Monoidal Infinity Categories I

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Monoidal categories:

HTT Appendix A.

A *monoidal category* is a category $\mathcal{C}$ equipped with a (coherently) associative "product" functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathbf{1}$. The associativity is expressed by demanding isomorphisms

→ associativity and units are not strict.

$$\eta_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$

and the requirement that $\mathbf{1}$ be unital is expressed by demanding isomorphisms

$$\alpha_A : A \otimes \mathbf{1} \to A$$

$$\beta_A : \mathbf{1} \otimes A \to A.$$

A strict monoidal category is a monoidal category such that η, α, β are all identities.

- The isomorphism $\eta_{A,B,C}$ depends ~~*functorially*~~ on the triple (A, B, C); in other words, η may be regarded as a natural isomorphism between the functors

$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

$$(A, B, C) \mapsto (A \otimes B) \otimes C$$

$$(A, B, C) \mapsto A \otimes (B \otimes C).$$

Similarly, α_A and β_A depend functorially on A.

- Given any quadruple (A, B, C, D) of objects of $\mathcal{C}$, the *MacLane pentagon*

$$\begin{tikzcd}
 & ((A \otimes B) \otimes C) \otimes D \arrow[dl, "\eta_{A,B,C} \otimes \mathrm{id}_D"'] \arrow[dr, "\eta_{A \otimes B, C, D}"] & \\
(A \otimes (B \otimes C)) \otimes D \arrow[d, "\eta_{A, B \otimes C, D}"] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\eta_{A,B,C \otimes D}"] \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\mathrm{id}_A \otimes \eta_{B,C,D}"] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

is commutative.

- For any pair (A, B) of objects of $\mathcal{C}$, the triangle

$$\begin{tikzcd}
(A \otimes \mathbf{1}) \otimes B \arrow[rr, "\eta_{A,\mathbf{1},B}"] \arrow[dr, "\alpha_A \otimes \mathrm{id}_B"'] & & A \otimes (\mathbf{1} \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \beta_B"] \\
 & A \otimes B &
\end{tikzcd}$$

is commutative.

Maclane's Coherence Theorem

$\Rightarrow$ All diagrams that can be written by η, α, β commute.

Monoidal Categories via Grothendieck opfibrations.

Let $p : \mathcal{C} \to \mathcal{D}$ a functor between ordinary categories, $d \in \mathcal{D}$ an object.

Define the fibre of p over d $\quad \mathcal{C}d$ as the following pullback

$$\begin{tikzcd}
\mathcal{C}d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d, "p"] \\
{[0]} \arrow[r, "d"'] & \mathcal{D}
\end{tikzcd}$$

$\mathcal{C}d$: objects : $c \in \mathcal{C}$ s.t. $p(c) = d$

morphisms : $c \to d$ in $\mathcal{C}$ s.t. $p(c \to d) = \mathrm{id}_d$.

$\mathcal{C}d$ is a subcategory of $\mathcal{C}$.

$p : \mathcal{C} \to \mathcal{D}$ as a collection of categories $\mathcal{C}d$ parametrised by $d \in \mathcal{D}$.

Aim: $\mathcal{C}_d$ depends covariantly on the objects $d \in \mathcal{D}$.
(no necessarily a functor)

Def 4.2 (Groth) $p: \mathcal{C} \to \mathcal{D}$ a functor, $f: c_1 \to c_2$ a morphism in $\mathcal{C}$ such that $p(f) = \alpha: d_1 \to d_2$.

The morphism f is p-cocartesian or p-cocartesian lift of α if it has the following property:

For any $h: c_1 \to c_3$ in $\mathcal{C}$ with image $\gamma = p(h): d_1 \to d_3$ and any $\beta: d_2 \to d_3$ such that $\gamma = \beta \circ \alpha$,

$\exists!\ g: c_2 \to c_3$ in $\mathcal{C}$ such that

$$\beta = g(p) \quad \text{and} \quad h = g \circ f.$$

$$\begin{tikzcd} \mathcal{C} \arrow[d, "p"'] & c_1 \arrow[r, "f"] \arrow[dr, "\forall h"'] & c_2 \arrow[d, dashed, "\exists! g"] \\ \mathcal{D} & & c_3 \end{tikzcd}
\qquad
\begin{tikzcd} d_1 \arrow[r, "p(f)"] \arrow[dr, "p(h)"'] & d_2 \arrow[d, "\forall \beta"] \\ & d_3 \end{tikzcd}$$

