$K_n(R)$ and agreement with K_1, K_2

Def (Quillen), R ring

$$K_n(R) := \pi_n(BGL(R))^+$$

- where $(-)^+$ is the plus construction w.r.t. the perfect normal subgroup $\underbrace{E(R)}_{[GL(R),GL(R)]} \trianglelefteq \pi_1 BGL(R) = GL(R)$

THM (Quillen)

$$\pi_n BGL(R)^+ = \begin{cases} GL(R)^{AB} & n=1 \\ \ker(U \twoheadrightarrow E(R)) & n=2 \end{cases}$$

$\rightsquigarrow$ universal central extension of $E(R)$

$\rightsquigarrow$ In particular Quillen's definition of K_n generalizes K_1, K_2 from previous lectures.

Pf (of Thm for $n=1$)

$$\pi_1 BGL(R)^+ = \frac{\pi_1 BGL(R)}{E(R)} = \frac{GL(R)}{E(R)} = GL(R)^{AB}$$

$E(R) = [GL(R), GL(R)]$. //

For $n=2$ need some group homology:

FACT $H_*(BG;\mathbb{Z})$ can be computed algebraically

so $H_*(BG;\mathbb{Z}) = \underbrace{H_*(G;\mathbb{Z})}_{\text{GROUP HOMOLOGY.}}$

• <u>GROUP HOMOLOGY</u>:

G a group, M a G-module (i.e. $\mathbb{Z}[G]$-module)

$H_n(G;M)$

<u>Def</u> $M_G := M/\langle \text{Ab. subgroup gen. by } x - gx \quad g\in G, x\in M\rangle$ co-invariants of M.

<u>Def</u> $H_n(G;M) := H_n((P_*)_G, d)$

where $\dots \to P_1 \xrightarrow{d} P_0 \to M \to 0$ is an exact sequence of G-modules with P_i projective $\mathbb{Z}[G]$-modules. (independent from the choice of P_*)

<u>RMK</u> $\mathbb{Z}$ is the trivial G-module.

<u>EXER</u>: (a) compute $H_*(C_2;\mathbb{Z})$

(b) compute $H_*(G;\mathbb{Z}[G])$

<u>PROP</u> M a G-module, then:

(1) $H_0(G;M) = M_G$

(2) $H_1(G;\mathbb{Z}) = G^{Ab}$

(3) $F(S)$: free group on the set S. Then

$$H_n(F(S);\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}[S] = \text{free ab. group on } S & n=1 \\ 0 & n\geq 2 \end{cases}$$

(h) if G is a perfect group, then

$$H_2(G;\mathbb{Z}) = \text{Schur multiplier of } G = \ker(\mathcal{U} \twoheadrightarrow G)$$

$\mathcal{U}$ UNIVERSAL CENTRAL EXT^{ns} of G.

pf (3): $G = F(S)$

$$0 \to I := \ker(\varepsilon) \to \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$
$$g \longmapsto 1$$

if G is a free group then I is a proj. (free) $\mathbb{Z}[G]$-module. Indeed:
(we just have to specify a base in I)

has a $\mathbb{Z}[G]$ basis $s \in S$

$$\bigoplus_S \mathbb{Z}[G] \xrightarrow{\Phi} I$$

has a $\mathbb{Z}$-base given by $\langle g \rangle - 1$

$$s \longmapsto \langle s \rangle - 1$$

1 in s component, 0 elsewhere.

Φ is a G-mod. homomorphism.

- Φ is surjective: show that $\langle g \rangle - 1$ is in $\mathrm{Im}(\Phi)$ by induction on the lenght of g (free group!)

(*)
$$\langle gs \rangle - 1 = \langle g \rangle \underbrace{(\langle s \rangle - 1)}_{\in \mathrm{Im}(\Phi)} + \underbrace{\langle g \rangle - 1}_{\in \mathrm{Im}(\Phi)}$$

by induction hyp.

$$\langle gs^{-1} \rangle - 1 = -\langle gs^{-1} \rangle \underbrace{(\langle s \rangle - 1)}_{\in \mathrm{Im}(\Phi)} + \underbrace{\langle g \rangle - 1}_{\in \mathrm{Im}(\Phi)}$$

use (*) to define $I \to \bigoplus_S \mathbb{Z}[G]$ inverse to ϕ

$\Rightarrow H_n(F(S);\mathbb{Z}) = H_n(0 \to 0 \to I_G \to \mathbb{Z}[G]_G) = 0$ for $n \geq 2$.

<u>Pf</u> (of (h)) G perfect group, $1 \to R \to F \to G \to 1$ ex. seq. of grps with F a free group.

Recall $\dfrac{[F,F]}{[R,F]} \twoheadrightarrow G$ universal central ext^{ns} of G.

<u>FACT</u>: if $1 \to N \to \mathcal{N} \to G \to 1$ is an exact seq. of groups then $\exists$ ex seq:

$$H_2(\mathcal{N}) \to H_2 G \to H_0(G, N^{ab}) \to H_1 \mathcal{N} \to H_1 G$$

(where $G \circlearrowright N^{ab}$ via conj: $G \times N^{ab} \to N^{ab}$, $(g,x) \mapsto \tilde{g} x \tilde{g}^{-1}$ for $\tilde{g} \in G$ s.t. $g = \tilde{g}N$)

<u>RMK</u> (Fact $\Leftarrow$ Hochschild-Serre S.S.)

Apply FACT to $1 \to R \to F \to G \to 1$ and get:

$$\underbrace{H_2 F}_{\substack{=0 \\ \text{by (3)}}} \to H_2 G \to \underbrace{H_0(G; R^{ab}) \to H_1 F}_{} \to \underbrace{H_1 G}_{\substack{=0 \text{ because} \\ G \text{ perfect}}}$$

$$(R^{ab})_G \longrightarrow F/[F,F]$$

$$\uparrow \cong$$

$$R/[R,F] \nearrow$$

has kernel $[F,F]/[R,F]$ because

$$\begin{tikzcd}
K_2 \arrow[r, hook] \arrow[d, hook] & {[F,F]}/{[R,F]} \arrow[r, two heads] \arrow[d, hook] & G \arrow[d, "(a)"', "\mathrm{Id}"] \\
R/[R,F] \arrow[r, hook] \arrow[d, two heads] & F/[R,F] \arrow[r, two heads] \arrow[d, two heads] & G \\
? \arrow[r, "(b)"] & F/[F,F] &
\end{tikzcd}$$

(a) iso $\Leftrightarrow$ (b) iso

$\square$

<u>lemma</u> $BE(R)^+$ is (up to homotopy) the universal cover of $BGL(R)^+$

<u>COR</u> $\pi_2 BGL(R)^+ = \ker(\text{universal central ext}^n \text{ of } E(R))$

<u>pf</u> $\pi_2 BGL(R)^+ \underset{\text{lemma}}{=} \pi_2 BE(R)^+ \underset{\substack{\pi_1 BE(R)^+ = 0 \\ \text{HUREWICZ}}}{=} H_2 BE(R)^+ \underset{\substack{\text{PLUS} \\ \text{CONST.}}}{=}$

$= H_2 BE(R) = H_2 E(R) \underset{\text{PROP (4)}}{=} \ker\left(\begin{matrix}\text{universal central} \\ \text{ext}^n \text{ of } E(R)\end{matrix}\right)$

$\square$