Stable ∞-categories

$\mathcal{C}$ will always be an ∞-cat.

Def: • An object $x \in \mathcal{C}$ is a zero obj. if it is initial and terminal

• $\mathcal{C}$ is pointed if it contains a zero obj

Rmk: $\mathcal{C}^{zero} \subseteq \mathcal{C}$ full subcategory spanned by zero obj. is contractible. ($\mathcal{C}^{zero} \to \Delta^0$ is an acyclic Kan fibration)

Def: $F : \mathcal{C} \to \mathcal{D}$ between two pointed ∞-cats. is <u>reduced</u> if it preserves zero objs

$$\mathrm{Fun}_*(\mathcal{C},\mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C},\mathcal{D})$$

Def: Let $\mathcal{C}$ be pointed, $0 \in \mathcal{C}$ a zero object. A triangle in $\mathcal{C}$ is a diagram of the form

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d] & y \arrow[d, "p"] \\ 0 \arrow[r] & z \end{tikzcd}$$

We say that this triangle is <u>exact</u> if it is a <u>pullback</u> ($\Rightarrow p$ admits a fibre)

We say that it is <u>coexact</u> if it is a <u>pushout</u>

(f admits a cofibre)

<u>Rmk</u>: A triangle consists of following data:

i) a pair of morphisms $f: x \to y$, $p: y \to z$ in $\mathcal{C}$

ii) A 2-simplex
$$\begin{tikzcd} & y \arrow[dr, "p"] & \\ x \arrow[ur, "f"] \arrow[rr] & & z \end{tikzcd}$$
representing a composition of f and p.

iii) A 2-simplex
$$\begin{tikzcd} & 0 \arrow[dr, "h"] & \\ x \arrow[ur] \arrow[rr, "h"'] & & z \end{tikzcd}$$
representing a null htpy of h.

$\Rightarrow$ A triangle is a functor
$$F: \Delta^1 \times \Delta^1 \to \mathcal{C}$$

Def: Let $\mathcal{C}$ be pointed. $\mathcal{C}$ is <u>stable</u> if

i) Every morphism admits a fiber & cofibre

ii) Every triangle is a pushout $\iff$ is a pullback

Def: A functor $F : \mathcal{C} \to \mathcal{D}$ between stable ∞-cat is <u>Exact</u> if it is reduced & preserves exact & coact triangles.

$\mathrm{Fun}^{Ex}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Constructions (Ω, Σ)

Const 1: Let $\mathcal{C}$ be pointed s.t. every morphism admits a fiber.

Denote by $\mathcal{M}^{\Omega} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ the full subcategory spanned by <u>exact</u> triangles of the form

$$\begin{tikzcd} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & y \end{tikzcd}$$

(pullback)

$$\begin{tikzcd} \mathcal{M}^{\Omega} \arrow[r, "d_0"] \arrow[r, "d_1"'] & \mathcal{C} \\ & \mathcal{C} \end{tikzcd}$$

d_i is given by composition with the inclusion $\Delta^{2\times 2} \times \Delta^{3+4} \hookrightarrow \Delta^1 \times \Delta^1$.

Fact: d_1 is a trivial fibration (Prop. 4.3.2.15 HTT)

Lemma: Trivial fibration have RLP (levelwise inclusions)
Hence, they admits sections

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ Y \arrow[r, equal] \arrow[ur, dashed] & Y \end{tikzcd} \quad \text{trivial}$$

Let $s_1 : \mathcal{C} \to \mathcal{M}^{\Omega}$ be a section of d_1 $(d_1 s_1 = Id)$

We define the loop functor

$$\Omega_{\mathcal{C}} = d_0 \circ s_1 : \mathcal{C} \longrightarrow \mathcal{M}^{\Omega} \longrightarrow \mathcal{C}.$$

Constr 2: If $\mathcal{C}$ is pointed and every morphism admits a cofibre

$M^{\Sigma} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by coexact triangle

$$\begin{tikzcd} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & y \end{tikzcd}$$

(pushout)

$$\begin{tikzcd} M^{\Sigma} \arrow[r, "d_0"] \arrow[dr, "d_1"'] & \mathcal{C} \\ & \mathcal{C} \end{tikzcd}$$

