Recap: Basics on ∞-categories I

Def A simplicial set is a functor
$X : \Delta^{op} \to \mathrm{Set}$

Δ = simplex category

<u>objects</u>: $[n] = (0<1<\cdots<n)$, $n \in \mathbb{N}$

<u>morphisms</u>: $[n] \to [m]$ order-preserving

Picture:

$$\begin{tikzcd} X_0 \arrow[r, "s_0"'] & X_1 \arrow[l, shift right=3, "d_1"'] \arrow[l, shift left=3, "d_0"] \arrow[r, shift right=1, "s_0"'] \arrow[r, shift right=4, "s_1"'] & X_2 \cdots \arrow[l, shift right=6, "d_2"'] \arrow[l, shift right=3, "d_1"'] \arrow[l, "d_0"'] \end{tikzcd}$$

+ simplicial identities

e.g.
$d_0 s_0 = id$
$d_1 s_0 = id$

sSet = category of simplicial sets

Def For $n \in \mathbb{N}$,

$$\Delta^n = \mathrm{Hom}_\Delta(-,[n]) \in \mathrm{sSet}$$

(standard n-simplex)

For $0 \leq i \leq n$,

Λ^n_i = obtained from Δ^n by removing interior and face opposite to ith vertex.

<u>E.g.</u>

$$\begin{tikzcd} & 1 \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dr] \\ 0 \arrow[ur] & & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dr] \\ 0 \arrow[rr] & & 2 \end{tikzcd}$$

$$\Lambda^2_0 \qquad\qquad \Lambda^2_1 \qquad\qquad \Lambda^2_2$$

Def For $\mathcal{C}$ a (small) category, the nerve of $\mathcal{C}$ is the simplicial set $N\mathcal{C}$ with

$$(N\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$$
$$\simeq \left\{ \begin{array}{l} \text{strings of } n \text{ composable maps} \\ c_0 \xrightarrow{f_1} c_1 \to \cdots \xrightarrow{f_n} c_n \quad \text{in } \mathcal{C} \end{array} \right\}$$

Prop $N : \mathrm{Cat} \to \mathrm{sSet}$ is fully faithful.

Prop Let $X \in \mathrm{sSet}$. Then $X \simeq N\mathcal{C}$ for some category $\mathcal{C}$ if and only if

(*) For all $n \in \mathbb{N}$ and $0 < i < n$, any map $f : \Lambda^n_i \to X$ extends uniquely to a map $\tilde{f} : \Delta^n \to X$:

$\forall\, 0 < i < n$
(inner horns)

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\forall f"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists!\, \tilde{f} \text{ (unique)}"'] & \end{tikzcd}$$

Def A simplicial set X is a Kan complex if

$(*1)$ For all $n \in \mathbb{N}$ and $0 \leq i \leq n$, any map $f : \Lambda^n_i \to X$ extends to a map $\tilde{f} : \Delta^n \to X$:

$\forall\, 0 \leq i \leq n$ (all horns)

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists \tilde{f}"'] & \end{tikzcd}$$

(non-unique in general)

Def An ∞-category (or quasicategory) is a simplicial set $\mathcal{C}$ such that: For all $n \in \mathbb{N}$ and $0 < i < n$, any map $f : \Lambda^n_i \to \mathcal{C}$ extends to a map $\tilde{f} : \Delta^n \to \mathcal{C}$:

$\forall\, 0 < i < n$ (inner horns)

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed, "\exists \tilde{f}"'] & \end{tikzcd}$$

(non-unique in general)

Example

For $X \in \mathrm{Top}$, we have $\mathrm{Sing}\, X \in \mathrm{sSet}$ such that

$$(\mathrm{Sing}\, X)_n = \mathrm{Hom}_{\mathrm{Top}}(\underbrace{|\Delta^n|}_{\text{topological } n\text{-simplex}}, X)$$

Prop. $\mathrm{Sing}\, X$ is Kan.

$$\begin{tikzcd}
\{\text{categories}\} \arrow[r, "N"] & \{\infty\text{-categories}\} \arrow[r, phantom, "\supseteq"] \arrow[d, phantom, "\cap" description] & \{\text{Kan complexes}\} \\
 & \{\text{simplicial sets}\} & \left\{\begin{matrix}\text{Topological} \\ \text{spaces}\end{matrix}\right\} \arrow[u, "\mathrm{Sing}"']
\end{tikzcd}$$

Basics on ∞-categories 2

1) Homotopy category of an ∞-category
2) Coherent nerve of a simplicially enriched category

Homotopy category of an ∞-category

Let C be an ∞-category.

