Talk 3: Functors and joins of ∞-categories

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Functors of ∞-categories

Definition 0.1. Let $K \in \mathrm{sSet}$ and $\mathcal{C}$ and ∞-category. Then a *functor* $F : K \to \mathcal{C}$ is a map of simplicial sets. A *natural transformation* $f : f_0 \to f_1$ of functors $f_i : K \to \mathcal{C}$, $i = 0, 1$ is a map (of simplicial sets?)

$$f : K \times \Delta^1 \to \mathcal{C}, \text{ such that } f|_{K\times\{i\}} = f_i$$

for $i = 0$ or 1.

We define the *Space of functors* $\mathrm{Fun}(K, \mathcal{C}) \in \mathrm{sSet}$ defined by

$$\mathrm{Fun}(K, \mathcal{C})_\bullet = \mathrm{map}_{\mathrm{sSet}}(K, \mathcal{C})_\bullet = \hom_{\mathrm{sSet}}(\Delta^\bullet \times K, \mathcal{C}).$$

We can use this to define a category of ∞-categories QCat. Since $N : \mathrm{Cat} \to \mathrm{QCat}$ is fully faithful we get that any functor in Cat gives rise to a functor in QCat. In general when we make construction of this we know in ordinary category theory, in the setting of ∞-categories, we want it to be compatible with the nerve functor. We see that our naive definition of functors is the correct one due to the following result:

Corollary 0.2 (Groth, Lem 2.2)**.** *Let A and B be ordinary categories. Then there exists a natural isomorphism of simplicial sets:*

$$N(Fun_{Cat}(A, B)) \cong Fun_{sSet}(NA, NB).$$

Proof. Let $[n] \in \Delta$. Then we have the following

$$\begin{aligned} N(\mathrm{Fun}_{\mathrm{Cat}}(A, B))_n &\cong \hom_{\mathrm{Cat}}([n], \mathrm{Fun}_{\mathrm{Cat}}(A, B)) \\ &\cong \hom_{\mathrm{Cat}}([n] \times A, B) \\ &\cong \hom_{\mathrm{sSet}}(N([n] \times A), N(B)) \\ &\cong \hom_{\mathrm{sSet}}(\Delta^n \times NA, NB) \\ &\cong \mathrm{Fun}_{\mathrm{sSet}}(NA, NB)_n. \end{aligned}$$

Here the first equality is the definition, second isomorphism is due to the fact that $- \times A : \mathrm{Cat} \to \mathrm{Cat}$ is left adjoint to $\mathrm{Fun}(A, -)$, third isomorphism is because N is fully faithful, the fourth is because it can be shown that N preserves products, the fifth isomorphism is due to $N([n]) \simeq \Delta^n$ and the last is the definition. $\square$

Another thing we often want when making new construction is that is an invariant notion, i.e. is again an ∞-category:

Theorem 0.3 (Groth Prop 2.5 (i))**.** *Let $K \in sSet$ and $\mathcal{C} \in QCat$. Then $Fun(K, \mathcal{C})$ is again an ∞-category.*

Main idea. Recall that a simplicial set $\mathcal{C}$ is an ∞-category iff there exist a lift in the diagram

$$\begin{array}{ccc} \Lambda^n_j & \longrightarrow & \mathcal{C} \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

Hence we wish to show that there exists a lift in

$$\begin{array}{ccc} \Lambda^n_j & \longrightarrow & \mathrm{Fun}(K, \mathcal{C}). \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

Due to the fact that $- \times K : \mathrm{sSet} \to \mathrm{sSet}$ is left adjoint to $\mathrm{Fun}(K, -)$ we get that this is equivalent to showing that there exists a lift in the following diagram

$$\begin{array}{ccc} \Lambda^n_j \times K & \longrightarrow & \mathcal{C}. \\ \big\downarrow & \nearrow & \\ \Delta^n \times K & & \end{array}$$

□

We want functor categories to be equivalent to eachother when the input are equivalent. So we will now consider the notion of categorical equivalence between first ∞-categories and then extend this notion to simplicial sets.

