

let \mathcal{C} be an ordinary cat.

The slice category \mathcal{C}/x , $\mathcal{C}/_x$ for an object $x \in \mathcal{C}$

$\text{Ob}(\mathcal{C}/x) = \text{morphisms } f: c \rightarrow x \text{ in } \mathcal{C}$

$\text{morph}(f', f) = g: c \rightarrow c' \text{ s.t.}$

$$\begin{array}{ccc} c & \xrightarrow{g} & c' \\ f \searrow & \alpha & \swarrow f' \\ & x & \end{array}$$

$\text{Ob}(\mathcal{C}/_x) = \text{functors } f: [0] \star [0] \rightarrow \mathcal{C} \text{ s.t.}$

$$f|_{[0]} = x$$

$\text{morph}(f', f) = \text{functors}$

$$g: [1] \star [0] \rightarrow \mathcal{C} \text{ s.t.}$$

$$g|_{[0]} = x$$

More generally: Given $p: A \rightarrow C$ a functor

$$\text{Ob}(C_p) = \text{funct } f: [0] \star A \rightarrow C$$

$$f|_A = p$$

$$\text{morph} = \text{functors } g: [1] \star A \rightarrow C$$

$$\text{st. } g|_A = p.$$

Prop: Let $p: A \rightarrow C$ be a functor. Then a colimit of p is an initial obj. in C_p .

A limit of p is a terminal obj. in C_p .

C_p is right adj. to $- \star A$

Recall \forall sSet S the functors

$S \star -$, $- \star S : \text{sSet} \rightarrow \text{sSet}_S$
preserves colimits.

Def/Prop: $\forall p: S \rightarrow X$ map of sSet it \exists a
simplicial set $X_{|p}$ s.t.

$$\text{hom}_{\text{sSet}}(K, X_{|p}) \cong \text{hom}_{\text{sSet}_S}(S \star K, X)$$

$$\text{hom}_{\text{sSet}}(K, X_{|p}) \cong \text{hom}_{\text{sSet}_S}(K \star S, X)$$

$$K = \Delta^n$$

$$(X_{|p})_n \cong \text{hom}_{\text{sSet}_S}(S \star \Delta^n, X)$$

In particular $\alpha: \Delta^0 \rightarrow X$

$$(X/\alpha)_n \cong \text{hom}_{\text{SSet}_{\Delta^0}}(\Delta^0 \star \Delta^n, X) \cong \text{hom}_{\text{SSet}_*}((\Delta^n)^{\Delta^0}, v), (X, \alpha) \quad \textcircled{1}$$

$$(X/\alpha)_n \cong \text{hom}_{\text{SSet}_*}((\Delta^n)^{\Delta^0}, v), (X, \alpha) \quad \textcircled{2}$$

In case $\textcircled{1}$ v is $\{n+1\}$

In case $\textcircled{2}$ v is $\{0\}$.

Prop: let $p: A \rightarrow B$ be a functor between ord. cat.

$$\rightarrow N(B/p) \cong N(B)/N(p)$$

Proof: $(N(B/p))_n \cong \text{hom}_{\text{Cat}}(\text{Inj}, B/p)$

$$\cong \text{hom}_{\text{Cat}_{A_1}}(\text{Inj} \star A, B)$$

N is fully faithful

$$\cong \text{hom}_{\text{SSet}_{N(A)}} (N(C_n) \# A, N(B))$$

$N(-) \# - \rightarrow$
 $\cong N(-) \# N(-)$
 $N(C_n) \cong \Delta^n$

$$\cong \text{hom}_{\text{SSet}_{N(A)}} (\Delta^n \# N(A), N(B))$$

$$\cong \text{hom}_{\text{SSet}} (\Delta^n, N(B) / N(P))$$

$$\cong (N(B) / N(P))_n$$

Remark: Given $p: S \rightarrow X$ morphism of sSet
Then X/p and X_p come with natural morphisms □.

$$X/p \xrightarrow{\quad} X \xleftarrow{\quad} X/p$$

in level n : $\alpha: \Delta^n \star S \rightarrow X \longmapsto \Delta^n \rightarrow \Delta^n \star S \xrightarrow{\alpha} X$