Terminology • Elements in C_0 are called vertices or objects of C

• Elements in C_1 are called edges or morphisms of C.

If $f \in C_1$, and $d_1 f = x$, $d_0 f = y$, we write $f : x \to y$.

$d_0 : C_1 \to C_0$ "target"

$d_1 : C_1 \to C_0$ "source"

Recall

By Yoneda, $C_n \simeq \mathrm{Hom}_{sSet}(\Delta^n, C)$ $\forall n$

e.g. $f \in C_1$ is $f : \Delta^1 \to C$.

Def Let $f, g : x \to y$ be two morphisms in C.
We say they are homotopic if $\exists \sigma : \Delta^2 \to C$ s.t.

boundary $\partial\sigma \overset{\text{def}}{:=} (d_0\sigma, d_1\sigma, d_2\sigma) = (g, f, id_x)$

$$\partial\sigma = \begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

We write $\sigma : f \to g$

Prop Let $x, y \in C$. Then the homotopy relation is an equivalence relation on the set of edges from x to y.

Proof Reflexivity Let $f : x \to y$ in C.
Then $\sigma = s_0 f$ we have

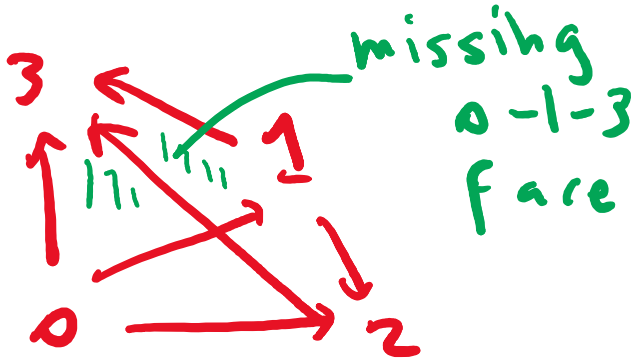
$$\partial\sigma = \begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "id"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

$d_0 s_0 f = f$
$d_1 s_0 f = f$
$d_2 s_0 f = s_0 d_1 f$
$= s_0 x = id_x$.

Symmetry Let $\sigma : f \to g$, that is

$$\partial\sigma = \begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "id"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

