Recall (Last week)

Def A monoidal ∞-category is a cocartesian fibration $p: \mathcal{M}^{\otimes} \to N(\Delta^{op})$ satisfying the Segal condition:

(*) for all $n \in \mathbb{N}$, the map

$$\mathcal{M}^{\otimes}_{[n]} \to (\mathcal{M}^{\otimes}_{[1]})^{\times n} = \mathcal{M}^{\times n}$$

induced by the maps

$$\iota_{\{i-1,i\}} : [1] \to [n] \qquad i = 1, \dots, n$$
$$0 \mapsto i-1$$
$$1 \mapsto i$$

is an equivalence of categories

This was motivated from:

Proposition

$$\left\{ \begin{array}{c} \text{monoidal} \\ \text{ordinary categories} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Grothendieck opfibrations} \\ \text{over } \Delta^{op} \text{ satisfying the} \\ \text{Segal condition} \end{array} \right\}$$

Today
- Algebra (or monoid) objects
- Symmetric monoidal ∞-categories
- Commutative algebra objects (commutative monoid)

§1 Algebra objects

Def Let $(\mathcal{M}, \otimes, 1)$ be a monoidal category.
An algebra (or monoid) on $\mathcal{M}$ is a triple (A, μ, η) where $A \in \mathcal{M}$,

$$\mu : A \otimes A \to A \qquad \eta : 1 \to A$$
(multiplication) (unit)

Satisfying:
i) associativity law
ii) right and left unit laws.

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Ex • An algebra on $(Set, \times, \{*\})$ is a monoid
• An algebra on $(Ab, \otimes, \mathbb{Z})$ is a ring.
• An algebra on $(RMod, \otimes_R, R)$ is an (associative unital) R-algebra
R commutative

Question How does this notion of algebra translates into our new approach to monoidal categories (via Groth. opfibrations satisfying segal) over Δ^{op}

Construction Let A be an algebra on M.
Let $p: M^{\otimes} \to \Delta^{op}$ be the associated Groth. opfibration.
objects $(M_1, .., M_n)$ $M_i \in M$, $n \in \mathbb{N}$

a map $(M_1, .., M_n) \to (L_1, .., L_k)$
is $(\alpha, \{f_i : M_{\alpha(i-1)+1} \otimes \cdots \otimes M_{\alpha(i)} \to L_i\}_{i=1}^{k})$
where $\alpha : [k] \to [n]$ in Δ

Consider for $n \in \mathbb{N}$, the maps

$$S_n : A^{\otimes n} \to A$$

by iterating $A \otimes A \to A$. (n=0 this is the unit $1 \to A$)

Define

$$\mathcal{A} : \Delta^{op} \to M^{\otimes}$$

$$[n] \mapsto (A, .., A) \quad (n \text{ times})$$

$$\begin{pmatrix} \alpha : [k] \to [n] \\ \text{in } \Delta \end{pmatrix} \mapsto (\alpha, \{S_{\alpha(i)-\alpha(i-1)}\}_{i=1}^{k})$$

This is a section of p.
But not all sections of p arise in this way!

Def An arrow $\alpha : [k] \to [n]$ in Δ is **convex** if it is injective and $\mathrm{im}\,\alpha$ is the interval $[\alpha(0), \alpha(k)]$

the interval $[\alpha(0), \alpha(k)]$

Remark If $\alpha : [k] \to [n]$ in Δ is convex, then a p-cocartesian lift of α is

$$(\alpha, \{id_{M_{\alpha(i)}}\}_{i=0}^{k}) : (M_1, .., M_n) \to (M_{\alpha(0)}, .., M_{\alpha(k)})$$

Let $\tilde{A} : \Delta^{op} \to M^{\otimes}$ be a section of p.
We want to endow $\tilde{A}[1] := A \in M^{\otimes}_{[1]} = M$ with an algebra structure.
Consider $d_1 = (d^1)^{op} : [2] \to [1]$ in Δ^{op}.
Let

$$\alpha = \tilde{A}(d_1) : \tilde{A}[2] \to A$$

$$\begin{array}{c} \| \; M^{\otimes}_{[2]} \simeq M \times M \\ (A_1, A_2) \in M \times M \end{array}$$