In this case d_0 is trivial fibration

$\Rightarrow$ let s_0 be a section of d_0 & we define

$Z_{\mathcal{C}}: \quad d_1 \circ s_0 : \mathcal{C} \to M^{\Sigma} \to \mathcal{C}$

Rmk: $x \in \mathcal{C} \rightsquigarrow \Sigma_{\mathcal{C}} x$ is a colimit of the diagram

$$0 \leftarrow x \rightarrow 0$$

$\Omega_{\mathcal{C}} x$ is a limit of the diagram

$$0 \leftarrow x \rightarrow 0$$

Moreover: $\forall$ coexact triangle

$$\begin{tikzcd} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & y \end{tikzcd}$$

yields to htpy eq.

$$\Sigma_{\mathcal{C}} x \xrightarrow{\simeq} y$$

is unique!

$\forall$ exact triangle

$$\begin{tikzcd} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & y \end{tikzcd} \rightsquigarrow x \xrightarrow{\simeq} \Omega_{\mathcal{C}} x$$

Rmk: If $\mathcal{C}$ is s.t. every morphism admits a fibre & cofibre $\Rightarrow$ $\Sigma_{\mathcal{C}}, \Omega_{\mathcal{C}}$ define an adj.

If $\mathcal{C}$ is stable $\Rightarrow$ • $\Sigma_{\mathcal{C}}, \Omega_{\mathcal{C}}$ are equivalence
• & inverse to each other.

Indeed: If $\mathcal{C}$ is stable $\Rightarrow$ $\mathcal{M}^{\Sigma} = \mathcal{M}^{\Omega}$

$$\Rightarrow \Sigma_{\mathcal{C}} \circ \Omega_{\mathcal{C}} = d_1 \circ \underbrace{s_1 \circ d_0 \circ s_1}_{\overset{\shortparallel}{Id}} = d_1 s_1 = Id.$$

By def of $\Sigma_{\mathcal{C}}$ the triangle

$$\begin{tikzcd} \Omega_{\mathcal{C}} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma_{\mathcal{C}} \Omega_{\mathcal{C}} x \end{tikzcd}$$

is coexact

By def. of $\Omega_{\mathcal{C}}$

$$\begin{tikzcd} \Omega_{\mathcal{C}} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & x \end{tikzcd}$$

is exact $\Rightarrow$ coexact

$\Rightarrow$ we have an equivalence $\Sigma_{\mathcal{C}} \Omega_{\mathcal{C}} x \xrightarrow{\sim} x$

"counit of the adj"

$$\begin{tikzcd} \Omega_{\mathcal{C}} \Sigma_{\mathcal{C}} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma_{\mathcal{C}} x \end{tikzcd} \quad \& \quad \begin{tikzcd} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma_{\mathcal{C}} x \end{tikzcd}$$

are both exact & coexact

$\Rightarrow x \xrightarrow{\sim} \Omega_{\mathcal{C}} \Sigma_{\mathcal{C}} x$

"unit of the adj".

Triangulated Struct. on $ho(\mathcal{C})$

Let $\mathcal{C}$ be stable ∞-cat.

<u>Recall:</u> A triangulated category, is an additive category C together with an auto eq. $\Sigma : C \to C$ and a collection of distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \quad \text{in } C$$

That have to satisfy axioms.

<u>Rmk:</u> $\Sigma_{\mathcal{C}} : ho(\mathcal{C}) \to ho(\mathcal{C})$ is well defined and is an equivalence.

Σ_e is such that we have a htpy equivalence

$$\mathrm{Map}_e(\Sigma_e x, y) \xrightarrow{\simeq} \Omega\, \mathrm{Map}_e(x,y)$$

$$\text{"}\ \mathrm{Map}_e(x, \Omega y) \simeq \mathrm{Map}_e\left(x, 0 \to \overset{0}{\downarrow} y\right)$$

$$\lim \begin{pmatrix} & \mathrm{Map}_e(x,0) \\ & \downarrow \\ \mathrm{Map}_e(x,0) \to & \mathrm{Map}_e(x,y) \end{pmatrix}$$

$$= \Omega\, \mathrm{Map}_e(x,y)\ \text{"}$$

$\Rightarrow$ $\Sigma_{\mathcal{C}}$ is an eq. on $ho(\mathcal{C})$ let z s.t. $\Sigma_{\mathcal{C}}^2 z \simeq x$

