Let $\mathcal{C}$ be an ordinary cat.

The slice category $C_{/x}$, $C_{x/}$ for an object $x \in \mathcal{C}$

$Ob(C_{/x})$ = morphisms $f : c \to x$ in $\mathcal{C}$

$morp(f', f)$ = $g : c \to c'$ s.t.

$$\begin{tikzcd} c \arrow[rr, "g"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowleft"] & c' \arrow[dl, "f'"] \\ & x & \end{tikzcd}$$

$Ob(C_{/x})$ = functors $f : [0] \star [0] \longrightarrow C$ s.t.

$$f|_{\emptyset \star [0]} = x$$

$morph(f', f)$ = functors $g : [1] \star [0] \longrightarrow C$ s.t.

$$g|_{\emptyset \star [0]} = x$$

More generally: Given $p: A \to C$ a functor

$Ob(C_{p/}) =$ fun $f: [0] \star A \to C$

$f|_A = p$

morph $=$ functors $g: [1] \star A \to C$

s.t. $g|_A = p$.

Prop: Let $p: A \to C$ be a functor. Then a colimit of $p$ is an initial obj. in $C_{p/}$.

A limit of $p$ is a terminal obj in $C_{/p}$.

$C_{p/}$ is right adj. to $- \star A$

Recall $\forall$ sSet $S$ the functors

$$S \star - \ , \ - \star S : sSet \longrightarrow sSet_{S/}$$

preserves colimits.

Def/Prop: $\forall\ p : S \to X$ map of sSet it $\exists$ a simplicial set $X_{/p}$ s.t.

$$\hom_{sSet}(K, X_{p/}) \cong \hom_{sSet_{S/}}(S \star K, X)$$

$$\hom_{sSet}(K, X_{/p}) \cong \hom_{sSet_{/S}}(K \star S, X)$$

$K = \Delta^n$

$$(X_{p/})_n \cong \hom_{sSet_{S/}}(S \star \Delta^n, X)$$

In particular $x : \Delta^0 \to X$

$$(X_{x/})_n \cong \hom_{sSet_{\Delta^0/}}(\Delta^0 \star \Delta^n, X) \cong \hom_{sSet_*}(([\Delta^n]^{\triangleleft}, v), (X, x))$$ (1)

$$(X_{/x})_n \cong \hom_{sSet_*}(((\Delta^n)^{\triangleright}, v), (X, x))$$ (2)

In case (1) $v$ is $\{n+1\}$

In case (2) $v$ is $\{0\}$.

---

Prop: let $p : A \to B$ be a functor between ord. cat.

$\rightarrow$ $N(B/p) \cong N(B)/N(p)$

Proof: $(N(B/p))_n \cong \hom_{Cat}([n], B/p)$

$$\cong \hom_{Cat_{A/}}([n] \star A, B)$$

$N$ is fully faithful

$$\cong \hom_{sSet_{N(A)/}}\big(N([n] \star A), N(B)\big)$$

$N(- \star -) \cong N(-) \star N(-)$

$N([n]) \cong \Delta^n$

$$\cong \hom_{sSet_{N(A)/}}\big(\Delta^n \star N(A), N(B)\big)$$

$$\cong \hom_{sSet}\big(\Delta^n, N(B)_{/N(p)}\big)$$

$$\cong \big(N(B)_{/N(p)}\big)_n$$

$\square$.

<u>Rmk</u>: Given $p : S \to X$ morphism of sSet

Then $X_{/p}$ and $X_{p/}$ come with natural morphisms

$$X_{/p} \longrightarrow X \longleftarrow X_{p/}$$

in level $n$: $\alpha : \Delta^n \star S \to X \longmapsto \Delta^n \to \Delta^n \star S \xrightarrow{\alpha} X$

$S \star \Delta^n \xrightarrow{\beta} X \longmapsto \Delta^n \to S \star \Delta^n \xrightarrow{\beta} X$

<u>Prop</u>: If $\mathcal{C}$ is an $\infty$-cat, $x \in \mathcal{C}_0$ is an obj

$\Rightarrow$ $\mathcal{C}_{x/} \longrightarrow \mathcal{C}$ is a left fibration

$\mathcal{C}_{/x} \longrightarrow \mathcal{C}$ is a right fibration

<u>Recall</u>: $f : X \to Y$ morphism of sSet is a right/left fibration if it has the RLP wrt

