CONSTRUCTIONS (PART 1)

JULIE
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FUNCTORS OF ∞-CATEGORIES

DEF $K \in sSet$, $\mathcal{C}$ an ∞-category

- A FUNCTOR $F: K \to \mathcal{C}$ is a map of ssets
- A NAT. TRANSFORMATION $F: F_0 \to F_1$ of functors is a map of ssets:

$$F: K \times \Delta^1 \to \mathcal{C} \quad \text{s.t.} \quad F|_{K\times\{0\}} \cong F_0 \,,\ F|_{K\times\{1\}} \cong F_1$$

- The SPACE OF FUNCTORS $\mathrm{Fun}(K,\mathcal{C}) \in sSet$ is defined:

$$(\mathrm{Fun}(K;\mathcal{C}))_n := \mathrm{Hom}_{sSET}(\Delta^n \times K ; \mathcal{C}) \qquad (\text{Denoted } \mathrm{Map}(K,\mathcal{C}) \text{ in Reak})$$

RMK we have an ordinary category QCAT consisting of ∞-categories and map of ssets

$N: \mathrm{Cat} \to \mathrm{QCAT}$ fully faithful functor

"GENERALISE": we want our notions to be compatible with the nerve functor.

Lemma (Gr 2.2.) $A, B \in \mathrm{CAT}$, then there exists a natural iso of simplicial sets:

$$N(\mathrm{Fun}_{CAT}(A,B)) \cong \mathrm{Fun}_{sSET}(NA, NB)$$

Proof

consider $[n]$ as a category, chain of bijections:

$$N(\mathrm{Fun}_{CAT}(A,B))_n = \mathrm{Hom}_{CAT}([n], \mathrm{Fun}_{CAT}(A,B))$$

$$\begin{tikzcd} \mathrm{CAT} \arrow[r, bend right, "\mathrm{Fun}(A,-)"'] \arrow[r, phantom, "\perp"] & \mathrm{CAT} \arrow[l, bend right, "-\times A"'] \end{tikzcd} \rightsquigarrow \cong \mathrm{Hom}_{CAT}([n]\times A, B)$$

$$\cong \mathrm{Hom}_{sSet}(N([m]\times A), NB)$$

N preserves products $\rightsquigarrow$ $$\cong \mathrm{Hom}_{sSet}(N([m])\times N(A), NB)$$

$$\cong \mathrm{Hom}_{sSet}(\Delta^m \times NA, NB)$$

$$= \mathrm{Fun}_{sSet}(NA, NB)_m \qquad \square$$

"STABLE NOTION": we want our constructions to give rise to ∞-cat when applied to ∞-cat.

<u>THM</u> (Gr. 2.5(i))

$K \in sSet$, $C \in QCat$, then $\mathrm{Fun}(K, C) \in QCat$.

$\rightsquigarrow$ Main tool to prove this: inner anodyne maps:

<u>THM</u> (R 15.7) $\overline{\mathrm{InnHorn}} \,\square\, \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$

<u>DEF</u> C a category with all small limits. A saturated class, is a class $\mathcal{A} \subseteq \mathrm{Mor}(C)$ s.t. contains all iso and it's closed under:

② cobase change
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "f'"] \\ \bullet \arrow[r] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad f \in \mathcal{A} \Rightarrow f' \in \mathcal{A}$$

③ countable composition: $X_0 \xrightarrow{f_0} X_1 \to \dots$ countable
$$\Rightarrow X_0 \to \mathrm{colim}_k X_k \in \mathcal{A}$$

④ transfinite comp.: As above but over general ordinal instead.

⑤ Retract in the sense of morphisms.

RMK The list is not minimal:

(4) $\Rightarrow$ (3), (2)(4) $\Rightarrow$ (5)

DEF $\mathrm{Cell} := \{\text{inclusions } \partial\Delta^m \subseteq \Delta^m \mid m \geq 0\} \subseteq \mathrm{Mor}(\mathrm{sSet})$

$\partial\Delta^m = \bigcup_{j \in [m]} \Delta^{[m]\setminus\{j\}}$

$\mathrm{InnHorn} := \{\text{inclusions } \Lambda^m_j \subseteq \Delta^m \mid 0<j<m,\ m \geq 2\} \subseteq \mathrm{Mor}(\mathrm{sSet})$

RMK $\overline{\mathrm{InnHorn}}$ is called the class of inner anodyne morphisms.

$\overline{\mathrm{Cell}}$ consists of all inclusions of simplicial sets.

PROP $C \in \mathrm{QCat}$ and $(A \hookrightarrow B) \in \overline{\mathrm{InnHorn}}$. Then for every map of simplicial sets $A \to C$, there exists a lift.