$$\Lambda^3_2 \subseteq \Delta^3$$

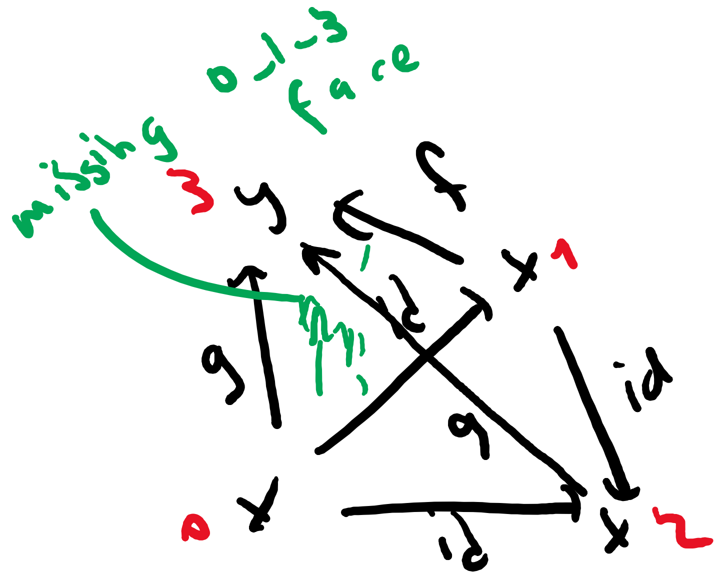


Define $\tau : \Lambda^3_2 \to C$

$$d_0\tau = \begin{tikzcd} & \overset{2}{x} \arrow[dr, "g"] & \\ {}_{1}x \arrow[ur, "id"] \arrow[rr, "f"'] & \sigma & y_{3} \end{tikzcd} \qquad d_1\tau = \begin{tikzcd} & \overset{2}{x} \arrow[dr, "g"] & \\ {}_{0}x \arrow[ur, "id"] \arrow[rr, "g"'] & s_0 g & y_{3} \end{tikzcd}$$

$$d_3\tau = \begin{tikzcd} & x^{1} \arrow[dr, "id_x"] & \\ {}_{0}x \arrow[ur, "id_x"] \arrow[rr, "id_x"'] & id_x & x_{2} \end{tikzcd}$$

$\tau =$



Since $\mathcal{C}$ is an ∞-category,

τ extends to a map $\tilde{\tau} : \Delta^3 \to \mathcal{C}$

Then $d_2 \tilde{\tau}$ has boundary

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "id"] \arrow[rr, "g"'] & & y \end{tikzcd} \quad \rightsquigarrow \quad d_2 \tilde{\tau} : g \to f.$$

<u>Transitivity</u> $f, g, h : x \to y$

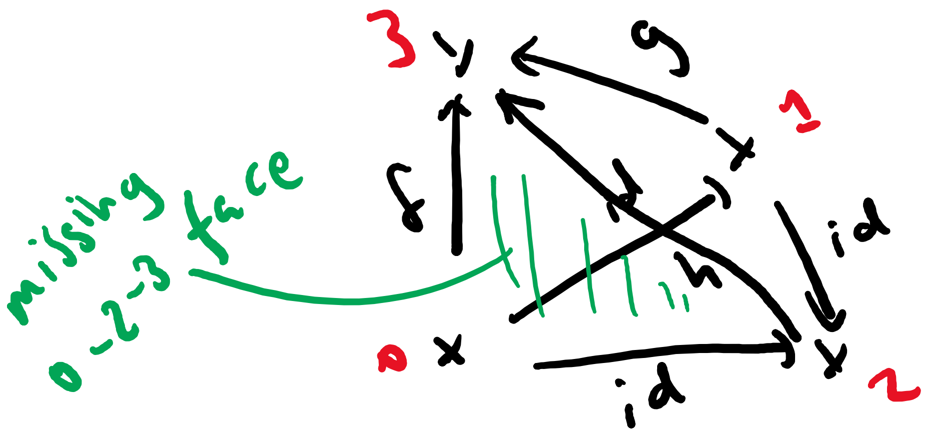
$\sigma : f \to g \qquad \sigma' : g \to h$

Define $\tau : \Lambda^3_1 \to \mathcal{C}$ by

$d_0 \tau = \sigma'$

$d_2 \tau = \sigma$

$d_3 \tau = id_x$



Extend τ to $\tilde{\tau} : \Delta^3 \to C$.
Then $d_1 \tilde{\tau} : f \to h$.

∎

<u>Notation</u> $[f]$ = htpy class of f.

Def Let $f : x \to y$ and $g : y \to z$ in C. A morphism $h : x \to z$ is called a composition of f and g if $\exists \sigma : \Delta^2 \to C$ such that

$$\partial\sigma = \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

Remark Given f and g, a composition h always exists: Extend the horn $\tau : \Lambda^2_1 \to C$

$$\tau = \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] & & z \end{tikzcd}$$

to $\sigma : \Delta^2 \to C$; then $h = d_1 \sigma$ is a composition of f and g (non-unique!)