Now, note that a p-cocartesian lift of d_1 with source $\tilde{A}[2]$ can be taken to be

$$\delta := (d_1, \{id_{A_1 \otimes A_2}\}) : \tilde{A}[2] = (A_1, A_2) \to A_1 \otimes A_2$$

So:

$$\begin{tikzcd} & A_1 \otimes A_2 \arrow[d, dashed, "\exists ! \mu"] \\ \tilde{A}[2] = (A_1, A_2) \arrow[ur, "\delta"] \arrow[r, "\tilde{A}(d_1)"'] & A \end{tikzcd}$$

since δ is p-cocartesian

$p \downarrow$

$$\begin{tikzcd} & {[1]} \arrow[d, equal, "id"] \\ {[2]} \arrow[ur, "d_1"] \arrow[r, "d_1"'] & {[1]} \end{tikzcd}$$

$\leadsto \mu : A_1 \otimes A_2 \to A$

Want $A_1 \simeq A \simeq A_2$.

Consider

$$\gamma_1 = \gamma_{\{0,1\}}^{op} : [2] \to [1]$$

$$\gamma_2 = \gamma_{\{1,2\}}^{op} : [2] \to [1]$$

$\gamma_{\{0,1\}}$, $\gamma_{\{1,2\}}$ are convex!

$$\gamma_2 := \gamma^{op}_{\{1,2\}} : [2] \to [1]$$

By remark above, we have p-cocartesian lifts of γ_1, γ_2 with the form

$$(A_1, A_2) \to A_1$$
$$(A_1, A_2) \to A_2$$

- If $\tilde{A}$ sends γ_1 and γ_2 to p-cocartesian arrows, then

$$\tilde{A}(\gamma_1) : (A_1, A_2) \to A \qquad p\text{-cocartesian}$$
$$\tilde{A}(\gamma_2) : (A_1, A_2) \to A$$

$$\Rightarrow \quad A_1 \simeq A \simeq A_2. \Rightarrow \mu : A \otimes A \to A.$$

Prop Let $p: M^{\otimes} \to \Delta^{op}$ be a monoidal structure on M. To give an algebra on M is equivalent to giving a section $\tilde{A}: \Delta^{op} \to M^{\otimes}$ of p s.t. $\tilde{A}$ sends convex arrows to p-cocartesian ones.

Def Let $p: M^{\otimes} \to N(\Delta^{op})$ be a monoidal ∞-category. An <u>(associative) algebra</u> on M is a section $A: N(\Delta^{op}) \to M^{\otimes}$ of p sending convex arrows to p-cocartesian ones.

(Future) **Example** An (associative) algebra on the ∞-category of spectra is a <u>A_∞-ring</u> (spectrum).

§2 Symmetric monoidal ∞-categories

(Analogous to the non-symmetric case: Replace Δ^{op} by Fin_*)

Def For $n \in \mathbb{N}$, $\langle n \rangle = \{0, \ldots, n\}$ be the pointed set with basepoint 0.
Define Fin_* to be category with

- objects $\langle n \rangle, n \in \mathbb{N}$

twist map
Have $t: \langle 2 \rangle \to \langle 2 \rangle$
$1 \mapsto 2$
$2 \mapsto 1$
(non-necessarily

Define Fin_* to be category with

- objects $\langle n \rangle$, $n \in \mathbb{N}$
- maps basepoint-preserving maps (non-necessarily order-preserving) $\begin{array}{l} 1 \leftrightarrow 2 \\ 2 \leftrightarrow 1 \end{array}$

So let M be a symmetric monoidal (ordinary) category. (That is, M is monoidal + "braiding"
$B_{x,y}: x \otimes y \to y \otimes x$ + "hexagon identity"
+ $B_{xy} \circ B_{yx} = id$).