$\Delta^n \rightarrow S \star \Delta^n \xrightarrow{\beta} X$

$S \star \Delta^n \xrightarrow{\beta} X$

Prop: If \mathcal{C} is an \mathcal{X} -cat, $x \in \mathcal{C}_0$ is an obj

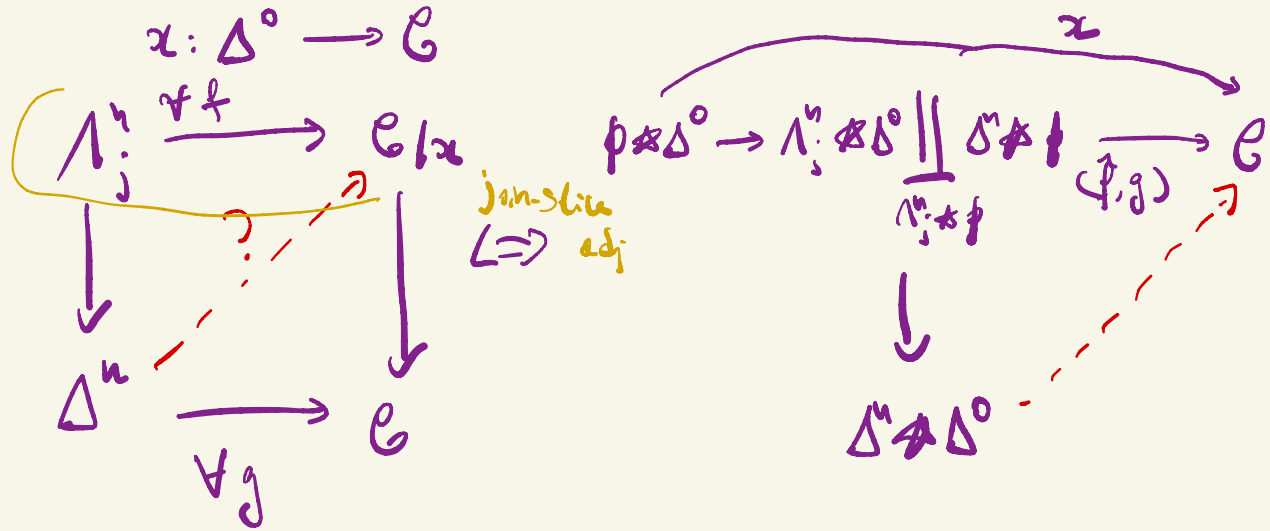
$\Rightarrow \mathcal{C}_{x/} \rightarrow \mathcal{C}$ is a left fibration

$\mathcal{C}/_x \rightarrow \mathcal{C}$ is a right fibration

Recall: $f: X \rightarrow Y$ morphism of set is a right/left fibration if it has the RLP wrt

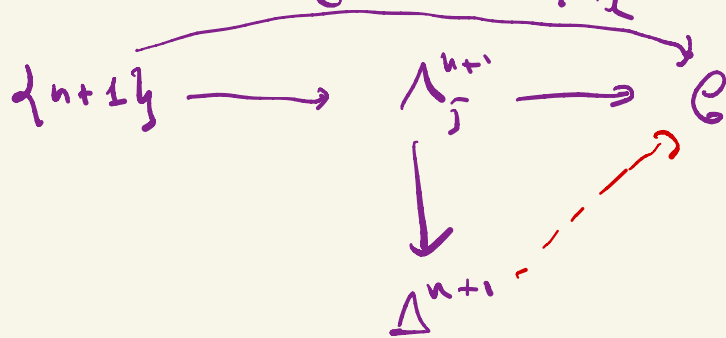
$$\Delta_k^n \hookrightarrow \Delta^n \quad \underline{0} \leq k \leq n \quad n \geq 1$$

Proof: $\mathcal{C}/\alpha \rightarrow \mathcal{C}$ is a right fibration.