Remark Let h be a composition of $f: x \to y$, $g : y \to z$, and $h' : x \to z$. Then h' is a composition of f and g if and only if $h \simeq h'$. (Exercise)

Proposition Let C be an ∞-category. There exists an ordinary category hC, called the homotopy category of C, such that:

i) The objects of hC are the vertices of C

ii) For $x, y \in C$, $\mathrm{Hom}_{hC}(x,y)$ is the set of htpy classes of edges $x \to y$

iii) Composition is given by

$$[g] \circ [f] = [h]$$

where h is <u>any</u> composition of f and g.

iv) For $x \in C$, the identity of x in hC is $[\mathrm{id}_x]$

Proof iii) Composition is well-defined:

Suppose $f \simeq f' : x \to y$, $g : y \to z$, $h : x \to z$ is a composition of f and g, and $h' : x \to z$ a composition of f' and g.

<u>Want</u> $h \simeq h'$.

<u>Have</u>

$\exists \rho : \Delta^2 \to C$

$$\partial \rho = \begin{tikzcd} & {}^{\color{red}1}x \arrow[d, "f"] \\ {}^{\color{red}0}x \arrow[ur, "id"] \arrow[r, "f'"'] & y^{\color{red}2} \end{tikzcd}$$

$\exists \sigma : \Delta^2 \to C$ $\quad \partial \sigma = \begin{tikzcd} & y^{\color{red}2} \arrow[dr, "g"] & \\ {}^{\color{red}1}x \arrow[ur, "f"] \arrow[rr, "h"'] & & z^{\color{red}3} \end{tikzcd}$

$\exists \sigma' : \Delta^2 \to C$

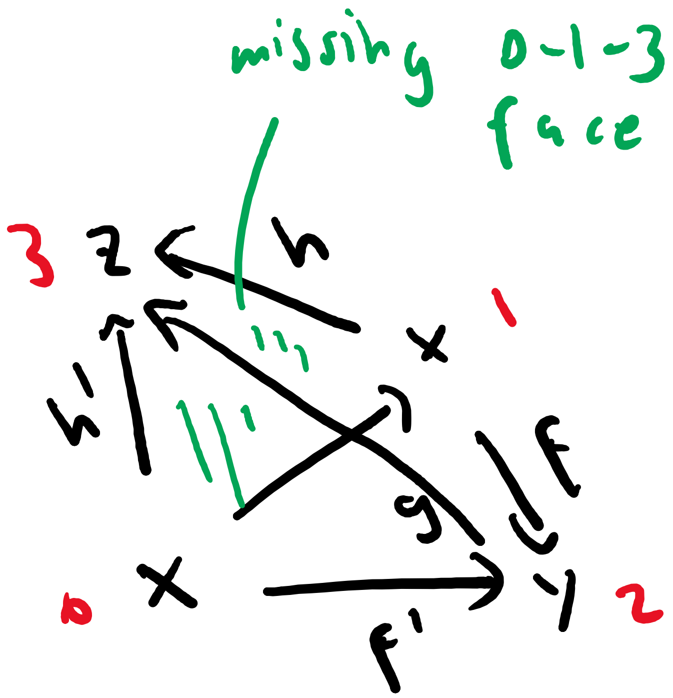
$$\partial \sigma' = \begin{tikzcd} & y^{\color{red}2} \arrow[dr, "g"] & \\ {}^{\color{red}0}x \arrow[ur, "f'"] \arrow[rr, "h'"'] & & z^{\color{red}3} \end{tikzcd}$$

Define $\tau : \Lambda^3_2 \to C$ by

$d_0\tau = \sigma$

$d_1\tau = \sigma'$

$d_3\tau = \rho$



Let $\tilde{\tau} : \Delta^3 \to C$ be an extension of τ. Then $d_2\tilde{\tau} : h' \to h$.

Similarly, if $f : x \to y$, $g \simeq g' : y \to z$, and h is a composition of f and g, and h' a composition of f and g', then $h \simeq h'$. It follows that composition is well-defined.

Similarly, composition is associative.

iii) For $x, y \in C$ and $f : x \to y$,

$$\partial(s_0 f) = \begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"'] & & y \end{tikzcd} \qquad \partial(s_1 f) = \begin{tikzcd} & y \arrow[dr, "id_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

$$\rightsquigarrow \quad [f] \circ \underbrace{[id_x]}_{\text{two sided inverse}} = [f] \qquad [id_y] \circ [f] = [f]$$

∎

Remark

$h : \underbrace{Cat_\infty}_{\substack{\text{category of small} \\ \infty\text{-categories} \\ \text{(full sub. of sSet)}}} \to Cat$ is left adjoint to $N : Cat \to Cat_\infty$

There is also $h : sSet \to Cat$ a left adjoint to $N : Cat \to sSet$

(Notion of htpy category of a simplicial set S)

Def A morphism $f: x \to y$ in C is an equivalence if $[f]$ is an isomorphism in hC.