Construction Define $M^{\otimes}$ as

objects $(M_1, .., M_n)$, $M_i \in M$, $n \in \mathbb{N}$

maps $(M_1, .., M_n) \to (L_1, .., L_k)$
is a pair $(\alpha, \{f_i\}_{i=1}^{k})$ where
$\alpha: \langle n \rangle \to \langle k \rangle$ in Fin_* and

$$f_i: \bigotimes_{j \in \alpha^{-1}(i)} M_j \to L_i \qquad i = 1, .., k$$

We have a canonical projection $p: M^{\otimes} \to Fin_*$ which is a Grothendieck opfibration satisfying the (new) Segal condition:

(**) For $n \in \mathbb{N}$, the map

$$M^{\otimes}_{\langle n \rangle} \xrightarrow{\sim} (M^{\otimes}_{\langle 1 \rangle})^{\times n} = \overbrace{M \times .. \times M}^{n \text{ times}}$$

induced by

$$\rho^j: \langle n \rangle \to \langle 1 \rangle \qquad j = 1, .., n$$
$$(\rho^j)^{-1}(1) = j$$

is an equivalence.

Conversely, giving a Grothendieck opfibration $p: \mathcal{C} \to Fin_*$ satisfying (**), consider
$\mathcal{C}_{\langle 1 \rangle} =: M$; we have

i) let $m: \langle 2 \rangle \to \langle 1 \rangle$, $m(1) = m(2) = 1$.
Then

i) Let $m : \langle 2 \rangle \to \langle 1 \rangle$, $m(1) = m(2) = 1$.
Then
$$(\otimes :=) m_! : \underbrace{\mathcal{C}_{\langle 2 \rangle} \simeq M \times M}_{\text{Segal}} \to \mathcal{C}_{\langle 1 \rangle} = M \quad \text{(tensor)}$$

ii) Let $\eta : \langle 0 \rangle \to \langle 1 \rangle$ (unique).
Then
$$\eta_! : \mathcal{C}_{\langle 0 \rangle} \to M \quad \text{(unit)}$$
$\mathcal{C}_{\langle 0 \rangle}$ is Segal terminal category

iii) Let $\tau : \langle 2 \rangle \to \langle 2 \rangle$ "twist map"
$1 \mapsto 2$
$2 \mapsto 1$

Then
$$\tau_! : \underbrace{\mathcal{C}_{\langle 2 \rangle} \simeq M \times M}_{\text{Segal}} \to M \times M \simeq \mathcal{C}_{\langle 2 \rangle}$$
(braiding)

Then $(M, m_!, \eta_!, \tau_!)$ is a symmetric monoidal category.

Prop
$$\left\{ \begin{array}{l} \text{symmetric} \\ \text{monoidal} \\ \text{category} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \text{Grothendieck opfib.} \\ \text{over } \mathrm{Fin}_* \text{ satisfying} \\ \text{Segal cond.} \end{array} \right\}$$

Def A <u>symmetric monoidal ∞-category</u> is a cocartesian fibration $p : M^{\otimes} \to N(\mathrm{Fin}_*)$ satisfying the Segal condition $(\ast\ast)$.

Examples • If $p : M^{\otimes} \to \mathrm{Fin}_*$ is symmetric monoidal, $N(p) : N(M^{\otimes}) \to N(\mathrm{Fin}_*)$ is a symmetric monoidal ∞-category.

• (Lurie, DAG III, 1.4) If $\mathcal{C}$ is an ∞-cat. with finite products there is a "cartesian" monoidal structure on $\mathcal{C}$ which is symmetric

"$\otimes$ is taking product"

e.g. Cat_∞

(Future) example • ∞-category of spectra
• ∞-category of "stable" ∞-categories
• ∞-category of "stable" presentable" ∞-cat

• ∞-category of "stable" ∞-categories
• ∞-category of "stable" "presentable" ∞-cat.
all have a symmetric monoidal structure.

Recall If (A, μ, η) is a (algebra) monoid in a symmetric monoidal category, then A is commutative if

$$\begin{tikzcd} A \otimes A \arrow[rr, "B_{A,A}"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\ & A \arrow[u, phantom, "\cong"] & \end{tikzcd}$$

Def A <u>commutative algebra</u> (<u>comm. monoid</u>) in a symmetric monoidal ∞-category $p: M^{\otimes} \to N(\mathrm{Fin}_*)$ is a section $A: N(\mathrm{Fin}_*) \to M^{\otimes}$ of p sending inert arrows to p-cocartesian arrows.

Def $\alpha: \langle n \rangle \to \langle k \rangle$ in Fin_* is <u>inert</u> if $\alpha^{-1}(i)$ is a singleton for every $i = 1, \dots, k$.