Note. a morphism $f: c_1 \to c_2$ is p-cocartesian iff the diagram

$$\begin{tikzcd} \mathrm{hom}_{\mathcal{C}}(c_2, c_3) \arrow[r, "f^*"] \arrow[d, "p"'] & \mathrm{hom}_{\mathcal{C}}(c_1, c_3) \arrow[d, "p"] \\ \mathrm{hom}_{\mathcal{D}}(p(c_2), p(c_3)) \arrow[r, "p(f)^*"'] & \mathrm{hom}_{\mathcal{D}}(p(c_1), p(c_3)) \end{tikzcd}$$

is a pullback, $\forall c_3 \in \mathcal{C}$.

g, h, β: $p(f)^*\beta = ph$

The pullback is the set consisting of pairs of morphisms (h, β) s.t. $p(f)^*\beta = ph$.

$(h, \beta) \underset{1-1}{\longleftrightarrow} g$.

Lemma 4.4. Let $f': c \to c'$ and $f'': c \to c''$ be p-cocartesian arrows with the same image $\alpha = p(f') = p(f'')$. Then there is a unique isomorphism $\phi: c' \to c''$ in the fiber $\mathcal{C}_{p(c')} = \mathcal{C}_{p(c'')}$ s.t. $\phi \circ f' = f''$.

Proof.

$$\begin{tikzcd} c \arrow[r, "f'"] \arrow[dr, "f''"'] & c' \arrow[d, dashed, shift left, "\exists! \phi"] \\ & c'' \arrow[u, dashed, shift left, "\exists! \phi'"] \end{tikzcd}
\qquad
\begin{tikzcd} d \arrow[r, "p(f')"] \arrow[dr, "p(f'')"'] & d' \arrow[d, equal, "id"] \\ & d' \end{tikzcd}$$

$\phi \circ \phi' = id_{c''}$, $\phi' \circ \phi = id_{c'}$

$\Rightarrow \phi$ is the unique isomorphism.

$\square$

For $\alpha: d_1 \to d_2$, the targets of p-co Cartesian lifts are isomorphic in $\mathcal{C}_{d_2}$.

$$\begin{tikzcd} & c_2 \arrow[d, "\cong" description] & \\ c_1 \arrow[r] \arrow[ur] & c_2 \arrow[r] & d_3 \end{tikzcd}$$

$$\begin{tikzcd} d_1 \arrow[r, "\alpha"] \arrow[rr, bend right] & d_2 \arrow[r] & d_3 \end{tikzcd}$$

Def 4.5 A functor $p: \mathcal{C} \to \mathcal{D}$ is a Grothendieck opfibration if for all $c_1 \in \mathcal{C}$ and for all morphism α in $\mathcal{D}$ with domain $p(c_1)$, there is a p-coCartesian lift $f: c_1 \to c_2$ of α.

$$\begin{tikzcd} \mathcal{C} \arrow[d, "p"'] & c_1 \arrow[r, dashed, "\exists f"] & c_2 \\ \mathcal{D} & p(c_1) \arrow[r, "\alpha"] & d \end{tikzcd}$$

Let $p: \mathcal{C} \to \mathcal{D}$ be a Grothendieck opfibration. Choose for each $c \in \mathcal{C}$ and each morphism $\alpha: p(c) \to d$ a p-coCartesian lift.

$$\begin{tikzcd} c \arrow[r] \arrow[d, dash] & c' \\ p(c) \arrow[r, "\alpha"'] & d \end{tikzcd}$$

$\alpha: d_1 \to d_2$ in $\mathcal{D}$. $\quad Pd_1 \to Pd_2$

Define $\alpha_!: Pd_1 \to Pd_2$, $c_1 \mapsto c_2$ where c_2 is the target of a p-coCartesian lift of $\alpha: d_1 \to d_2$.

$$\begin{tikzcd} \mathcal{C} \arrow[d, "p"] & c_1 \arrow[r, dashed] & c_2 \\ \mathcal{D} & d_1 \arrow[r, "\alpha"'] & d_2 \end{tikzcd}$$

Extend the definition to morphisms, so that $\alpha_! : C_{d_1} \to C_{d_2}$ defines a functor.

$$\begin{tikzcd} c_1 \arrow[r] \arrow[d, "f"'] & c_2 \arrow[d, dashed, "\exists!\ \alpha_!(f)"] \\ c_1' \arrow[r] & c_2' \end{tikzcd}$$

$$d_1 \xrightarrow[\alpha]{} d_2 .$$

$\to$ $\alpha_!(f\circ g) = \alpha_!(f)\,\alpha_!(g)$ follows from the uniqueness of such isomorphism.

