K-THEORY OF FINITE FIELDS

Def A complex vector bundle over a top. space X is $E \xrightarrow{p} X$ map where each fiber is f.d. $\mathbb{C}$-vector space and $\exists$ $X = \bigcup_{i \in I} U_i$ open cover s.t. with g ones and vector space iso over each $x \in U_i$.

$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "g"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowright"] & U_i \times \mathbb{C}^n \arrow[dl, "\pi_1"] \\ U_i & \end{tikzcd}$$

RMK If each $p^{-1}(x)$ has dim $= n$ we say E has rank n.

EX: trivial one.

Def Iso of vector bundles is a ones g s.t. $g: p_1^{-1}(x) \to p_2^{-1}(x)$ is iso of vector spaces.

$$\begin{tikzcd} E_1 \arrow[rr, "g"] \arrow[dr, "p_1"'] & \arrow[d, phantom, "\circlearrowright"] & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Def

1) Directs sum $E_1 \oplus E_2 \xrightarrow{p} X$ $\quad p^{-1}(x) = p_1^{-1}(x) \oplus p_2^{-1}(x)$
2) Tensor product $E_1 \otimes E_2 \xrightarrow{p} X$ $\quad p^{-1}(x) = p_1^{-1}(x) \otimes_{\mathbb{C}} p_2^{-1}(x)$

Def The <u>COMPLEX TOP. K-THEORY</u> of X is:

$KU(X) :=$ group completion of the abelian monoid of iso classes of vector bundles on X with $\oplus$.

Rmk $KU(X)$ is a ring under $\otimes$

KU is (contr.) functorial: $X \xrightarrow{f} Y$

$$KU(X) \xleftarrow{f^*} KU(Y)$$
$$[f^*V]-[f^*W] \longleftarrow\!\shortmid [V]-[W]$$

Ex $KU(\text{point}) = \mathbb{Z}$

Def $X = (X, x_0)$ pointed. The reduced K-Theory of X is:

$$\widetilde{KU}(X) := \ker(KU(X) \to \underbrace{KU(x_0)}_{\mathbb{Z}})$$

- $U_n :=$ unitary group of size $n = \{A \in M_n(\mathbb{C}) \mid AA^* = 1\}$ (top. induced by $\mathbb{C}^{n^2}$)

$$U_n \hookrightarrow U_{n+1}$$
$$A \longmapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}$$

$U := \bigcup_{n \geq 1} U_n$ infinite unitary group.

let BU_n and BU the class. space. Then:

(1) X paracompact then $[X, BU_n] \cong \Phi_n(X) =$ set of n-cpx bundles on X.

(2) X compact pointed CW-complex then:

$$[(X,x_0),(BU,0)] = \widetilde{KU}(X) = \operatorname{colim}_{n\to\infty} \phi_n(X)$$

in particular $\pi_n BU = \widetilde{KU}(S^n)$

(3) BU is an H-group (group up to homotopy)

$\exists\, BU \times BU \xrightarrow{\oplus} BU$, $BU \xrightarrow{(\,)^{-1}} BU$ inverse

unit map $pt \to BU$ which make BU into a group object up to homotopy.

(4) $\pi_i BG = \begin{cases} \pi_{i-1} G & i \geq 1 \\ 0 & i = 0 \end{cases}$ $\quad (G = U_n, U)$

- <u>BOTT PERIODICITY</u>

$$\mathcal{O}(1) := (\mathbb{C}^2 - \{0\}) \times_{\mathbb{C}^*} \mathbb{C} = (\mathbb{C}^2 - \{0\}) \times \mathbb{C} \big/ \text{diagonal action of } \mathbb{C}^*.$$

HOPF BUNDLE

$$\begin{array}{c} \mathcal{O}(1) \\ \big\downarrow \text{ line bundle} \\ \mathbb{CP}^1 = \text{lines in } \mathbb{C}^2 = \mathbb{C}^2 - \{0\} \big/ \mathbb{C}^* \end{array}$$

<u>THM</u> (PROJECTIVE LINE BUNDLE FORMULA)

Let X be cpt ptd CW cpx, then ISO:

$$\widetilde{KU}(X) \longrightarrow \widetilde{KU}(\mathbb{CP}^1 \wedge X) \subseteq KU(\mathbb{CP}^1 \times X)$$

$$V \longmapsto (\mathcal{O}(1) - 1) \otimes V$$

$\mathbb{CP}^1 \times \mathbb{C} \to \mathbb{CP}^1$

COR (BOTT PERIODICITY)

$$\pi_n BU = \begin{cases} \mathbb{Z} & n>0 \text{ even} \\ 0 & \text{else} \end{cases}$$

Pf

$n=0$ ✓

$n \geq 1$:

$$\pi_n BU = \widetilde{KU}(S^n) \xrightarrow[\cong]{\text{THM}} \widetilde{KU}(\underbrace{\mathbb{CP}^1 \wedge S^n}_{S^{n+2}}) = \pi_{n+2} BU$$