Example • A commutative algebra on the ∞-category of spectra is called and <u>E_∞-ring</u> (spectrum).

§3. Organising those collections into ∞-categories

Def Let $p: M^{\otimes} \to N(\Delta^{op})$ and $q: N^{\otimes} \to N(\Delta^{op})$ be monoidal ∞-categories. A functor $F: M^{\otimes} \to N^{\otimes}$ over $N(\Delta^{op})$ is

i) <u>lax monoidal</u> if F sends p-cocartesian lifts of convex arrows in $N(\Delta^{op})$ to q-cocartesian arrows.

ii) <u>monoidal</u> if F sends all p-cocartesian

generalisation of concepts in

us

ii) monoidal if F sends all p-cocartesian arrows in $M^{\otimes}$ to q-cocartesian arrows in $N^{\otimes}$ — generalisation of concepts in 1-categories

Define $\mathrm{Fun}_{N(\Delta^{op})}(M^{\otimes}, N^{\otimes})$ to be the ∞-category of functors $M^{\otimes} \to N^{\otimes}$ over $N(\Delta^{op})$:

$$\begin{tikzcd}
\mathrm{Fun}_{N(\Delta^{op})}(M^{\otimes}, N^{\otimes}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \mathrm{Fun}(M^{\otimes}, N^{\otimes}) \arrow[d, "q_*"] \\
\Delta^{o} \arrow[r, "p"] & \mathrm{Fun}(M^{\otimes}, N(\Delta^{op}))
\end{tikzcd}$$

③ is ∞-cat

② is inner fib

① inner fib. since so is q.

Let $\mathrm{Fun}^{\mathrm{Lax}}(M^{\otimes}, N^{\otimes})$ be the full subcategory of $\mathrm{Fun}_{N(\Delta^{op})}(M^{\otimes}, N^{\otimes})$ spanned by the lax monoidal functors. And $\mathrm{Fun}^{\mathrm{Mon}}(M^{\otimes}, N^{\otimes}) \subseteq \mathrm{Fun}^{\mathrm{Lax}}(M^{\otimes}, N^{\otimes})$ spanned by monoidal functors.

Def Define a simplicially enriched category $\mathrm{Cat}_{\infty}^{\Delta,\mathrm{Mon}}$ to have

- objects monoidal ∞-categories
- mapping spaces

$$\mathrm{Map}_{\mathrm{Cat}_{\infty}^{\Delta,\mathrm{Mon}}}(M^{\otimes}, N^{\otimes}) = \mathrm{core}\, \mathrm{Fun}^{\mathrm{Mon}}(M^{\otimes}, N^{\otimes})$$

largest Kan complex contained

Define the ∞-category of monoidal ∞-categories by

$$N_{\Delta}(\mathrm{Cat}_{\infty}^{\Delta,\mathrm{Mon}}) =: \mathrm{Cat}_{\infty}^{\mathrm{Mon}}.$$

For a (fixed) monoidal ∞-category $p : M^{\otimes} \to N(\Delta^{op})$, define the ∞-category of algebra objects on M

$$\mathrm{Alg}_{A_\infty}(M^{\otimes}) := \mathrm{Fun}^{\mathrm{Lax}}(N(\Delta^{op}), M^{\otimes})$$

$\mathrm{Alg}_{A_\infty}(M^{\otimes}) := \mathrm{Fun}(N(\Delta^{op}), M^{\otimes})$

Sometimes $\mathrm{Alg}(M)$

or $\mathrm{Mon}(\mathrm{Cat}_\infty)$

$N(\Delta^{op}) \xrightarrow{id} N(\Delta^{op})$

($\Delta^{op} \xrightarrow{id} \Delta^{op}$ is a monoidal category corresponding to the terminal category)

similarly, define:

i) symmetric (lax) monoidal functors between sym. mon. ∞-cat.

ii) The ∞-category of sym. mon. ∞-cat.

iii) For $p: M^{\otimes} \to N(\mathrm{Fin}_*)$ symmetric monoidal,

$$\mathrm{Alg}_{E_\infty}(M^{\otimes}) = \infty\text{-cat. of commutative algebra objects on } M.$$

$= \mathrm{CMon}(\mathrm{Cat}_\infty)$