Equivalence of ∞-categories

Here we are following Rezk's notion of categorical equivalences (section 18), but can be shown to be equivalent Groth definition 1.35.

Definition 0.4. A *categorical inverse* to a functor $F : \mathcal{C} \to \mathcal{D}$ between two ∞-categories is a functor $G : \mathcal{D} \to \mathcal{C}$ such that $GF \simeq id_{\mathcal{C}}$ and $FG \simeq id_{\mathcal{D}}$.

We say that a functor F between ∞-categories is an *categorical equivalence* if it admits a categorical inverse. We call two ∞-categories *categorical equivalent* if there exists a categorical equivalence between them.

Proposition 0.5 (Rezk Lemma 18.5). *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then F is a categorical equivalence iff the induced functor*

$$F^* : Fun(\mathcal{D}, \mathcal{E}) \to Fun(\mathcal{C}, \mathcal{E})$$

is a categorical equivalence of ∞-categories for any $\mathcal{E} \in QCat$.

Proof. $\Rightarrow$: Let $G : \mathcal{D} \to \mathcal{C}$ denote the categorical inverse to F. Then G induces a functor

$$G^* : \text{Fun}(\mathcal{C}, \mathcal{E}) \to \text{Fun}(\mathcal{C}, \mathcal{E}),$$

by postcomposing with G. We see that G^* is a categorical inverse to F^*, since

$$\eta :\Rightarrow id_{\mathcal{C}} \rightsquigarrow \eta^* : G^* F^* \Rightarrow id_{\text{Fun}(\mathcal{D},\mathcal{E})}$$
$$\theta :\Rightarrow id_{\mathcal{D}} \rightsquigarrow \theta^* : F^* G^* \Rightarrow id_{\text{Fun}(\mathcal{C},\mathcal{E})},$$

which are natural isomorphisms.

$\Leftarrow$: Assume

$$F^* : \text{Fun}(\mathcal{D}, \mathcal{E}) \to \text{Fun}(\mathcal{C}, \mathcal{E})$$

is a categorical equivalence for any $\mathcal{E} \in$ QCat, with G^* the categorical inverse. Since a natural transformation $F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ induce natural transformation $h(F) \Rightarrow h(G) : h\mathcal{C} \to h\mathcal{D}$, we get that

$$hF^* : h\text{Fun}(\mathcal{D}, \mathcal{E}) \to h\text{Fun}(\mathcal{C}, \mathcal{E})$$

is an equivalence of ordinary categories.

Write $\mathcal{C}^{\simeq} \subset \mathcal{C}$ for the subcategory of $\mathcal{C}$ with $ob\mathcal{C}^{\simeq} = ob\mathcal{C}$ and with all isomorphisms. Then $\pi_0\mathcal{C}^{\simeq}$ are the isomorphism classes, hence $\pi_0\mathcal{C}^{\simeq} \simeq \pi_0 h\mathcal{C}$, since $f : X \to Y \in \mathcal{C}^{\simeq}$ iff $hf : hX \to hY \in h\mathcal{C}^{\simeq}$. So we get that F^* induces a bijection

$$hF^* : \pi_0 h\text{Fun}(\mathcal{D}, \mathcal{E})^{\simeq} \xrightarrow{\sim} \pi_0 h\text{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}$$

which gives us

$$F^* : \pi_0\text{Fun}(\mathcal{D}, \mathcal{E})^{\simeq} \xrightarrow{\sim} \pi_0\text{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}.$$

Set $\mathcal{E} := \mathcal{C}$

$$F^* : \pi_0\text{Fun}(\mathcal{D}, \mathcal{C})^{\simeq} \xrightarrow{\sim} \pi_0\text{Fun}(\mathcal{C}, \mathcal{C})^{\simeq}.$$

Then there exists $G \in \pi_0\text{Fun}(\mathcal{D}, \mathcal{C})^{\simeq}$ which satiesfies $F^*(G) \simeq GF \simeq id_{\mathcal{C}}$ in $\text{Fun}(\mathcal{C}, \mathcal{C})_1$ (since F^* is postcomposing with F).