Def An ∞-category C is an ∞-groupoid if every morphism of C is an equivalence.

Prop Let C be an ∞-category. Then C is an ∞-groupoid if and only if C is a Kan complex.

Proof "easy part" Suppose C is Kan. Let $f: x \to y$ in C.

Consider $\tau : \Lambda^2_0 \to C$.

outer horn

$$\tau = \begin{tikzcd} & y \\ x \arrow[ur, "f"] \arrow[rr, "id"'] & & x \end{tikzcd}$$

$0 \quad 1 \quad 2$

Since C is Kan, τ extends to $\tilde{\tau}: \Delta^2 \to C$. Then $[d_0 \tilde{\tau}]$ is a left inverse of $[f]$.

Similarly, $[f]$ has a right inverse, hence an inverse.

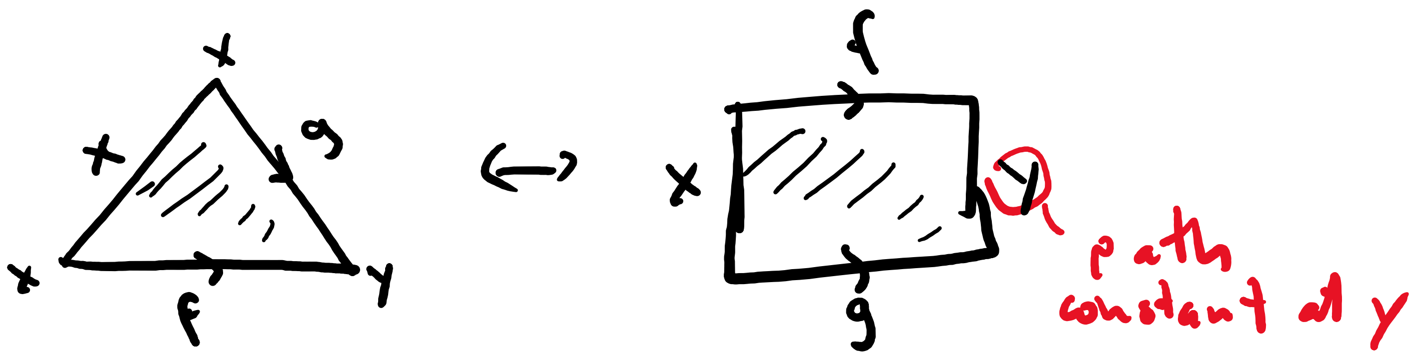
"Hard part: ∞-groupoid $\Rightarrow$ Kan" Omitted.

Examples • Let C be a category. Then $h(NC) = C$: if $f, g \in (NC)_1$ (i.e. f and g are maps in C) then $f \simeq g$ if and only if $f = g$:

$$\partial\sigma = \begin{tikzcd} & y \arrow[dr, "g"] \\ x \arrow[ur, "id"] \arrow[rr, "f"'] & & y \end{tikzcd} \quad \rightsquigarrow \quad f = g \circ id = g$$

So NC is an ∞-groupoid (a Kan complex) if and only if C is a groupoid.

- Let $X \in \mathrm{Top}$. Then $h\,\mathrm{Sing}\,X$ has:
  - objects are points $x \in X$
  - For $x, y \in X$, $\mathrm{Hom}_{h\mathrm{Sing}X}(x,y)$ is the set of pointed homotopy classes of paths from x to y:



iii) Composition is given by concatenation of paths:

$$[g] \circ [f] = [f * g]$$

any composition

Thus, $h\,\mathrm{Sing}\,X = \pi_{\leq 1}(X) =$ fundamental groupoid of X.