Under The identification $\Delta^m \star \Delta^0 \cong \Delta^{m+1}$
 $\phi \star \Delta^0$ corresponds to the vertex $\{n+1\}$
 and the disc. is $\cong \Lambda_j^{n+1}$

→ the second diagram is equivalent to the diagram



Since \mathcal{C} is an ∞ -cat the red arrow exists
 $\forall 0 < j \leq n$

Moreover $\mathcal{C}/\alpha \rightarrow \mathcal{C} \rightarrow *$ is an inner fibration

⇒ \mathcal{C}/α is an ∞ -cat. □

Initial & Terminal obj

Def. Let x be an object in an ∞ -cat \mathcal{C}

Then x is

- INITIAL if $\forall f: \Delta^n \rightarrow \mathcal{C}$

s.t. $f|_{\partial\Delta^n} = x \quad \exists$ an extension

$$f': \Delta^n \rightarrow \mathcal{C}.$$

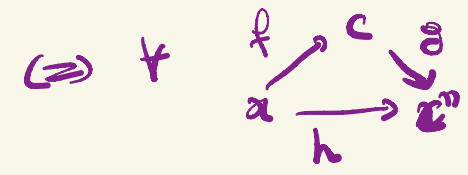
- TERMINAL if $\forall f: \Delta^n \rightarrow \mathcal{C}$

s.t. $f|_{\partial\Delta^n} = x \quad \exists$ an extension

$$f': \Delta^n \rightarrow \mathcal{C}.$$

Rmk: For $n=1$ INITIAL $\Leftrightarrow \forall c \in C_0 \exists f: a \rightarrow c$
 $n=2$

$a \in C$ initial iff
 $\text{Map}(a, c)$ is contractible
 $\text{Map}(a, c) \rightarrow \text{Map}(\Delta^1, C)$
 $\downarrow \quad \downarrow (\delta, t)$
 $\text{pt} \rightarrow C \times C$



\exists an ext $\Rightarrow [h] = [f] \circ [g]$

If we take $f = \text{Id}_a$
 $\Rightarrow [h] = [g]$

$\Rightarrow \text{Map}_{h(c)}(a, c) = \text{pt} \quad \forall c \in C_0$

Prop: $a \in C_0$ is initial $\Leftrightarrow C_0 \rightarrow C$ is a trivial fibration

Recall: $f: X \rightarrow Y$ is a trivial fibration if it has the LLP wrt $d\Delta^n \rightarrow \Delta^n \quad \forall n$.

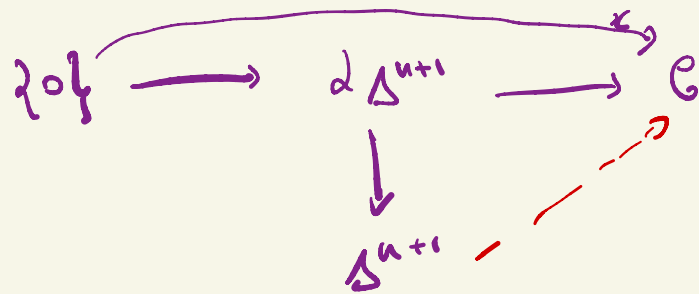
Proof:

$$\begin{array}{ccc}
 \mathcal{D}\Delta^n & \xrightarrow{f} & \mathcal{C}_2 \\
 \downarrow & \nearrow ? & \downarrow \\
 \Delta^n & \xrightarrow{g} & \mathcal{C}
 \end{array}
 \Leftrightarrow$$

$$\begin{array}{ccc}
 \Delta^0 \star \phi & \xrightarrow{\quad} & (\Delta^0 \star \mathcal{D}\Delta^n) \parallel (\phi \star \Delta^n) \xrightarrow{(f, g)} \mathcal{C} \\
 & & \downarrow \phi \star \mathcal{D}\Delta^n \\
 & & \Delta^0 \star \Delta^n
 \end{array}$$

(A curved arrow labeled α connects $\Delta^0 \star \phi$ to \mathcal{C} in the top row.)