$$\mathrm{Map}_{ho(\mathcal{C})}(x,y) = \pi_0\left(\mathrm{Map}_{\mathcal{C}}(x,y)\right) \simeq \pi_0\left(\mathrm{Map}_{\mathcal{C}}(\Sigma_{\mathcal{C}}^2 z, y)\right)$$

$$\simeq \pi_0\left(\Omega^2\left(\mathrm{Map}_{\mathcal{C}}(z,y)\right)\right) \simeq \pi_2\left(\mathrm{Map}_{\mathcal{C}}(z,y)\right)$$

<u>Def:</u> We say that

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma_{\mathcal{C}} x$$ is a dist. triangle in $ho(\mathcal{C})$

if there exists a diagram

$$\begin{tikzcd} x \arrow[r, "\tilde{f}"] \arrow[d] & y \arrow[r] \arrow[d, "\tilde{g}"] & 0 \arrow[d] \\ 0 \arrow[r] & z \arrow[r, "\tilde{h}"] & w \end{tikzcd}$$

in $\mathcal{C}$ sA

i) $\tilde{f}, \tilde{g}$ represents $[f], [g]$.

ii) every square is coexact ($\Rightarrow$ the outer rectangle is coexact

$$\Sigma_{\mathcal{C}} x \xrightarrow{\simeq} w$$

iii) $\tilde{h} : z \to w$ represents the htpy class of the composition: $z \to \Sigma_{\mathcal{C}} x \xrightarrow{\simeq} w$.

Thm: Let $\mathcal{C}$ be a stable ∞-cat. Then $\Sigma_{\mathcal{C}}$ and the class of dist. triangles just defined endow $ho(\mathcal{C})$ with a triangulated structure.

Proof: (HA Lurie)

Lemma: If

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d] & y \arrow[r] \arrow[d, "g"] & o \arrow[d] \\ o \arrow[r] & z \arrow[r, "h"] & w \end{tikzcd}$$

is a diagram in $\mathcal{C}$

$$\Rightarrow \quad x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{-h'} \Sigma_{\mathcal{C}} x$$

is dist. in $ho(\mathcal{C})$

h' is the composition of h and $\Sigma_{\mathcal{C}} x \xrightarrow{\sim} w$.

$-h'$ is h' composed with $-id \in \underset{ho(\mathcal{C})}{Map}(\Sigma_{\mathcal{C}} x, \Sigma_{\mathcal{C}} x)$

II axiom of triang. cat. $\therefore$ $\forall$ every morphism f is part of a dist. triangle

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma_{\mathcal{C}} x .$$

$\mathcal{C}$ is stable $\Rightarrow$ f admits a cofibre

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d] & y \arrow[r] \arrow[d, "p"] & 0 \arrow[d] \\
0 \arrow[r] & z \arrow[r, "n"'] & w
\end{tikzcd}$$

The two squares are pushout $\Rightarrow$ so is the outer rectangle

$\Rightarrow$ $w \cong \Sigma_{\mathcal{C}} x$

$\square$.

Properties of stable ∞-cat

Let $\mathcal{C}, \mathcal{D}$ be stable.

Fact: $F: \mathcal{C} \to \mathcal{D}$ functor. Tfae: i) F is exact
ii) F preserves zero obj & pushout
iii) " & pullback

Thm: $\mathcal{C}$ has finite limits & colimits
$\Rightarrow$ $F: \mathcal{C} \to \mathcal{D}$ is exact $\Leftrightarrow$ it preserve fin colimits
$\Leftrightarrow$ " fin limits.

Notation: $\mathrm{Cat}^{Ex} \subseteq \mathrm{Cat}_*^{\text{fincolim}}$

finitely cocomplete pointe ∞-cat

Thm: $\mathcal{C}$ pointed. Tfae: i) $\mathcal{C}$ is stable
ii) Every morphism admits cofibre and Σe is an eq.

iii) Every morphism admits a fibre & Ω_e is an eq.

