 \gamma \in I & & \gamma \in I
 \end{array}$$

(\*\*)

Van Kampen:  $\pi_1$  applied to (\*\*) is a pushout in groups:

$$\begin{array}{ccc}
 0 = \pi_1 X_0 & \longrightarrow & \pi_1 X^+ \\
 \uparrow \text{pushout in grp} & & \uparrow \Rightarrow \pi_1 X^+ = 0 \\
 0 & \longrightarrow & 0
 \end{array}$$

$\leadsto H_0 X \rightarrow H_0 X_0 \rightarrow H_0 X^+ \rightarrow 0$  ( $X_0, X^+$  connected)  
 $H_1 X \xrightarrow{\cong} H_1 X_0 \xrightarrow{\cong} H_1 X^+ = 0 \quad \text{as all } \pi_1 = 0$

$$\begin{array}{ccccc}
 & & H_2 X & & \\
 & & \downarrow & \searrow \textcircled{b} & \\
 \oplus H_2 S^2 & \xrightarrow{H_2 \tilde{\tau}^2} & H_2 X_0 & \longrightarrow & H_2 X^+ \longrightarrow 0 \\
 \gamma \in I & & \downarrow & & \\
 \text{By choice of } \tilde{\tau}^2 = \nu \tilde{\delta}^2 & & \oplus H_2 S^2 & & \\
 & & \gamma \in I & & 
 \end{array}$$

ⓐ:  $\text{IS} \Rightarrow H_2 \tilde{\tau}^2$  inj. and now end columns are exact.

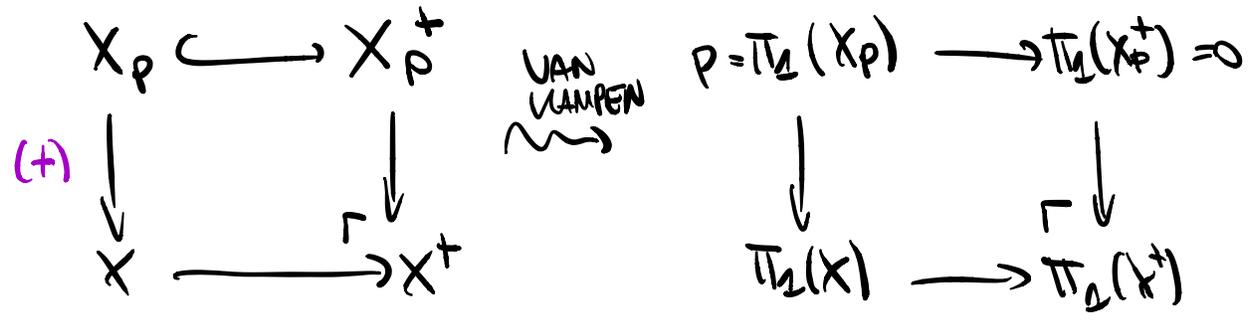
ⓓ  $\text{IS} \Leftrightarrow \textcircled{b}$   $\text{IS} \Rightarrow H_2(X) \xrightarrow{\cong} H_2 X^+ \text{ IS, for } H_n(X) \text{ } n \geq 3 \text{ due by (**)}$

**CASE 2**  $P \trianglelefteq \pi_2 X$  perfect normal subgroup.

$\tilde{X}^2$  universal cover of  $X$ ,  $\pi_2 X \trianglelefteq \tilde{X}$  as deck transformations and  $X = \tilde{X} / \pi_2(X)$

Let  $X_P = \tilde{X}^2 / P$  then  $\pi_2(X_P) = P$ , by case 1  
 $\downarrow$   
 $X = \tilde{X}^2 / \pi_2 X$   
 $X_P^+$  exists with  $H_*(X_P) \cong H_*(X_P^+)$  and  $\pi_2(X_P^+) = 0$

Define  $X^+$  as the pushout:



$$\pi_2 X^+ = \pi_2 X_P^+ *_{\pi_2 X_P} \pi_2 X \Rightarrow \pi_2 X^+ = \pi_2 X / P$$

$\hat{H}_*$  excision (on MV) applied to (+) using  $H_* X_P \cong H_* X_P^+$  yields  $H_* X \cong H_* X^+$  □

## DEF (QUILLEN)

$$K_n(\mathbb{R}) = \pi_n(BGL(\mathbb{R})^+) \quad n \geq 2 \text{ where}$$

- $BGL(\mathbb{R})$  is the classifying space of the group  $GL(\mathbb{R})$
- $(BGL(\mathbb{R}))^+$  is its plus construction.

## CLASSIFYING SPACE OF A DISCRETE GROUP

$G$ : discrete group

DEF  $BG$  classifying space of  $G$  is a pointed connected CW-complex s.t.

$$\pi_n(BG) = \begin{cases} G & n=1 \\ 0 & n \neq 1 \end{cases}$$

PROP Let  $EG \rightarrow BG$  be the universal cover of  $BG$ .

Then  $\pi_n(EG) = 0 \quad \forall n$

(since  $\tilde{X} \rightarrow X$  covering space of connected CW-spaces

then  $\pi_0 \tilde{X} = \pi_0 X = 0$ , and  $\pi_n \tilde{X} \cong \pi_n X \quad n \geq 2$

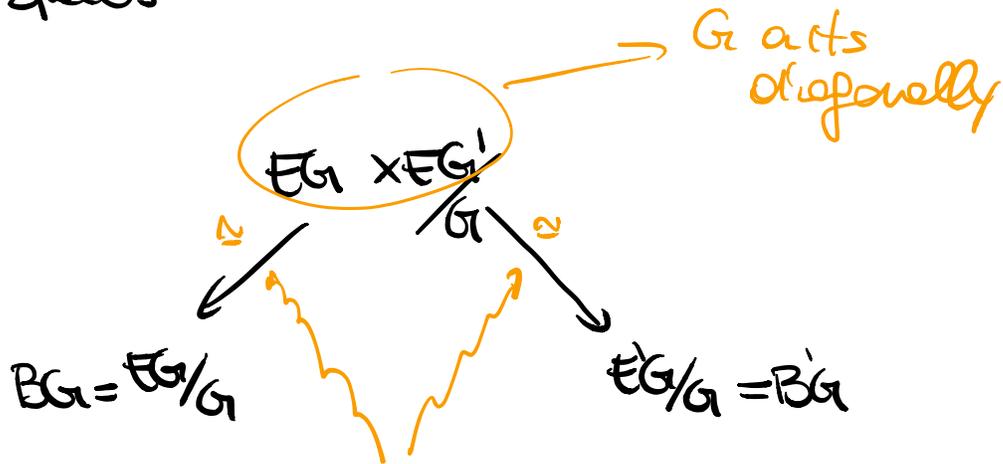
by homotopy lifting property.)

$\pi_1(BG) = G$  acts freely on  $EG$  and

$$EG/G \xrightarrow{\cong} BG$$

LEMMA Any two classifying spaces for the group  $G$  are homotopy equivalent.

$\square$  if  $EG \rightarrow BG$ ,  $EG' \rightarrow BG$  two classifying spaces



Homotopy equivalent since they are  
 locally trivial with contractible  
 fibres  $EG, EG'$

$\square$