$\alpha : d_1 \to d_2$ in D. and $\beta : d_2 \to d_3$ is another morphism

$$C_{d_1} \xrightarrow{\alpha_!} C_{d_2} \xrightarrow{\beta_!} C_{d_3} \qquad\qquad C_{d_1} \xrightarrow{(\beta\circ\alpha)_!} C_{d_3}.$$

check $\beta_! \circ \alpha_! \cong (\beta\circ\alpha)_!$

<u>Exercise</u> The composition of two p-cocartesian morphisms is p-cocartesian.

Proof. Suppose $f : C_1 \to C_2$, $g : C_0 \to C_1$ are p-cocartesian.

$$\begin{tikzcd} \hom(C_2,C_3) \arrow[r, "f^*"] \arrow[d, "p"'] \arrow[rr, bend left, "(f\circ g)^*"] & \hom(C_1,C_3) \arrow[r, "g^*"] \arrow[d, "p"] & \hom(C_0,C_3) \arrow[d, "p"] \\ \hom_D(p(C_2),p(C_3)) \arrow[r, "p(f)^*"'] \arrow[rr, bend right, "p(f\circ g)^*"'] & \hom_D(p(C_1),p(C_3)) \arrow[r, "p(g)^*"'] & \hom_D(p(C_0),p(C_3)) \end{tikzcd}$$

$\Rightarrow$ The outer square is a pullback, so $f\circ g$ is also p-cocartesian. $\square$

<u>Essential Example</u>.

Let $\mathcal{M}$ be a monoidal category with monoidal pairing $\otimes : \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ and monoidal unit $\mathbb{1} \in \mathcal{M}$.

Form a new category $\mathcal{M}^{\otimes}$.

<u>Objects</u>: finite sequences of objects in $\mathcal{M}$,

$(M_1, \dots, M_n)$, $n \geq 0$, $M_i \in \mathcal{M}$.

<u>Morphisms</u>: $(\alpha, \{f_i\}_i) : (M_1, \dots, M_n) \to (L_1, \dots, L_k)$ consists of a morphism $\alpha : \langle k\rangle \to \langle n\rangle$ in [illegible] together with

Morphisms: $(\alpha, \{f_i\}) : (M_1, \ldots, M_n) \to (L_1, \ldots, L_k)$ consists of a morphism $\alpha : [k] \to [n]$ in Δ together with morphisms $f_i : \underbrace{M_{\alpha(i-1)+1} \otimes \cdots \otimes M_{\alpha(i)}}_{\text{ranges from } \alpha(i-1)+1 \text{ to } \alpha(i)} \to L_i$, $i = 1, \ldots, k$

$\mathbb{1}$

Convension: If there is an index i s.t. $\alpha(i-1) = \alpha(i)$, then

$$M_{\alpha(i-1)+1} \otimes \cdots \otimes M_{\alpha(i)} = \mathbb{1}, \quad f_i : \mathbb{1} \to L_i.$$

Composition: Compositions in Δ and $\mathcal{M}$ together with associativity constraint of the monoidal structure on $\mathcal{M}$.

Remark: The identity object is easily seen to be given by $(id_{[n]}, \{id_{M_i}\}_i)$

The associated Grothendieck opfibration $p : \mathcal{M}^{\otimes} \to \Delta^{op}$ is given by $(M_1, \ldots, M_n) \mapsto [n]$.

the projection.

show p is a Grothendieck opfibration:

$$\begin{array}{ccc} (M_1 \ldots M_n) & \xrightarrow{(\alpha, \{f_i\}_i)} & (L_1 \ldots L_k) \\ \Big\downarrow & & \\ [n] & \xrightarrow{\alpha^{op}} & [k] \end{array}$$

L_i = any object in the isomorphism class of $M_{\alpha(i-1)+1} \otimes \cdots \otimes M_{\alpha(i)}$

$\Rightarrow$ f_i are isomorphisms

$[k] \longrightarrow [n]$.