Proof: Ch 2 HA

Stabilization

$$\mathrm{Cat}^{Ex} \subseteq \mathrm{Cat}_*^{\text{fin colim}}$$

is characterized by Σ_e being an equivalence

Q: Is this inclusion $\mathrm{Cat}^{Ex} \subseteq \mathrm{Cat}_*^{\text{fin colim}}$ part of an adjunction?

Can we stabilize an finitely cocomplete pointed ∞-cat?

$$\mathrm{Cat}^{Ex} \subseteq \mathrm{Cat}_*^{\text{fin lim}}$$

Constr.: If $\mathcal{C}$ is pointed with finite colimits $\Sigma_{\mathcal{C}}$ is well defined

$$\mathcal{C}[\Sigma^{-1}] = \operatorname{colim}\left(\mathcal{C} \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C} \to \cdots\right)$$

Constr.: $\mathcal{C}$ pointed with finite limits $\Rightarrow \Omega_{\mathcal{C}}$ is well defined.

$$\mathcal{C}[\Omega^{-1}] = \lim\left(\mathcal{C} \xleftarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xleftarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xleftarrow{\Omega_{\mathcal{C}}} \cdots\right)$$

Def: Let $\mathcal{C}$ be ∞-cat. A prespectrum obj. in $\mathcal{C}$ is a functor

$X : N(\mathbb{Z} \times \mathbb{Z}) \to \mathcal{C}$ s.t. $\forall\, i \neq j$ $X(i,j)$ is zero obj in $\mathcal{C}$

$$PSp(\mathcal{C}) \subseteq \mathrm{Fun}(N(\mathbb{Z}\times\mathbb{Z}), \mathcal{C})$$

$\forall n$ we have a proj. $PSp(\mathcal{C}) \to \mathcal{C}$

$$X \mapsto X(n,n) := X(n).$$

$\forall n$ we have a diagram

$$\begin{tikzcd} X(n) \arrow[r] \arrow[d] & X(n,n+1) = 0 \arrow[d] \\ 0 = X(n+1,n) \arrow[r] & X(n+1) \end{tikzcd}$$

We have morphisms

$\alpha_n : \Sigma_{\mathcal{C}} X(n) \to X(n+1)$

$\beta_n : X(n) \to \Omega_{\mathcal{C}} X(n+1).$

<u>Def:</u> $Sp(\mathcal{C}) \subseteq PSp(\mathcal{C})$ is the full subcat. of prespectrum obj s.t. β_n is an equivalence $\forall n$

Prop: If $\mathcal{C}$ is pointed and with finite limits

$$Sp(\mathcal{C}) \simeq \mathcal{C}[\Omega^{-1}].$$

Idea: $\{X_n\}$ together with eq. $X_n \simeq \Omega_e X_{n+1}$ $\forall n$

Def: $X : N(\mathbb{Z}\times\mathbb{Z}) \to \mathcal{C}$

$$(n,n) \mapsto X_n \qquad n \geq 0$$

$$\searrow 0 \qquad n < 0 \qquad \square$$

Thm: If $\mathcal{C}$ has finite limits and is pointed

$\Rightarrow$ $Sp(\mathcal{C})$ is stable.

Proof: [1.4.2.18 HA].

The ∞-cat of Spectra

Let $\mathcal{S}$ be the ∞-cat of spaces i.e. $\mathcal{S} = N_\Delta(\mathrm{Kan})$

$\mathcal{S}_* \subseteq \mathrm{Fun}(\Delta^1, \mathcal{S})$ the full subcategory of morphisms

$f: x \to y$ s.t. x is terminal in $\mathcal{S}$.

<u>Def</u>: A spectrum is an obj in $\mathrm{Sp}(\mathcal{S}_*) := \mathrm{Sp}$.

<u>Thm</u>: Sp is stable.

$\mathcal{S}_*^{fin}$ = smallest subcat. of $\mathcal{S}$ which cont terminal object $*$ and is closed under finite colimits.

Σ is well defined $\mathcal{S}_*^{fin} \to \mathcal{S}_*^{fin}$.

Define: $S_\infty^{fin} = \operatorname{colim}\left(S_*^{fin} \xrightarrow{\Sigma} S_*^{fin} \xrightarrow{\Sigma} \cdots \right)$

Fact: Sp is stable $\Leftarrow$ S_∞^{fin} is stable

We just "show" that S_∞^{fin} is stable.