Set $\mathcal{E} := \mathcal{D}$:

$$F^* : \pi_0\text{Fun}(\mathcal{D}, \mathcal{D})^{\simeq} \xrightarrow{\sim} \pi_0\text{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}.$$

Then there exists $F \in \pi_0\text{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$ such that

$$F^*(id_{\mathcal{D}}) \simeq F \simeq Fid_{\mathcal{C}} \simeq FGF \simeq F^*(FG).$$

Since F^* is a bijection, this implies that $id_{\mathcal{D}} \simeq FG$.

□

We use this equivalent definition of categorical equivalence between ∞-categories to extend the notion to simplicial sets.

Definition 0.6. A map $f : X \to Y$ between simplicial sets is called a *categorial equivalence* if for every ∞-category $\mathcal{C}$, the induced map of ∞-categories

$$\text{Fun}(f, \mathcal{C}) : \text{Fun}(Y, \mathcal{C}) \to \text{Fun}(X, \mathcal{C})$$

admits a categorical inverse.

Proposition 0.7 (Groth proposition 2.5 (ii)). *Let $F : \mathcal{C} \to \mathcal{D}$ be a categorical equivalence between ∞-categories. Then Fun$(K, \mathcal{C}) \to$ Fun$(K, \mathcal{D})$ is a categorical equivalence for any $K \in$ sSet.*

Proof. Let $G : \mathcal{D} \to \mathcal{C}$ denote the categorical inverse to F. Then as earlier we can prove that the induced map

$$G_* : \text{Fun}(K, \mathcal{D}) \to \text{Fun}(K, \mathcal{C})$$

given by precomposing with G, is categorical inverse to

$$F_* : \text{Fun}(K, \mathcal{C}) \to \text{Fun}(K, \mathcal{D}).$$

□

This is actually an if and only if statement, since if the induced functor on the functor spaces is a categorical equivalence for any simplicial set K, this inparticular holds for $K = \Delta^0$, which implies

$$\mathcal{C} \simeq \text{Fun}(\Delta^0, \mathcal{C}) \to \text{Fun}(\Delta^0, \mathcal{D}) \simeq \mathcal{D}$$

is an equivalence of categories.

Joins

This is again a notion of ordinary categories, which we want to extend to ∞-categories.

Definition 0.8. Let $\mathcal{C}, \mathcal{D}$ be two ordinary categories. Then the *join* $\mathcal{C} \star \mathcal{D}$ of $\mathcal{C}$ and $\mathcal{D}$ is the category with $\mathbf{ob}(\mathcal{C} \star \mathcal{D}) = \text{ob}(\mathcal{C}) \coprod \text{ob}(\mathcal{D})$ and with

$$\text{Hom}_{\mathcal{C}\star\mathcal{D}}(x, y) = \begin{cases} \text{Hom}_{\mathcal{C}}(x, y), & x, y \in \mathcal{C} \\ \text{Hom}_{\mathcal{D}}(x, y), & x, y \in \mathcal{D} \\ *, & x \in \mathcal{C}, y \in \mathcal{D} \\ \emptyset, & x \in \mathcal{D}, y \in \mathcal{C}, \end{cases}$$

with composition such that the inclusions

$$\mathcal{C} \hookrightarrow \mathcal{C} \star \mathcal{D}, \ \ \mathcal{D} \hookrightarrow \mathcal{C} \star \mathcal{D}$$

are functors.

Two important examples are the *right join* $\mathcal{C}^{\triangleleft} := [0] \star \mathcal{C}$ and the *left join* $\mathcal{C}\triangleright := \mathcal{C} \star [0]$. These are obtained by attaching a new terminal, respectively initial, object to the category $\mathcal{C}$.

We can make the same kind of definition for simplicial sets.

Definition 0.9. Let $X, Y \in \text{sSet}$. Then we define the *join* of X and Y as the simplicial set $X \star Y$ defined by

$$(X \star Y)_n := X_n \cup Y_n \cup \left(\bigcup_{i+j+1=n} X_i \times Y_i \right), \quad n \geq 0,$$

where we define $X_{-1} = Y_{-1} = *$ a point.