The Grothendieck opfibration $p : \mathcal{M}^{\otimes} \to \Delta^{op}$ has the property that the fibre $\mathcal{M}^{\otimes}_{[n]}$ is canonically equivalent to the n-fold product of $\mathcal{M}^{\otimes}_{[1]} \simeq \mathcal{M}$.

Let $l_{\{i-1, i\}} : [1] \to [n]$ be the inclusion, unique morphism with image $\{i-1, i\}$
$0, 1 \mapsto i-1, i$.

$l^{op}_{\{i-1,i\}} : [n] \to [1]$ $\quad \Delta^{op}$.

$M_{\alpha(i-1)+1} \otimes \cdots \otimes M_{\alpha(i)}$ $\quad i = 1$

$\Rightarrow$ $(l_i)_! : \mathcal{M}^{\otimes}_{[n]} \to \mathcal{M}^{\otimes}_{[1]} = \mathcal{M}$, $i = 1, \ldots, n$.

$(M_1 \ldots M_n) \mapsto M_{\alpha(i-1)+1} = M_i$.

$\Rightarrow$ The Segal map:

$(\mathcal{M}^{\otimes}_{[1]})^{\times n}$

$$G = ((l_1)_!, \ldots, (l_n)_!) : \mathcal{M}^{\otimes}_{[n]} \xrightarrow[\text{equivalence}]{\simeq} \underbrace{\mathcal{M} \times \cdots \times \mathcal{M}}_{n \text{ times}} \quad (\ast)$$

n times

equivalence.

We say that the Grothendieck opfibration $p : \mathcal{M}^{\otimes} \to \Delta^{op}$ satisfies the Segal condition if the Segal maps are equivalences.

Monoidal structure given by Grothendieck opfibration

① Can recover the monoidal product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ up to equivalence from $p : \mathcal{M}^{\otimes} \to \Delta^{op}$. $\mathcal{M}^{\otimes}_{[1]} = \mathcal{M}$.

② More generally, any Grothendieck opfibration $p : \mathcal{D} \to \Delta^{op}$ satisfying the Segal condition defines a monoidal structure on the fibre $\mathcal{D}_{[1]} = \mathcal{C}$.

Sketch of proof of ②: $\mathcal{D}_{[0]}$ is equivalent to the terminal object $\mathbb{1}$. So it has a single object up to equivalence.

The right hand-side of the Segal map is the empty product

(a) $s_0 : [1] \to [0]$ gives a functor $\mathcal{D}_{[0]} \to \mathcal{D}_{[1]} = \mathcal{C}$ which is identified as the identity object 1.

(b). $\otimes : \mathcal{C} \times \mathcal{C} \simeq \mathcal{D}_{[2]} \to \mathcal{D}_{[1]}$.

$$d^1 : [1] \to [2].$$
$$\simeq \{0,2\}$$

Associativity using different factorization of $[1] \to [3]$.

$$d^2 d^1 = d^1 d^1 : [1] \to [3].$$

$$\begin{tikzcd}
\{0,3\} \arrow[r, "d^1"] \arrow[d, "d^1"'] & \{0,1,3\} \arrow[d, "d^2"] \\
\{0,2,3\} \arrow[r, "d^1"'] & \{0,1,2,3\}
\end{tikzcd}$$

[1] [2] [2] [3]

$$(d_1)_! \circ (d_2)_! \cong (d_1)_! \circ (d_1)_! : \mathcal{D}_{[3]} \to \mathcal{D}_{[1]}$$

$$\begin{tikzcd}
M_1\otimes(M_2\otimes M_3) & \mathcal{D}_{[1]} & \mathcal{D}_{[2]} \arrow[l, "(d_1)_!"'] & (L_1, L_2) \arrow[d] \arrow[dr] & \\
& & & M_{a(1-1)+1} = M_1 & M_{a(2-1)+1}\otimes M_{a(2)} = M_2\otimes M_3 \\
(M_1\otimes M_2, M_3) \arrow[uu, mapsto] & \mathcal{D}_{[2]} \arrow[uu, "(d_1)_!"] & \mathcal{D}_{[3]} \arrow[l, "(d_1)_!"] \arrow[uu, "(d_2)_!"'] & (M_1, M_2, M_3) &
\end{tikzcd}$$

$$L_i \cong M_{a(i-1)+1}\otimes\cdots\otimes M_{a(i)}$$

Commutativity of Maclane's pentagon

$$M_1\otimes(M_2\otimes M_3) \cong (M_1\otimes M_2)\otimes M_3.$$

$$\begin{tikzcd}
& ((A\otimes B)\otimes C)\otimes D \arrow[dl] \arrow[dr] & \\
A\otimes(B\otimes C)\otimes D \arrow[d] & & (A\otimes B)\otimes(C\otimes D) \arrow[d] \\
\cdot \arrow[rr] & & \cdot
\end{tikzcd}$$

Similarly by different factorisations of $[1] \to [4]$.