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & C \\ B \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

USE OF INNER ANODYNE MAPS

DEF The PUSHOUT PRODUCT of two morphisms $f: A \to B$, $g: K \to L$ of simplicial sets is the morphism:

$$f \,\square\, g : (A \times L) \coprod_{A \times K} (B \times K) \longrightarrow B \times L$$

$$\begin{tikzcd} A \times K \arrow[r, "f \times \mathrm{Id}_K"] \arrow[d, "\mathrm{Id}_A \times g"'] & B \times K \arrow[d] \arrow[ddr, bend left, "\mathrm{Id} \times g"] & \\ A \times L \arrow[r] \arrow[drr, bend right, "f \times \mathrm{Id}_L"'] & (A \times L) \coprod_{A \times K} (B \times K) \arrow[dr, dashed, "f \,\square\, g"] & \\ & & B \times L \end{tikzcd}$$

proof (that $C \in \mathrm{QCat} + K \in \mathrm{sSet} \Rightarrow \mathrm{Fun}(K, C) \in \mathrm{QCat}$)

$\Lambda^n_e \overset{i}{\hookrightarrow} \Delta^n \in \mathrm{InnHorn} \subseteq \overline{\mathrm{InnHorn}}$ $\quad 0<e<n$

$\emptyset \overset{j}{\hookrightarrow} K \in \overline{\mathrm{Cell}}$

$\Rightarrow i \,\square\, j : (\Lambda^n_e \times K) \coprod_{\Lambda^n_e \times \emptyset} (\Delta^n \times \emptyset) \simeq \Lambda^n_e \times K \longrightarrow \Delta^n \times K \in \overline{\mathrm{InnHorn}}$

therefore we can use the property of inner anodyne maps: for any $C \in \mathrm{QCat}$, and $f : \Lambda^n_e \times K \to C$ there exists a lift:

$$\begin{tikzcd} \Lambda^n_e \times K \arrow[r] \arrow[d, "i \square j"'] & C \\ \Delta^n \times K \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

using the adjunction $\begin{tikzcd} \mathrm{sSet} \arrow[r, bend right, "\mathrm{Fun}(K,-)"'] \arrow[r, phantom, "\perp"] & \mathrm{sSet} \arrow[l, bend right, "- \times K"'] \end{tikzcd}$

we get that this is equivalent to the existence of a lift:

$$\begin{tikzcd} \Lambda^n_e \arrow[r] \arrow[d] & \mathrm{Fun}(K, C) \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

which is precisely the definition of ∞-cat. $\square$

EQUIVALENCES OF ∞-CATEGORIES

(Following Resk, equivalent to Groth's)

"INVARIANT NOTION": we want our new constructions to respect some sort of equivalences.

DEF Let $F: \mathcal{C} \to \mathcal{D}$ functor bet. ∞-categories

- A CATEGORICAL INVERSE of F consists of:
  A functor $G: \mathcal{D} \to \mathcal{C}$
  two natural equivalences $GF \cong \mathrm{Id}$ & $FG \cong \mathrm{Id}$
- F is a CATEGORICAL EQUIVALENCE if it admits a categorical inverse.
- $\mathcal{C}$ an $\mathcal{D}$ are called EQUIVALENT if there exists a categorical equivalence between them.

DEF We can generalize to maps of arbitrary ssets:
$F: X \to Y$ bet. ssets is a CATEGORICAL EQUIVALENCE if for every $\mathcal{C} \in \mathrm{Qcat}$ the induced map:

$$F^*: \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$$

is a categorical equivalence of ∞-categories.

DEF $X \in$ sset, on the set $\coprod_{n \geq 0} X_n$ define equiv. relation generated by: $\sigma \sim \sigma f$ $\forall \sigma \in X_n$, $\forall n, m$ $\forall f: [m] \to [n]$

An equivalence class is called a CONNECTED COMPONENT of X ($\pi_0 X$ for the set of equiv. classes).

RMK Connected components are path components:
X is connected if $\pi_0 X$ is a singleton.

PROP (R 19.6) (The generalization to sSet is correct)
$F: \mathcal{C} \to \mathcal{D}$ functor between ∞-categories is a categorical equivalence iff the induced functor
$$F^*: \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$
is a categorical equivalence $\forall \mathcal{E} \in \mathrm{Qcat}$.

PROOF

"$\Rightarrow$" $G: \mathcal{D} \to \mathcal{C}$ the categorical inverse to F.
Then $G^*: \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{E})$
is categorical inverse to F^*:

$\eta: GF \cong \mathrm{Id}_{\mathcal{C}} \rightsquigarrow \eta^*: G^*F^* \cong \mathrm{Id}_{\mathrm{Fun}(\mathcal{C}, \mathcal{E})}$
$\mu: FG \cong \mathrm{Id}_{\mathcal{D}} \rightsquigarrow \mu^*: F^*G^* \cong \mathrm{Id}_{\mathrm{Fun}(\mathcal{D}, \mathcal{E})}$
USING COMPOS. IS FUNCTORIAL.