Coherent Nerve

Def A simplicially enriched category (simplicial category) is a category enriched over sSet: It consists of

i) A collection of objects $ob(C)$

ii) For $x,y \in ob(C)$, a simplicial set $Map_C(x,y) \in sSet$

iii) For $x,y,z \in ob(C)$, a composition law

$$\circ : Map_C(Y,Z) \times Map_C(X,Y) \longrightarrow Map_C(X,Z)$$

map in sSet

product in sSet : $(S \times T)_n = S_n \times T_n$

iv) For $x \in ob(C)$, a vertex $id_x \in Map_C(x,x)_0$

These data must satisfy

A) composition is associate

u) For $x,y \in ob(C)$, both composites

$$Map_C(X,Y) \times \{id_X\} \hookrightarrow Map_C(X,Y) \times Map_C(X,X) \xrightarrow{\circ} Map_C(X,Y)$$

$$\{id_Y\} \times Map_C(X,Y) \hookrightarrow Map_C(Y,Y) \times Map_C(X,Y) \xrightarrow{\circ} Map_C(X,Y)$$

are the projection

Recall For C a category,

$$(NC)_\bullet = \mathrm{Hom}_{Cat}([\bullet], C)$$

Want "thickening" $C[\Delta^n] \in sCat$ of $[n] \in Cat$.

category of simplicially enriched cat.

Def For $n \in \mathbb{N}$, $C[\Delta^n] \in sCat$:

i) objects are $0, 1, \dots, n$ (same as $[n]$)

ii) For $i, j \in C[\Delta^n]$,

$$\mathrm{Map}_{C[\Delta^n]}(i,j) = N P_{ij} \in sSet$$

where P_{ij} = poset of finite linearly ordered subsets S of $\{i < \dots < j\}$ such that $i, j \in S$

$= \{ S \subseteq \{i, \dots, j\} \mid i, j \in S \}$ ordered by inclusion

iii) Composition is induced by the union operation: If $i, j, k \in C[\Delta^n]$, then

$$\mathrm{Map}(j,k) \times \mathrm{Map}(i,j) \to \mathrm{Map}(i,k)$$

is given on vertices by $(S, T) \mapsto S \cup T$.

iv) For $i \in C[\Delta^n]$, $id_i = \{i\} \in \mathrm{Map}(i,i)_0$.

$$C[\Delta^0] = 0$$

$$C[\Delta^1] = 0 \longrightarrow 1$$

$$C[\Delta^2] = \begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\Uparrow"] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd}$$

htpy encoded by inclusion
$\{0,2\} \subseteq \{0,1,2\}$
$= \{1,2\} \circ \{0,1\}$

Def The coherent nerve of a simplicially enriched category C is $N_\Delta C \in sSet$ given by

$$(N_\Delta C)_\bullet = \mathrm{Hom}_{sCat}(\underline{C[\Delta^\bullet]}, C)$$

$[n] \to [m] \rightsquigarrow C[\Delta^n] \to C[\Delta^m]$

Prop Let C be a simplicially enriched category such that $\forall\, x,y \in C$, $\mathrm{Map}_C(x,y)$ is Kan. Then $N_\Delta C$ is an ∞-category.

Proof omitted. Lurie's HTT Prop. 1.1.5.10.

Example sSet is simplicially enriched: For $X, Y \in$ sSet,

$$\mathrm{Map}_{sSet}(X,Y) := \mathrm{Fun}(X,Y) := \mathrm{Hom}_{sSet}(\Delta^\bullet \times X, Y) \in sSet$$

Let Kan be the full simplicially enriched subcategory of sSet spanned by Kan complexes.

Fact $X \in sSet$, $K \in$ Kan $\Rightarrow$ $\mathrm{Fun}(X,K)$ is Kan.

$\rightsquigarrow$ N_ΔKan is an ∞-category, called the ∞-category $\mathcal{S}$ of spaces

(e.g. ∞-categorical Yoneda Lemma:
For $X \in sSet$, have a Yoneda embedding

$$X \to \mathrm{Fun}(X^{op}, \mathcal{S})$$

which is fully faithful)