$\square$

Moreover, check that the monoidal structure defined from $\mathcal{M}^{\otimes} \to \Delta$ recovers the original monoidal category structure on $\mathcal{M}$.

$$\boxed{\text{monoidal category} \longleftrightarrow \text{Grothendieck opfibration satisfying the Segal condition.}}$$

$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A \times B, C)$

Advantages

- Sometimes easier to specify $\mathcal{C}^{\otimes}$ than to specify the bifunctor $\otimes$, in the sense that it requires fewer arbitrary choices.

- Requirements concerning the functor $\mathcal{C}^{\otimes} \to \Delta^{op}$ are a bit simpler than the usual definition of a monoidal category. Complicated diagrams, such as MacLane's pentagon are consequences of the simplicial identities. (More evident in the setting of ∞-categories where there are increasing complexity).

Monoidal ∞-categories via coCartesian fibrations

Lemma 4.11. Let $p: \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. A morphism $f: C_1 \to C_2$ in $\mathcal{C}$ is p-coCartesian if and only if the following functor is an isomorphism

$$\mathcal{C}_{f/} \longrightarrow \mathcal{C}_{C_1/} \times_{\mathcal{D}_{p(C_1)/}} \mathcal{D}_{p(f)/}$$

objects $\mathcal{C}_{f/}$ $\quad \mathrm{Hom}(C_2, -)$

functor $\bar{f}: [1] \star [0] \to \mathcal{C}$ s.t. $F|_{[1]} = (f: C_1 \to C_2)$

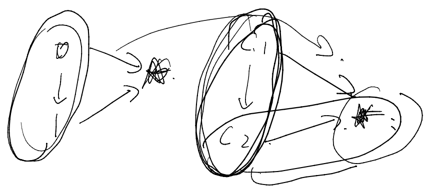
$\bar{F}: [1] \to \mathcal{C}$.

$\mathrm{hom}(C_1, -)$

$$\begin{tikzcd}
\mathrm{hom}_{\mathcal{C}}(C_2, C_3) \arrow[r, "f^*"] \arrow[d, "p"'] & \mathrm{hom}_{\mathcal{C}}(C_1, C_3) \arrow[d, "p"] & h \arrow[d, dashed] \\
\mathrm{hom}_{\mathcal{D}}(p(C_2), p(C_3)) \arrow[r, "p(f)^*"] & \mathrm{hom}_{\mathcal{D}}(p(C_1), p(C_3)) & p(h)
\end{tikzcd}$$

$\beta \longmapsto p(f)^*\beta$

(β, h)



both a weak equivalence and a fibration, characterised by

(Groth)

Def 4.12 Let $p: \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. A morphism $f: C_1 \to C_2$ in $\mathcal{C}$ is p-coCartesian or p-coCartesian lift of $d = p(f)$ if the following map is an acyclic Kan fibration.

$$\mathcal{C}_{f/} \longrightarrow \mathcal{C}_{C_1/} \times_{\mathcal{D}_{p(C_1)/}} \mathcal{D}_{p(f)/}$$

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta^n \arrow[r] & Y
\end{tikzcd}$$

Analogue of Grothendieck opfib.

(Groth)

Definition 4.13. A functor $p: \mathcal{C} \to \mathcal{D}$ between ∞-categories is a **coCartesian fibration** if the following two properties are satisfied.

(i) The functor p is an inner fibration (Definition 1.37).

(ii) For every object $c_1 \in \mathcal{C}$ and every morphism $\alpha: p(c_1) \to d$ in $\mathcal{D}$ there

Definition 4.13. A functor $p\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is a **coCartesian fibration** if the following two properties are satisfied.

(i) The functor p is an inner fibration (Definition 1.37).

(ii) For every object $c_1 \in \mathcal{C}$ and every morphism $\alpha\colon p(c_1) = d_1 \to d_2$ in $\mathcal{D}$, there is a p-coCartesian lift $f\colon c_1 \to c_2$ of α.