"$\Leftarrow$" Assume $F^*: \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$
cat. equiv. for any $\mathcal{E} \in \mathrm{Qcat}$ with cat. inverse H.
Now use:

Lemma: $F: \mathcal{C} \to \mathcal{D}$ cat. equiv. of Qcat, then
$h(F): h\mathcal{C} \to h\mathcal{D}$ is an equiv. of categories.
(Nat. iso $f \xrightarrow{\cong} g \rightsquigarrow$ induces nat. iso $h(f) \xrightarrow{\cong} h(g)$)

we get that
$$hF^*: h\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\cong} h\mathrm{Fun}(\mathcal{C}, \mathcal{E})$$
is an equivalence of ordinary categories.

DEF Write $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$ for the CORE of $\mathcal{C}$: sub ∞-cat.
Is the subcomplex consisting of those elements all of whose edges are isomorphisms. ($\mathrm{ob}\,\mathcal{C} = \mathrm{Ob}(\mathcal{C}^{\simeq})$)

RMK $\pi_0(\mathcal{C}^{\simeq})$ = iso classes of objects of $\mathcal{C}$.
moreover note that $(f: X \to Y) \in \mathcal{C}^{\simeq} \Leftrightarrow (hf: X \to Y) \in h\mathcal{C}^{\simeq}$
we get $\pi_0 \mathcal{C}^{\simeq} \simeq \pi_0 h\mathcal{C}^{\simeq}$
so F^* induces bijections

$$\pi_0 hF^* : \pi_0 h\mathrm{Fun}(\mathcal{D}, \mathcal{E})^{\simeq} \xrightarrow{\cong} \pi_0 h\mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}$$

$$\Rightarrow \pi_0 F^* : \pi_0 \mathrm{Fun}(\mathcal{D}, \mathcal{E})^{\simeq} \xrightarrow{\cong} \pi_0 \mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}$$

PUT $\mathcal{E} = \mathcal{C}$: $\pi_0 F^* : \pi_0 \mathrm{Fun}(\mathcal{D}, \mathcal{C})^{\simeq} \xrightarrow{\cong} \pi_0 \mathrm{Fun}(\mathcal{C}, \mathcal{C})^{\simeq}$

$$G \longmapsto \mathrm{Id}_{\mathcal{C}}$$

$\rightsquigarrow \exists G \in \pi_0 \mathrm{Fun}(\mathcal{D}, \mathcal{C})^{\simeq}$ s.t. $F^*(G) = GF \simeq \mathrm{Id}_{\mathcal{C}}$

PUT $\mathcal{E} = \mathcal{D}$: $\pi_0 F^* : \pi_0 \mathrm{Fun}(\mathcal{D}, \mathcal{D})^{\simeq} \xrightarrow{\cong} \pi_0 \mathrm{Fun}(\mathcal{C}; \mathcal{D})^{\simeq}$

$$F^*(\mathrm{Id}_{\mathcal{D}}) \cong \mathrm{Id}_{\mathcal{D}} F \cong F \cong F\,\mathrm{Id}_{\mathcal{C}} \cong FGF \cong F^*(FG)$$

and since F^* is a bijection : $\mathrm{Id}_{\mathcal{D}} \cong FG$ $\square$

PROP (G.2.5 (ii))

$$\left[\begin{array}{c} F: \mathcal{C} \to \mathcal{D} \text{ is} \\ \text{a cat. equiv. of} \\ \infty\text{-CAT} \end{array}\right] \Leftrightarrow \left[\begin{array}{c} F_*: \mathrm{Fun}(K; \mathcal{C}) \to \mathrm{Fun}(K; \mathcal{D}) \\ \text{is a cat. equiv. of } \infty\text{-cat} \\ \forall K \in \mathrm{sSet} \end{array}\right]$$

PROOF

"$\Rightarrow$" same idea as before : G cat. inverse to F,
then G_* is cat. inverse to F_*.

"$\Leftarrow$" $K = \Delta^0$ $\quad \mathcal{C} \cong \mathrm{Fun}(\Delta^0; \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(\Delta^0; \mathcal{D}) \simeq \mathcal{D}$

$\square$

DEF (∞-category of ∞-categories)

Let Cat_∞^Δ be the simplicial category defined by:

- objects: ∞-categories
- $Map(\mathcal{C},\mathcal{D}) := Fun(\mathcal{C},\mathcal{D})^{\simeq}$ sSet.

applying the coherent nerve we get an ∞-category

$$Cat_\infty := N_\Delta(Cat_\infty^\Delta)$$

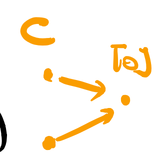
which is the ∞-CATEGORY OF ∞-CATEGORIES.