$\Lambda^n_k \hookrightarrow \Delta^n \qquad 0 < k \leq n \qquad n \geq 1$

**Proof:** $\mathcal{C}_{/x} \to \mathcal{C}$ is a right fibration.

$x : \Delta^0 \longrightarrow \mathcal{C}$

$$
\begin{tikzcd}
\Lambda^n_j \arrow[r, "\forall f"] \arrow[d] & \mathcal{C}_{/x} \arrow[d] \\
\Delta^n \arrow[r, "\forall g"'] \arrow[ur, dashed, "?"] & \mathcal{C}
\end{tikzcd}
$$

join-slice $\Longleftrightarrow$ adj

$$
\begin{tikzcd}
\emptyset \star \Delta^0 \arrow[r] \arrow[rr, bend left, "x"] & \Lambda^n_j \star \Delta^0 \coprod_{\Lambda^n_j \star \emptyset} \Delta^n \star \emptyset \arrow[r, "{(f,g)}"] \arrow[d] & \mathcal{C} \\
& \Delta^n \star \Delta^0 \arrow[ur, dashed] &
\end{tikzcd}
$$

Under the identification $\Delta^n \star \Delta^0 \cong \Delta^{n+1}$

$\emptyset \star \Delta^0$ corresponds to the vertex $\{n+1\}$

and the disj. is $\cong \Lambda^{n+1}_j$

$\Rightarrow$ the second diagram is equivalent to the diagram

$$\begin{tikzcd}
\{n+1\} \arrow[r] \arrow[rr, bend left, "x"] & \Lambda^{n+1}_j \arrow[r] \arrow[d] & \mathcal{C} \\
& \Delta^{n+1} \arrow[ur, dashed] &
\end{tikzcd}$$

Since $\mathcal{C}$ is an $\infty$-cat the red arrow exists

$\forall \quad 0 < j \leq n$

Moreover $\quad \mathcal{C}_{/x} \to \mathcal{C} \to *$ is an inner fibration

$\Rightarrow \quad \mathcal{C}_{/x}$ is an $\infty$-cat. $\quad \square$

# Initial & Terminal obj

**Def:** Let $x$ be an element in an $\infty$-cat $\mathcal{C}$.
Then $x$ is

- **INITIAL** if $\forall f : \partial\Delta^n \to \mathcal{C}$ s.t. $f|_{\{0\}} = x$ $\exists$ an extension $f' : \Delta^n \to \mathcal{C}$.

- **TERMINAL** if $\forall f : \partial\Delta^n \to \mathcal{C}$ s.t. $f|_{\{n\}} = x$ $\exists$ an extension $f' : \Delta^n \to \mathcal{C}$.

**Rmk:** For $n=1$ INITIAL $\iff \forall\, c \in \mathcal{C}_0\ \exists\, f : x \to c$

$n=2$ $\iff \forall$

$$\begin{tikzcd} & c \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & c'' \end{tikzcd}$$

$\exists$ an ext $\Rightarrow [h] = [f] \circ [g]$

If we take $f = \mathrm{Id}_x$

$\Rightarrow [h] = [g]$

$\Rightarrow \mathrm{Map}_{h(\mathcal{C})}(x,c) = pt \quad \forall c \in \mathcal{C}_0$.

$x \in \mathcal{C}$ initial iff

$\mathrm{Map}(x,c)$ is contractible

$$\begin{tikzcd} \mathrm{Map}(x,c) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\Delta^1, \mathcal{C}) \arrow[d, "(s,t)"] \\ \{pt\} \arrow[r] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

**Prop:** $x \in \mathcal{C}_0$ is initial $\iff \mathcal{C}_{x/} \to \mathcal{C}$ is a trivial fibration

**Recall:** $f : X \to Y$ is a trivial fibration if it has the RLP wrt $\partial\Delta^n \to \Delta^n$ $\forall n$.

Proof:

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "f"] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^n \arrow[r, "g"'] \arrow[ur, dashed, "?"] & \mathcal{C} \end{tikzcd} \quad \Longleftrightarrow$$

$$\begin{tikzcd} \Delta^0 \star \emptyset \arrow[r] \arrow[rr, bend left, "x"] & (\Delta^0 \star \partial\Delta^n) \coprod_{\emptyset \star \partial\Delta^n} (\emptyset \star \Delta^n) \arrow[r, "(f,g)"'] \arrow[d] & \mathcal{C} \\ & \Delta^0 \star \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Using the identification $\Delta^0 \star \Delta^n \cong \Delta^{n+1}$

