# Constructions Quasicategories

November 4, 2020

Whenever we are making a construction with $\infty$-categories we will generally define it for sSet and we want them to satisfy four general principles:

- *Generalize:* The construction is compatible with the nerve functor.
- *Coherence:* The construction is a coherent version of the classical construction.
- *Closed:* The construction applied to Qcat gives Qcat as output.
- *Invariance:* Equivalent Qcat give equivalent outputs.

In what follows $X$ and $Y$ are arbitrary simplicial sets, $C$ and $D$ are Qcat, while $A$ and $B$ are ordinary categories.

**Product** $(X \times Y)_n = X_n \times Y_n$

- Closed: product of Qcat is a Qcat.
- Generalize: $N(A \times B) \cong NA \times NB$

**Arbitrary coproduct** $(\coprod_{i \in I} X_i)_n = \coprod_{i \in I}((X_i)_n)$

- Closed: product of Qcat is a Qcat.
- Generalize: $N(A \times B) \cong NA \times NB$

**Subcategory** $C' \subseteq C$ subcomplex is an $\infty$-subcategory if for every inner horn $\Lambda^n_k$: if $f : \Delta^n \to C$ is such that $f(\Lambda^n_k) \subseteq C'$ then $f(\Delta^n) \subseteq C'$.

- Generalize: applied to usual categories gives the usual notion.
- $C'$ is a full subcategory when the following holds: a simplex of $C$ is in $C'$ if and only if all its vertices are in $C'$.

**Opposite Qcat** $X^{\text{op}} = X \circ \text{op}$ where $\text{op} : \Delta \to \Delta$ is the involution that reverse the order ($[n] \to [n]$ sends $x$ to $n - x$).

- Generalize: $(NC)^{\text{op}} = N(C^{\text{op}})$.
- Closed: since $(\Lambda^n_k)^{\text{op}} = \Lambda^n_{n-k}$ it's still inner.

**Homotopy category** $hC$ is an ordinary category:

- $\text{Obj}(hC) = C_0$.
- $\text{Hom}_{hC}(x,y) = \text{Hom}_C(x,y)/$ homotopy
- $[f] \circ [g] = [f \circ g]$
- $f$ is an equivalence in $C$ (also called isomorphism in $C$) iff $[f]$ is an isomorphism in $hC$.
- $h(C \times D) \cong hC \times hD$

**Connected components**

$$\pi_0 X \cong (\coprod_{n \geq 0} X_n)/_\sim \cong X_0/\sim_1$$

- $\sim$: related by a simplicial operator.
- $\sim_1$: connected by an edge.

**Core of a Qcat** $C^{\simeq} \subseteq C$ is the subcomplex of those elements all whose edges are isomorphisms (namely the "maximal" quasigroupoid in $C$). For ordinary categories is the subcategory on all the objects and isomorphisms.

- Generalize: $N(C^{\simeq}) = (NC)^{\simeq}$
- $\pi_0(C^{\simeq})$ is isomorphism classes of objects of $C$.

**Space of functors**

$$\text{Fun}(X,Y)_n = \text{Hom}_{\text{sSet}}(\Delta^n \times X, Y)$$

with simplicial operators induced by $\Delta^n$. The functors are the vertices, the natural transformations the edges.

- Generalize: $N(\text{Fun}_{\text{Cat}}(A,B)) \cong \text{Fun}(NA, NB)$.
- Closed: if $Y$ is a qCat then $\text{Fun}(X,Y)$ is a qCat.
- Defines a functor: $\text{Fun} : \text{sSet}^{\text{op}} \times \text{sSet} \to \text{sSet}$.
- $\text{Fun}(X, \_)$ is right adjoint to $\_ \times X$.