$C_d = \Delta^0 \times_d \mathcal{C}$.

$$\begin{tikzcd} C_d \arrow[r] \arrow[d, "\text{inner}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\text{inner}"] \\ \Delta^0 \arrow[r] & D \end{tikzcd}$$

inner fibrations are stable under pull back.

$\Rightarrow$ C_d is also an ∞-cat.

a simplicial set S is an ∞-category $\Leftrightarrow$ $S \to \Delta^0$ is an inner fibration.

(DAG II, Lurie)

$\{i, i+1\} : [1] \to [n]$.

Definition 1.1.2. A *monoidal ∞-category* is a coCartesian fibration of simplicial sets $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ with the following property:

($*$) For each $n \geq 0$, the associated functors $\mathcal{C}^{\otimes}_{[n]} \to \mathcal{C}^{\otimes}_{\{i,i+1\}}$ determine an equivalence of ∞-categories

$$\mathcal{C}^{\otimes}_{[n]} \to \mathcal{C}^{\otimes}_{\{0,1\}} \times \ldots \times \mathcal{C}^{\otimes}_{\{n-1,n\}} \simeq (\mathcal{C}^{\otimes}_{[1]})^n.$$

The segal condition in $n=0$ $\Rightarrow$ $C^{\otimes}_{[0]}$ is a contractible Kan complex.

$s' : \mathcal{C}^{\otimes}_{[0]} \to \mathcal{C}^{\otimes}_{[1]}$ picks out an object $\mathbb{1}$ as the identity

$\otimes : \mathcal{C}^{\otimes}_{[1]} \times \mathcal{C}^{\otimes}_{[1]} \xrightarrow{\simeq} \mathcal{C}^{\otimes}_{[2]} \xrightarrow{d'} \mathcal{C}^{\otimes}_{[1]}$.

$C^{\otimes}_{[0]} \simeq \Delta^0$ the terminal object,

so $C^{\otimes}_{[0]}$ is a quasigroupoid and hence a Kan complex.

$C^{\otimes}_{[0]} \to \Delta^0$ is a trivial fibration

(Section 37.10, Stuffs of quasicategories, Rezk)

Example 1. $\mathcal{C}$ is a monoidal category, $\mathcal{C}^{\otimes}$ as defined before.

The induced map $N(C^{\otimes}) \to N(\Delta)^{op}$ is a monoidal ∞-category

(check by the isomorphism $N(B/P) \cong N(B)/N(P)$)

Example 2 (DAG II, 1.2)

1.2 Cartesian Monoidal Structures

Let $\mathcal{C}$ be an ordinary category which admits finite products. Then $\mathcal{C}$ has the structure of a monoidal category, with the bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ given by the Cartesian product. We will refer to this monoidal structure on $\mathcal{C}$ as the *Cartesian monoidal structure*. Our goal in this section is to give an analogous construction in the ∞-categorical setting.

Definition 1.2.1. Let $\mathcal{C}$ be an ∞-category. We will say that a monoidal structure on $\mathcal{C}$ is *Cartesian* if the following conditions are satisfied:

(1) The unit object $1_{\mathcal{C}} \in \mathcal{C}$ is final.

(2) For every pair of objects $C, D \in \mathcal{C}$, the canonical maps

$$C \simeq C \otimes 1_{\mathcal{C}} \leftarrow C \otimes D \to 1_{\mathcal{C}} \otimes D \simeq D$$

exhibit $C \otimes D$ as a product of C and D in the ∞-category $\mathcal{C}$.

If $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathbf{\Delta})^{op}$ is a Cartesian monoidal structure on an ∞-category $\mathcal{C} = \mathcal{C}^{\otimes}_{[1]}$, then we can construct a functor $\pi : \mathcal{C}^{\otimes} \to \mathcal{C}$, which is given informally as follows. To an object $C \in \mathcal{C}^{\otimes}_{[n]}$, corresponding to an n-tuple $(C_1, \ldots, C_n) \in \mathcal{C}^n$, the functor π associates the object $\pi(C) = \prod_{1 \leq i \leq n} C_i$. We will give rigorous construction of π below (Proposition 1.2.4); first, we axiomatize its properties.

Example 3 (Simplicial Model Categories)

Definition of Simplicial Category (Simplicial Homotopy Theory, Jardine).