First we note that we have canonical inclusions $X \hookrightarrow X \star Y$ and $Y \hookrightarrow X \star Y$ by sending $X_n \mapsto X_n \cup Y_{-1}$ and $Y_n \mapsto X_{-1} \cup Y_n$. Again, as when we considered the construction of functors, we see that our definition of join is compatible with the nerve functor.

Proposition 0.10 (Groth 2.13)**.** *Let cC and $\mathcal{D}$ be two categories. Then*

$$N(\mathcal{C} \star \mathcal{D}) \simeq N(\mathcal{C}) \star N(\mathcal{D}).$$

Note that on the left hand side we have the join of categories and on the right hand side it is the join of simplicial sets. Further note that the equivalence is of simplicial sets.

Proof. We want to construct a map $N(\mathcal{C} \star \mathcal{D}) \to N(\mathcal{C}) \star N(\mathcal{D})$. First we note that an object $x \in N(\mathcal{C} \star \mathcal{D})_n$ is a functor $[n] \to \mathcal{C} \star \mathcal{D}$, where we can consider these as the strings (or substrings within)

$$s_0 \to s_1 \to \cdots \to s_n,$$

where each $s_i \in \mathcal{C}$ or $\mathcal{D}$. Since there in the join $\mathcal{C} \star \mathcal{D}$ is no map from an object in $\mathcal{D}$ to an object in $\mathcal{C}$, we get that the string of objects at most can 'jump' from $\mathcal{C}$ to $\mathcal{D}$ once. Assume that the 'jump' happens at $1 < k < n$, so we can consider the string in $N(\mathcal{C} \star \mathcal{D})$ as a k-simplex A in $N(\mathcal{C})$ and a $(n-k-1)$-simplex B in $N(\mathcal{D})$. Then $(A, B) \in N(\mathcal{C}) \times N(\mathcal{D})_n$ since they are a point in

$$N(\mathcal{C})_n \cup N(\mathcal{D})_n \cup \left(\bigcup_{i+j+1=n} N(\mathcal{C})_i \times N(\mathcal{D})_j \right)$$

since $k + (n-k-1) + 1 = n$.

This can be shown to be a bijection since we run through all possible strings which go from $\mathcal{C}$ to $\mathcal{D}$, and $N(\mathcal{C})_n$ are the case where all are in $\mathcal{C}$ and $N(\mathcal{D})_n$ is the case where the entire string is in $\mathcal{D}$. □

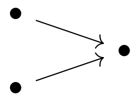
Example 0.11. First we see that when considered as ∞-categories, we get that the join $[n] \star [m] \simeq [n+m+1]$, since we take the n string and then attach the m string

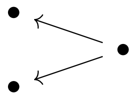
$$0_n \to 1 \to \cdots \to n \to 0_m \to 1 \to \cdots \to m.$$

This gives us

$$\Delta^{n+m+1} \simeq N([n+m+1]) \simeq N([n] \star [m]) \simeq N([n]) \star N([m]) \simeq \Delta^n \star \Delta^m.$$

Example 0.12. The join of simplicial sets is not symmetric: Consider the case $K = \Delta^0 \coprod \Delta^0$, $L = \Delta^0$. Then

$K \star L =$ 

$L \star K =$  ,

where the first diagram is an example of a right cone and the latter an example of a left cone.

Definition 0.13. Let $K \in$ sSet. Then the *right cone* is the simplicial set $K^{\triangleright} := K \star \Delta^0$, and the *left cone* is the simplicial set $K^{\triangleleft} := \Delta^0 \star K$.

Theorem 0.14 (Groth 2.15)**.**

i) If $\mathcal{C}$ and $\mathcal{D}$ are two ∞-categories, then $\mathcal{C} \star \mathcal{D}$ an ∞-category.

ii) If $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{D} \to \mathcal{D}'$ are equivalences of ∞-categories, then the induced map $F \star G : \mathcal{C} \star \mathcal{D} \to \mathcal{C}' \star \mathcal{D}'$ again an equivalence of ∞-categories.