JOINS

DEF $\mathcal{C},\mathcal{D} \in CAT$. The JOIN of $\mathcal{C}$ and $\mathcal{D}$ is the category $\mathcal{C} \star \mathcal{D}$ with:

- $OBJ(\mathcal{C}\star\mathcal{D}) := OBJ(\mathcal{C}) \amalg OBJ(\mathcal{D})$
- $$Map_{\mathcal{C}\star\mathcal{D}}(x,y) = \begin{cases} Map_{\mathcal{C}}(x,y) & x,y \in \mathcal{C} \\ Map_{\mathcal{D}}(x,y) & x,y \in \mathcal{D} \\ * & x \in \mathcal{C}, y \in \mathcal{D} \\ \emptyset & x \in \mathcal{D}, y \in \mathcal{C} \end{cases}$$
- composition law s.t. the inclusions $\mathcal{C} \hookrightarrow \mathcal{C}\star\mathcal{D} \hookleftarrow \mathcal{D}$ are functors.

EX RIGHT CONE $\mathcal{C}^{\triangleright} := \mathcal{C} \star [0]$ (Attaching a new terminal object)



LEFT CONE : $\mathcal{C}^{\triangleleft} := [0] \star \mathcal{C}$ (Attaching a new initial object)

DEF $X, Y \in sSet$, the JOIN of X and Y: $X \star Y \in sSET$ is defined by:

$$(X \star Y)_n := X_n \cup Y_n \cup \Big(\bigcup_{i+j+1=n} X_i \times Y_j \Big) \qquad n \geq 0 \quad i,j \geq 0$$

RMK: canonical inclusions

$$X \hookrightarrow X \star Y \hookleftarrow Y$$

PROP (compatibility with nerve) (G.2.13)

$\mathcal{C}, \mathcal{D} \in CAT$. Then

$$\underbrace{N(\mathcal{C} \star \mathcal{D})}_{\text{JOIN OF CATEGORIES}} \underbrace{\cong}_{\text{ISO SSET.}} N(\mathcal{C}) \underbrace{\star}_{\text{JOIN OF SSET}} N(\mathcal{D})$$

PROOF

Want to construct map $N(\mathcal{C} \star \mathcal{D}) \to N(\mathcal{C}) \star N(\mathcal{D})$

an object in $N(\mathcal{C} \star \mathcal{D})_n$ has one of the following forms:

1) $s_0 \to \dots \to s_n$ $\qquad s_i \in \mathcal{C}$

2) $\underbrace{s_0 \to \dots \to s_k}_{\text{in } \mathcal{C}} \to \underbrace{s_{k+1} \to \dots \to s_n}_{\text{in } \mathcal{D}}$

3) $s_0 \to \dots \to s_n$ $\qquad s_i \in \mathcal{D}$

Since in $\mathcal{C} \star \mathcal{D}$ there are no maps from obj. in $\mathcal{D}$ to objects in $\mathcal{C}$. Consider case 2:

$$c = \underbrace{s_0 \to \dots \to s_k}_{= A \in N(\mathcal{C})_k} \to \underbrace{s_{k+1} \to s_n}_{= B \in N(\mathcal{D})_{n-k-1}}$$

then $(A, B) \in N(\mathcal{C})_n \star N(\mathcal{D})_n$ since $k + (n-k-1) + 1 = n$.

Case 1 e 3: $s_0 \to \dots \to s_n$ is an n-simplex in $N(\mathcal{C})_n$ or $N(\mathcal{D})_n$.

This is a bijection $\square$

THM (G. 2.15) [Stable and invariant]

[STABLE]: $\mathcal{C}, \mathcal{D} \in QCAT \Rightarrow \mathcal{C} \star \mathcal{D} \in QCAT$

[INVARIANT]: if $F: \mathcal{C} \to \mathcal{C}'$, $G: \mathcal{D} \to \mathcal{D}'$ equiv. of ∞-cat, then the same is true for:

$$F \star G : \mathcal{C} \star \mathcal{D} \to \mathcal{C}' \star \mathcal{D}'$$

RMK Join is a functor $\star : sSet \times sSet \to sSet$.

EX · $[n] \star [m] \cong [n+m+1]$

$$0_n \to \dots \to n \to 0_m \to \dots \to m$$

· $\Delta^{n+m+1} \cong N([n+m+1]) \cong N([n] \star [m]) \cong N([n]) \star N([m])$
$\cong \Delta^n \star \Delta^m$

RMK The join of sSet is not symmetric: consider

$K = \Delta^0 \amalg \Delta^0$, $L = \Delta^0$

$K \star L$ $\qquad\qquad$ $L \star K$

DEF for $K \in sSet$ define:

RIGHT CONE: $K^{\triangleright} := K \star \Delta^0$

LEFT CONE: $K^{\triangleleft} := \Delta^0 \star K$.