Using the identification $\Delta^0 \star \Delta^n \cong \Delta^{n+1}$



Which has a solution because $a \in C_0$ is initial. \square

Cor: let $C^{int} \subseteq C$ be the ∞ -cat spanned by initial objects (e.g. $\alpha: \Delta^n \rightarrow C$ is an n -simplex in $C^{init} \Leftrightarrow$ all the vertices of α are in C^{int})

Then C^{init} is either empty or contractible

$C^{init} \rightarrow \Delta^0$ is a trivial fibration.

proof:

$$\begin{array}{ccc} \Delta^n & \xrightarrow{f} & \mathcal{C}^{\text{init}} \\ \downarrow & \nearrow \tilde{f} & \\ \Delta^n & & \end{array}$$

$f(\cdot)$ is initial
 $\Rightarrow \exists \tilde{f}: \Delta^n \rightarrow \mathcal{C}^{\text{init}}$

In part. order

$$\begin{array}{ccc} \Delta^n & \xrightarrow{f} & \mathcal{C}^{\text{init}} \\ \downarrow & \nearrow h & \downarrow \\ \Delta^n & \xrightarrow{f_g} & \Delta^0 \end{array}$$

the lifting is
 the same as
 an extension
 of f .

Def: let $p: K \rightarrow \mathcal{C}$ be a functor with \mathcal{C} an ~~in~~-cat.

A colimit of p is an initial obj in \mathcal{C}_p

A limit of p is a terminal obj in \mathcal{C}_p

Rmk: Given $p: K \rightarrow \mathcal{C}$, $\tilde{p}: K \star \Delta^0 \rightarrow \mathcal{C}$ is
 a colimit of p if p

$$\begin{array}{ccccc}
 K \star \{0\} & \longrightarrow & K \star \Delta^n & \longrightarrow & \mathcal{C} \\
 & & \downarrow \circlearrowleft & \nearrow \exists & \\
 & & K \star \Delta^n & &
 \end{array}$$

Ex: $K = \emptyset \Rightarrow p: \emptyset \rightarrow \mathcal{C} \Rightarrow \mathcal{C}_{p_1} = \mathcal{C}$ & colimit of
 p is just an initial obj. of \mathcal{C} .

Def: $K = \Lambda_0^2 \Rightarrow (\Lambda_0^2)^\triangleright \cong \Delta^1 \times \Delta^1$

A colimit of $\Lambda_0^2 \rightarrow \mathcal{C}$ is a pushout in \mathcal{C}
 limit $\Lambda_0^2 \rightarrow \mathcal{C}$ pullback in \mathcal{C} .

Prop: Given $p: K \rightarrow \mathcal{C}$, then the full subcategory spanned by colimits of p , $\mathcal{C}_{p_1}^{\text{colim}} \subseteq \mathcal{C}_{p_1}$ is either empty or contractible.

Prop: Let $\tilde{p}: K^\Delta \rightarrow \mathcal{C}$ and let $p = \tilde{p}|_K$

Then \tilde{p} is a colimit of p

$\Leftrightarrow \mathcal{C}_{\tilde{p}_1} \rightarrow \mathcal{C}_{p_1}$ is a trivial fibration.

Proof: \tilde{p} is a colimit of $p \Leftrightarrow$ is initial in \mathcal{C}_{p_1}

$\Leftrightarrow (\mathcal{C}_{p_1})_{\tilde{p}_1} \rightarrow \mathcal{C}_{p_1}$ is a trivial fibration

\parallel

$\mathcal{C}_{\tilde{p}_1}^{\text{colim}}$

□.

Prop: $p: S \rightarrow C$, $\pi: C/p \rightarrow C$

i) Let $x \in (C/p)_0$ s.t. $\pi(x)$ is initial in C
 $\Rightarrow x$ is initial in C/p .

ii) $x \in (C/p)_0$ s.t. $\pi(x)$ is terminal in C
 $\Rightarrow x$ is terminal in C/p .

Slogan: ∞ -colimits are homotopy colimits.

Thm (4.2.4.4, Lurie HTT)

Let $F: \overline{\delta} \rightarrow C$ sSet-enriched functor
between (Kan complex)-enriched
categories

(C is a simplicial model category)

Then a cone $\tilde{F}: J \star [0] \rightarrow C$
is a htpy colimit \Leftrightarrow

$$N_{\Delta}(\tilde{F}): N_{\Delta}(J) \star \delta^0 \rightarrow N_{\Delta}(C)$$

is a ∞ -colimit.

"□".