7. PLUS CONSTRUCTION

15/09/19

THM (Plus construction) Let X be a connected, pointed CW-complex, $P \trianglelefteq \pi_1(X)$ a perfect normal subgroup. Then $\exists$ a pointed connected CW-complex X^+ and a pointed cont. map $X \to X^+$ such that:

(1) $\pi_1 X \to \pi_1 X^+$ is $\pi_1 X \twoheadrightarrow \pi_1 X / P$

(2) $H_m X \xrightarrow{\cong} H_m X^+$ $\forall m$

(3) $\forall$ pointed map $X \xrightarrow{f} Y$ s.t. $(\pi_1 f)(P) = 1 \in \pi_1 Y$ $\exists$ pointed cont. map $X^+ \xrightarrow{\tilde{f}} Y$ unique up to homotopy s.t.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & X^+ \arrow[d, dashed, "\tilde{f}"] \\ & Y \end{tikzcd}$$

commutes up to homotopy.

CONSTRUCTION OF X^+:

$\boxed{\text{Case 1}}$ $P = \pi_1 X$ perfect

choose $I = \{\gamma : (S^1, s_0) \to (X, x_0)\}$ s.t. $[\gamma] \in P = \pi_1 X$ generates P as a group

define X_0 as a pushout:

$$\begin{tikzcd} X \arrow[r, hook] & X_0 \\ \bigvee_{\gamma \in I} S^1 \arrow[u, "\Gamma = \bigvee_{\gamma \in I} \gamma"] \arrow[r, hook] & \bigvee_{\gamma \in I} D^2 \arrow[u, "(*)"'] \end{tikzcd}$$

last time:

$\pi_1 X \to \pi_1 X_0$ is $\pi_1 X \twoheadrightarrow \pi_1 X / P$, $\pi_1 X_0 = 0$

$\rightsquigarrow$ the homology is still too big:

$$\tilde{H}_*(X) = \operatorname{coker}\left(H_*(x_0) \to H_*(X)\right) = \begin{cases} H_n(X) & n \geq 1 \\ H_0(X)/\mathbb{Z} & n = 0 \end{cases}$$

$\hookrightarrow$ reduced homology for $X = (X, x_0)$ pointed

long exact sequence:

$$\begin{tikzcd}
 & & \tilde{H}_{n+1}(X_0) \arrow[dll, bend left] \\
H_n \bigvee_{\gamma \in I} S^1 \arrow[r] & \tilde{H}_n(X) \oplus \cancel{\tilde{H}_n\left(\bigvee_{\gamma \in I} D^2\right)} \arrow[r] & \tilde{H}_n X_0 \arrow[dll, bend left] \\
H_{n-1} \bigvee_{\gamma \in I} S^1 & &
\end{tikzcd}$$

$\bigvee_{\gamma \in I} S^1 = \bigoplus_{\gamma \in I} \tilde{H}_n S^1$

CONTRACTIBLE

(***)

$\Rightarrow$ $H_0 X \xrightarrow{\cong} H_0 X_0 = \mathbb{Z}$

$H_1 X \cong H_1 X_0$ $\quad (\pi_1(X) = \pi_1(X_0) = 0$

$H_n X \cong H_n X_0$ $\quad n \geq 3$

$$\begin{tikzcd}
0 \arrow[r] & H_2 X \arrow[r] & H_2 X_0 \arrow[r, two heads] & \bigoplus_{\gamma \in I} \underbrace{H_1 S^1}_{\mathbb{Z}\gamma} \arrow[r] & 0 \\
 & & \pi_2 X_0 \arrow[u, "\cong"] \arrow[ur, "\tilde{\gamma}"] & &
\end{tikzcd}$$

HUREWICZ!
$\pi_0 X_0 = \pi_1 X_0 = 0$

1 in γ component and 0 - elsewhere

this map is surjective re $\exists$ a left $\tilde{\gamma}$ for every $\gamma \in I$.