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[rr, bend left, "x"] & \partial\Delta^{n+1} \arrow[r] \arrow[d] & \mathcal{C} \\
& \Delta^{n+1} \arrow[ur, dashed] &
\end{tikzcd}$$

which has a solution because $x \in \mathcal{C}_0$ is initial. $\square$

<u>Cor:</u> Let $\mathcal{C}^{init} \subseteq \mathcal{C}$ be the $\infty$-cat spanned by initial objects (e.g. $\alpha: \Delta^n \to \mathcal{C}$ is an $n$-simplex in $\mathcal{C}^{init}$ $\iff$ all the vertices of $\alpha$ are in $\mathcal{C}^{init}$)

Then $\mathcal{C}^{init}$ is either empty or contractible

$\mathcal{C}^{init} \longrightarrow \Delta^0$ is a trivial fibration.

proof:

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\forall f"] \arrow[d] & \mathcal{C}^{init} \\ \Delta^n \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

$f(0)$ is initial $\Rightarrow \exists \hat{f} : \Delta^n \to \mathcal{C}^{init}$

In particular

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\forall f"] \arrow[d] & \mathcal{C}^{init} \arrow[d] \\ \Delta^n \arrow[r, "\forall g"'] \arrow[ur, "h"] & \Delta^0 \end{tikzcd}$$

the lifting is the same as an extension of $f$. $\square$

---

**Def**: let $p: K \to \mathcal{C}$ be a functor with $\mathcal{C}$ an $\infty$-cat.

A colimit of $p$ is an initial obj in $\mathcal{C}_{p/}$

A limit of $p$ is a terminal obj in $\mathcal{C}_{/p}$

**Rmk:** Given $p: K \to \mathcal{C}$, $\tilde{p}: K \star \Delta^0 \to \mathcal{C}$ is a colimit of $p$ if

$$\begin{tikzcd}
K \star \{0\} \arrow[r] \arrow[rr, bend left, "p"] & K \star \partial\Delta^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{C} \\
 & K \star \Delta^n \arrow[ur, "\exists"'] &
\end{tikzcd}$$

**Ex:** $K = \emptyset \Rightarrow p: \emptyset \to \mathcal{C} \Rightarrow \mathcal{C}_{p/} = \mathcal{C}$ & a colimit of $p$ is just an initial obj. of $\mathcal{C}$.

**Def:** $K = \Lambda^2_0 \Rightarrow (\Lambda^2_0)^{\triangleright} \cong \Delta^1 \times \Delta^1$

A colimit of $\Lambda^2_0 \to \mathcal{C}$ is a pushout in $\mathcal{C}$

limit $\Lambda^2_0 \to \mathcal{C}$ pullback in $\mathcal{C}$.

**Prop:** Given $p: K \to \mathcal{C}$, then the full subcategory spanned by colimits of $p$, $\mathcal{C}_{p/}^{\mathrm{colim}} \subseteq \mathcal{C}_{p/}$ is either empty or contractible.

**Prop:** Let $\tilde{p}: K^{\triangleright} \to \mathcal{C}$ and let $p = \tilde{p}|_K$.

Then $\hat{p}$ is a colimit of $p$

$\iff$ $\mathcal{C}_{\tilde{p}/} \longrightarrow \mathcal{C}_{p/}$ is a trivial fibration.

**Proof:** $\hat{p}$ is a colimit of $p$ $\iff$ is initial in $\mathcal{C}_{p/}$

$$\iff \begin{array}{c} (\mathcal{C}_{p/})_{\tilde{p}/} \longrightarrow \mathcal{C}_{p/} \text{ is a trivial fibration} \\ \cong \\ \mathcal{C}_{\tilde{p}/} \end{array}$$

$\square$.

**Prop**: $p : S \to \mathcal{C}$, $\pi : \mathcal{C}_{/p} \to \mathcal{C}$

i) If $x \in (\mathcal{C}_{/p})_0$ s.t. $\pi(x)$ is initial in $\mathcal{C}$

$\Rightarrow$ $x$ is initial in $\mathcal{C}_{/p}$.

ii) $x \in (\mathcal{C}_{p/})_0$ s.t. $\pi(x)$ is terminal in $\mathcal{C}$

$\Rightarrow$ $x$ is terminal in $\mathcal{C}_{p/}$.

---

**Slogan**: $\infty$-colimits are homotopy colimits.

**Thm** (4.2.4.1, Lurie HTT)

Let $F : \mathcal{J} \to \mathcal{C}$ sSet-enriched functor between (Kan cplx)-enriched categories

($\mathcal{C}$ is a simplicial model category)

Then a cocone $\tilde{F} : J \star [0] \longrightarrow \mathcal{C}$

is a htpy colimit $\Leftrightarrow$

$$N_\Delta(\tilde{F}) : N_\Delta(J) \star \Delta^0 \longrightarrow N_\Delta(\mathcal{C})$$

is a $\infty$-colimit.

"$\square$".