DEFINITION 2.1. *A category $\mathcal{C}$ is a simplicial category if there is a mapping space functor*

$$\mathbf{Hom}_{\mathcal{C}}(\cdot,\cdot) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{S}$$

with the properties that for A and B objects in $\mathcal{C}$

(1) $\mathbf{Hom}_{\mathcal{C}}(A,B)_0 = \hom_{\mathcal{C}}(A,B)$;

(2) *the functor $\mathbf{Hom}_{\mathcal{C}}(A,\cdot) : \mathcal{C} \to \mathbf{S}$ has a left adjoint*

$$A \otimes \cdot : \mathbf{S} \to \mathcal{C}$$

which is associative in the sense that there is a isomorphism

$$A \otimes (K \times L) \cong (A \otimes K) \otimes L,$$

natural in $A \in \mathcal{C}$ and $K, L \in \mathbf{S}$;

(3) *The functor $\mathbf{Hom}_{\mathcal{C}}(\cdot, B) : \mathcal{C}^{op} \to \mathbf{S}$ has left adjoint*

$$\mathbf{hom}_{\mathcal{C}}(\cdot, B) : \mathbf{S} \to \mathcal{C}^{op}.$$

sSet.

$\mathrm{Map}(X,Y)_n = \mathrm{Hom}(X \times \Delta^n, Y)$

Of course, the adjoint relationship in (3) is phrased

$$\hom_{\mathbf{S}}(K, \mathbf{Hom}_{\mathcal{C}}(A,B)) \cong \hom_{\mathcal{C}}(A, \mathbf{hom}_{\mathcal{C}}(K,B)) .$$

Def. A <u>simplicial model category</u> is a closed model category that is also a simplicial category.

<u>Examples</u> sSet

A pair of adjunctions (1.2, Groth)

$$s\mathcal{C}at \underset{\mathfrak{C}[-]}{\overset{N_\Delta(\ )}{\rightleftarrows}} sSet$$

$\mathfrak{C}[-] : sSet \to s\mathcal{C}at$

$N_\Delta(-) : s\mathcal{C}at \to sSet$ (coherent nerve functor)

Def. A simplicial category is <u>locally fibrant</u> if all simplicial mapping spaces are Kan complexes.

Cor 1.3 (Groth) The coherent nerve of a locally fibrant simplicial category is an ∞-category.

<u>Example</u>: Given a simplicial model category A, A^{cf} as the full subcategory consisting of all fibrant and cofibrant objects, then $N_\Delta(A^{cf})$ is an ∞-category.

(Lurie, DAG II, Prop 1.6.5)

Assumptions

(a) The closed monoidal structure is compatible with the enrichment in that $\otimes$ and the adjunction expressing the closedness are simplicial.
(b) The monoidal pairing $\otimes\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a *left Quillen bifunctor*.
(c) The monoidal unit $\mathbb{S} \in \mathcal{M}$ is cofibrant.

Simplicial version of $\mathcal{M}^\otimes$ for simplicial model category $\mathcal{M}$. $\mathcal{M}^\otimes \to \Delta^{op}$

$(M_1 \dots M_n) \mapsto [n]$

$\mathcal{M}^\otimes$ as constructed before, where $\otimes : \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ comes from $\otimes$ for the simplicial category $\mathcal{M}$ (the left adjoint to the mapping space functor).

Need to define mapping spaces.

Construct the mapping space for two objects $(M_1 \cdots M_m)$ and $(L_1 \cdots L_n)$,

$$\coprod_{\alpha: [n] \to [m]} \prod_{i=1}^{n} \mathrm{Map}_{\mathcal{M}}(M_{\alpha(i-1)+1} \otimes \cdots \otimes M_{\alpha(i)}, L_i)$$

$\mathcal{M}_{cf}^{\otimes} \subseteq \mathcal{M}^{\otimes}$.

$\to$ full subcategory consisting of finite strings of fibrant and cofibrant objects

Consequence of the assumption

$\downarrow$ locally fibrant simplicial category

Cor. 1.3 $\Rightarrow$ $N_\Delta(\mathcal{M}_{cf}^{\otimes})$ is an ∞-category. Apply $N_\Delta(-)$ to $\mathcal{M}_{cf}^{\otimes} \to \Delta^{op}$.

$\underline{N_\Delta(\mathcal{M}_{cf}^{\otimes}) \to N_\Delta(\Delta^{op}) = N(\Delta^{op})}$.

is an ∞-monoidal category (proved in section 1.6, DAG II, Lurie.)