$\tilde{\gamma} : (S^1, s_0) \to (X_0, x_0)$

Define X^+ as the pushout:

$$\begin{tikzcd} X_0 \arrow[r, hook] & X^+ \\ \bigvee_{\gamma\in I} S^2 \arrow[u, "\tilde{\Gamma}=\bigvee_{\gamma\in I}\tilde{\gamma}"] \arrow[r, hook] \arrow[ur, phantom, "(**)"] & \bigvee_{\gamma\in I} D^3 \arrow[u] \end{tikzcd}$$

Van Kampen: π_1 applied to $(**)$ is a pushout in groups:

$$\begin{tikzcd} 0 = \pi_1 X_0 \arrow[r] & \pi_1 X^+ \\ 0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\text{pushout in Grp}"] & 0 \arrow[u] \end{tikzcd} \Rightarrow \pi_1 X^+ = 0$$

$\rightsquigarrow H_0 X \to H_0 X_0 \to H_0 X^+ \to 0$ (X_0, X^+ connected)

$H_1 X \xrightarrow{\cong} H_1 X_0 \xrightarrow{\cong} H_1 X^+ = 0$ as aff $\pi_1 = 0$

$$\begin{tikzcd} & H_2 X \arrow[d, tail] \arrow[dr, "b"] & & \\ \bigoplus_{\gamma\in I} H_2 S^2 \arrow[r, "H_2\tilde{\Gamma}"] \arrow[dr, "a"] & H_2 X_0 \arrow[r] \arrow[d, two heads] & H_2 X^+ \arrow[r] & 0 \\ & \bigoplus_{\gamma\in I} H_1 S^1 & & \end{tikzcd}$$

By choice of $\tilde{\Gamma} = \bigvee \tilde{\gamma}$

a: iso $\Rightarrow H_2\tilde{\Gamma}$ inj. and row and columns are exact.

a iso $\Leftrightarrow$ b iso

$\Rightarrow H_2(X) \xrightarrow{\cong} H_2 X^+$ iso, for $H_n(X)$ $n \geq 3$ done by CCS. $(***)$

$\boxed{\text{CASE 2}}$ $P \trianglelefteq \pi_1 X$ perfect normal subgroup.

$\tilde{X}$ universal cover of X, $\pi_1 X \curvearrowright \tilde{X}$ as deck transformations and $X = \tilde{X}/\pi_1(X)$
$\downarrow$
X

Let
$$\begin{array}{c} X_P = \tilde{X}/P \\ \downarrow \\ X = \tilde{X}/\pi_1 X \end{array}$$

then $\pi_1(X_P) = P$, by case 1 X_P^+ exists with $H_*(X_P) \xrightarrow{\cong} H_*(X_P^+)$ and $\pi_1(X_P^+) = 0$

Define X^+ as the pushout:

$$\begin{tikzcd} X_P \arrow[r, hook] \arrow[d, "(+)"'] & X_P^+ \arrow[d] \\ X \arrow[r] & X^+ \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \quad \overset{\text{VAN KAMPEN}}{\rightsquigarrow} \quad \begin{tikzcd} P = \pi_1(X_P) \arrow[r] \arrow[d] & \pi_1(X_P^+) = 0 \arrow[d] \\ \pi_1(X) \arrow[r] & \pi_1(X^+) \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

$$\pi_1 X^+ = \pi_1 X_P^+ \underset{\pi_1 X_P}{*} \pi_1 X \Rightarrow \pi_1 X^+ = \pi_1 X / P$$

$\tilde{H}_*$ excision (on MV) applied to (+) using $H_* X_P \xrightarrow{\cong} H_* X_P^+$ yields $H_* X \xrightarrow{\cong} H_* X^+$

$\square$

DEF (QUILLEN)

$K_n(R) = \pi_n(BGL(R)^+)$ $n \geq 1$ where

- $BGL(R)$ is the classifying space of the group $GL(R)$
- $(BGL(R))^+$ is it's plus construction.

CLASSIFYING SPACE OF A DISCRETE GROUP

G: discrete group

Def BG classifying space of G is a pointed connected CW-complex s.t.

$$\pi_n(BG) = \begin{cases} G & n=1 \\ 0 & n \neq 1 \end{cases}$$

RMK Let $EG \to BG$ be the universal cover of BG.

Then $\pi_n(EG) = 0$ $\forall n$

(since $\tilde{X} \to X$ covering space of connected CW-spaces then $\pi_0 \tilde{X} = \pi_0 X = 0$, and $\pi_n \tilde{X} \xrightarrow{\cong} \pi_n X$ $n \geq 2$ by homotopy lifting property.

$\pi_1(BG) = G$ acts freely on EG and

$EG/G \xrightarrow{\cong} BG$)

Lemma Any two classifying spaces for the group G are homotopy equivalent.

Pf if $EG \to BG$, $EG' \to B'G$ two classifying spaces

$$\begin{tikzcd}
 & EG \times EG'/G \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\
BG = EG/G & & EG'/G = B'G
\end{tikzcd}$$

G acts diagonally

Homotopy equivalent since they are locally trivial with contractible fibre EG', EG

$\square$