$\mathbb{CP}^1 \wedge S^n = S^2$

$n=1$: $\pi_1 BU = \pi_0 U = 0$ (U conn.)

because

$$U_n \xleftarrow[\text{SURJ. CONT.}]{e^A \leftarrow\!\shortmid A} \{A \in M_n(\mathbb{C}) \mid A^* = -A\} \quad \text{CONNECT.}$$

$$\pi_2 BU = \widetilde{KU}(\mathbb{CP}^1) \xrightarrow{\cong} \mathrm{Pic}(\mathbb{CP}^1) \cong \mathbb{Z} \quad \text{gen. by } \mathcal{O}(1)$$

$$V \longmapsto \Lambda V$$

ADAMS OPERATIONS :

$\exists!$ functorial ring homomorphism $\psi^k : KU(X) \to KU(X)$ s.t. $\psi^k(\mathcal{L}) = \mathcal{L}^{\otimes k}$ for all line bundles $\mathcal{L}$ on X.

RMK: ψ^k induced by cont. maps $BU \xrightarrow{\psi^k} BU$

EX $\psi^9 : \widetilde{KU}(S^2) \xrightarrow{\cdot 9} \widetilde{KU}(S^2) = \mathbb{Z}$

$$\mathcal{O}(1)-1 \longmapsto \mathcal{O}(9)-1$$

$$\begin{array}{ccc} \mathcal{O}(1) & & \mathcal{O}(q) \\ \mathrm{Pic}(\mathbb{CP}^1) & \longrightarrow & \mathrm{Pic}(\mathbb{CP}^m) \end{array}$$

$$\mathcal{U} = \widetilde{KU}(S^{2m}) \xrightarrow{\psi^q = q^m} \widetilde{KU}(S^2)$$

gen. by $(\mathcal{O}(1)-1)^{\otimes m} \longmapsto (\mathcal{O}(q)-1)^{\otimes m} = [q(\mathcal{O}(1)-1)]^{\otimes m} =$

$$= q^m(\mathcal{O}(1)-1)^{\otimes m}$$

THE MAP $BGL(\mathbb{F}_q) \longrightarrow BU$

G finite group. A cpx rep. V of G defines a vector bundle on BG, namely

$$V \times_G EG \longrightarrow BG = EG/G$$

$$(v,x) \longmapsto x$$

hence a cont. map $[V]: BG \to BU_m \hookrightarrow BU$ if $m = \dim_{\mathbb{C}} V$

A virtual rep. $V-W$ of G defines a map:

$$BG \xrightarrow[[V]-[W]]{} BU$$ using that BU is an H-group

BROWER LIFTING:

G finite, A $\mathbb{F}_q$-linear rep. V of G has associated Brower eff: a virtual cpx

rep^n $W_1 - W_2$ alternated:

$g \in G$ $\quad ev(g)$ = eigenvalues of $(V \xrightarrow{g} V)$

for $\lambda \in ev(g)$ $\quad \mu(\lambda)$ = mult. of λ

$\bigcup_{g \in G} ev(g) \subseteq \mathbb{F}_{q^l}$ $\quad$ (finite alg. ext. of $\mathbb{F}_q$)

choose embedding $\rho : \mathbb{F}_{q^l}^* = C_{q^l-1} \cong$ group of q^l-1-th roots of 1 in $\mathbb{C}^*$.

$g \longmapsto \chi_V^{\mathbb{F}_q}(g) = \sum_{\lambda \in ev(g)} \mu(\lambda)\, \rho(\lambda)$ is class function

$\Rightarrow \chi_V^{\mathbb{F}_q} = \sum_{i=1}^{r} c_i \chi_i$ where $\chi_1, \dots, \chi_r$ are the irred. cpx characters of G.

<u>FACT</u> $c_i \in \mathbb{Z}$

<u>COR</u> $\chi_V^{\mathbb{F}_q} = \chi_{W_1} - \chi_{W_2}$ for cpx G-rep^n W_1, W_2

$$BG \xrightarrow{W_1 - W_2} BU$$

Applied to $GL_n(\mathbb{F}_q) \circlearrowright \mathbb{F}_q$ this gives

$BGL_n(\mathbb{F}_q) \xrightarrow{\partial_n} BU$ comp when $n \to \infty$

get

$$\begin{tikzcd} BGL(\mathbb{F}_q) \arrow[rr, "\partial"] \arrow[dr] & & BU \\ & BGL(\mathbb{F}_q)^+ \arrow[ur, "\pi_1 BU = 0"'] & \end{tikzcd}$$

THM (QUILLEN)

$BGL(\mathbb{F}_q)^+ \longrightarrow BM \xrightarrow{1-x^q} BM$ is htpy fibration