To do so we need to show that every pushout square is a pullback.

Fact: Every pushout in S_∞^{fin} is obtained from a pushout in S_*^{fin}

$$\begin{tikzcd} w \arrow[r] \arrow[d] & x \arrow[d] \\ y \arrow[r] & z \end{tikzcd}$$

Blakers-Massey thm: If we have a pushout of spaces

$$\begin{tikzcd} w \arrow[r] \arrow[d] & x \arrow[d] \\ y \arrow[r] & z \end{tikzcd}$$

where $w \to y$ is r-connected

$w \to x$ is s-connected

$w \to \operatorname{hopull}\left(\begin{tikzcd} & x \arrow[d] \\ y \arrow[r] & z \end{tikzcd} \right)$ is $r+s-1$ connected.

Since Σ reverses the connectivity a pushout square in S_*^{fin} is a pullback.

Excisive functors

Def: Let $\mathcal{C}, \mathcal{D}$ be pointed ∞-cat. s.t. $\mathcal{C}$ has finite colimits $\mathcal{D}$ has finite limits.

$F: \mathcal{C} \to \mathcal{D}$ a functor is EXCISIVE if sends pushouts to pullbacks.

$$\mathrm{Ex}_*(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

Fact: $\forall$ pointed ∞-cat. with fin colimits $\mathcal{D}$

$$\mathrm{Fun}_*^{\mathrm{fincolim}}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}$$

$$F \longmapsto F(S^0)$$

If $\mathcal{D}$ is stable

$$\Rightarrow \quad \mathrm{Exc}_*(\mathcal{S}^{fin}_*, \mathcal{D}) \simeq \mathrm{Fun}^{fin\,colim}_*(\mathcal{S}^{fin}_*, \mathcal{D}) \simeq \mathcal{D}.$$

"$\mathrm{Exc}_*(\mathcal{S}^{fin}_*, -)$: stable ∞-cat $\longrightarrow$ stable ∞-cat

"is" the identity"

!

If $\mathcal{C}$ has finite limits $\Rightarrow$ $\mathrm{Exc}_*(\mathcal{S}^{fin}_*, \mathcal{C}) \not\simeq \mathcal{C}$

But. $\mathrm{Exc}_*(\mathcal{S}^{fin}_*, \mathcal{C})$ admits finite limits (computed levelwise)

$$\mathrm{Exc}_*(\mathcal{S}^{fin}_*, -) : \mathrm{Cat}_*^{finlim}$$

Prop: If $\mathcal{C} \in \mathrm{Cat}_*^{\mathrm{fin\,lim}} \Rightarrow E := E \times_* (S^{\mathrm{fin}}_*, \mathcal{C})$ is stable.

Proof: We need to show that Ω_E is an equivalence.

But: $\Omega_E(F)(-) = (\Omega_{\mathcal{C}}(F))(-)$

So we can define $\Sigma_E := E \to E$, $F \mapsto F \circ \Sigma_{S^{\mathrm{fin}}_*}$

Σ_E is an inverse of Ω_E

Idea:

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^{n+1} \end{tikzcd} \quad \text{i.e. pushout in } S^{\mathrm{fin}}_* \longrightarrow \begin{tikzcd} F(S^n) \arrow[r] \arrow[d] & * \arrow[d] \\ 0 \arrow[r] & F(S^{n+1}) \end{tikzcd}$$

pullback bc F is excisive.

$\Rightarrow$ We have an eq.

$$F(\underline{S^n}) \xrightarrow{\eta} \Omega_e F(S^{n+1}) = \Omega_e F(\Sigma_{S^{fin}_*}(S^n))$$

$$= (\Omega_e \circ F \circ \Sigma_{S^{fin}})(S^n)$$

$$= (\Omega_E \circ \Sigma_E)(F)(\underline{S^n})$$

$\mathrm{Exc}(S^{fin}_*, -)$ is another way of stabilizing "$\square$".

a pointe complete ∞-cat.

By formal argument $\mathrm{Exc}(S^{fin}_*, \mathcal{C}) \simeq Sp^{\Omega}(\mathcal{C})$

$$F \longmapsto n \mapsto F(S^n)$$

$\uparrow$ "$\square$".