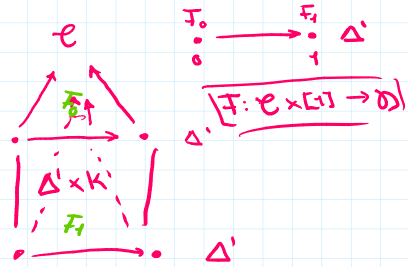
Functors of ∞-categories

Def: $K \in sSet$, $\mathcal{C}$ an ∞-category.

- A functor $F: K \to \mathcal{C}$ is a map of simplicial sets
- A natural transformation $F: F_0 \to F_1$ of functors $F_i: K \to \mathcal{C}$ is a map
$$F: K \times \Delta^1 \to \mathcal{C} \quad \text{s.t.} \quad F|_{K\times\{i\}} \cong F_i, \quad i = 0 \text{ or } 1$$
- Space of functors $Fun(K,\mathcal{C}) \in sSet$:
$$Fun(K,\mathcal{C})_\bullet := map_{sSet}(K,\mathcal{C})_\bullet := hom_{sSet}(\Delta^\bullet \times K, \mathcal{C})$$

$$\begin{tikzcd} sSet \arrow[r, bend right, "Fun(K{,}\mathcal{C})"'] \arrow[r, phantom, "\perp"] & sSet \arrow[l, bend right, "-\times K"'] \end{tikzcd}$$



$\overline{F}: \mathcal{C} \times [1] \to \mathcal{D}$

$x \to y \in K_1$

$$\begin{tikzcd} F_0(x) \arrow[r] \arrow[d] \arrow[dr] & F_0(y) \arrow[d] \\ F_1(x) \arrow[r] & F_1(y) \end{tikzcd}$$

$\rightsquigarrow$ Category QCat consisting of ∞-categories.

$N: Cat \to QCat$ fully faithful gives us that any $F: \mathcal{C} \to \mathcal{D}$ in Cat gives rise to $N(F): N(\mathcal{C}) \to N(\mathcal{D})$

Want compatible with nerve functor

Cor: (G. 2.2). $A, B \in Cat$. Then there exists a natural isomorphism of simplicial sets
$$N(Fun_{cat}(A,B)) \cong Fun_{sSet}(NA, NB)$$

Pf: Consider $[n]$ as a category.

$$\begin{aligned} N(Fun_{cat}(A,B))_n &= hom_{cat}([n], Fun_{cat}(A,B)) \\ &\cong hom_{cat}([n]\times A, B) \\ &\cong hom_{sSet}(N([n]\times A), NB) \\ &\cong hom_{sSet}(N([n])\times NA, NB) \\ &\cong hom_{sSet}(\Delta^n \times NA, NB) \\ &= Fun_{sSet}(NA, NB)_n \end{aligned}$$
$\square$

$$\begin{tikzcd} Cat \arrow[r, bend right, "Fun(A{,}-)"'] \arrow[r, phantom, "\perp"] & Cat \arrow[l, bend right, "-\times A"'] \end{tikzcd}$$

N preserves products

"Stable notion" (We want our constructions to give rise to ∞-categories when applied to ∞-cats)

Thm (G, 2.5 (i)) Let $K \in sSet$ & $\mathcal{C} \in QCat$. Then $Fun(K,\mathcal{C}) \in QCat$.

The main tool to prove this:

Thm (R 15.7) $\overline{InnHorn} \,\square\, Cell \subset \overline{InnHorn}$

Will not prove, but will explain what it means.

Def: $\mathcal{C}$ a category w. all small limits. A saturated class is a class $A \subset mor(\mathcal{C})$ which satisfies

1) It contains all isomorphisms

It is closed under:

2) Cobase change
3) Composition
4) Transfinite composition
5) Coproducts
6) Retract

4) Transfinite composition
5) Coproducts
6) Retract

Given a class of morphisms S, it's <u>saturation</u> $\bar{S}$ is the smallest saturated class containing S.

- Cobase change: $\begin{tikzcd} \bullet \arrow[r, "g"] \arrow[d, "f"'] & \bullet \arrow[d, "f'"] \\ \bullet \arrow[r] & \bullet \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$, $f \in \mathcal{A} \Rightarrow f' \in \mathcal{A}$

- Countable composition: $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots$ countable, $f_i \in \mathcal{A}$

$\Rightarrow X_0 \longrightarrow \operatorname{colim}_k X_k \in \mathcal{A}$

- Transfinite composition: As above, but over a general ordinal instead of $\mathbb{N}$.

- Retract: f retract of g if there exists a diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "id"] & \bullet \arrow[r] \arrow[d, "g"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] \arrow[rr, bend right, "id"'] & \bullet \arrow[r] & \bullet \end{tikzcd} \qquad g \in \mathcal{A} \Rightarrow f \in \mathcal{A}$$

Note: The list is not minimal:

- (4) $\Rightarrow$ (3)
- (2), (4) $\Rightarrow$ (5)

Def: $\text{Cell} := \{ \text{Inclusions } \partial\Delta^n \subset \Delta^n \mid n \geq 0 \} \subset \text{mor}(\text{sSet})$

$\bigcup_{j \in [n]} \Delta^{[n] \setminus \{j\}}$

$\text{InnHorn} := \{ \text{Inclusions } \Lambda^n_j \subset \Delta^n \mid 0 < j < n, \ n \geq 2 \} \subset \text{mor}(\text{sSet})$

← So the inner horns

Def: $\overline{\text{InnHorn}}$ is called the class of <u>inner anodyne morphisms</u>

Prop: Let $\mathcal{C} \in \text{QCat}$ and $(A \hookrightarrow B) \in \overline{\text{InnHorn}}$. Then for every map $A \to \mathcal{C}$, there exists a lift

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathcal{C} \\ B \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Rem: It can be shown that $\overline{\text{Cell}}$ consists of all inclusions of simplicial sets.

We still need "$\square$".

Def: The <u>pushout product</u> of two morphisms $f: A \to B$, $g: K \to L$ of simplicial sets is the morphism

$$f \square g : (A \times L) \coprod_{A \times K} (B \times K) \longrightarrow B \times L$$

Picture:

$$\begin{tikzcd} A \times K \arrow[r, "f \times id_K"] \arrow[d, "id_A \times g"'] & B \times K \arrow[d] \\ {} & {} \end{tikzcd}$$

$$
\begin{tikzcd}
A\times K \arrow[r, hook] \arrow[d, "\mathrm{id}_A\times g"'] & B\times K \arrow[d] \\
A\times L \arrow[r] \arrow[ur, phantom, "\ulcorner"] & A\times L \coprod\limits_{A\times K} (B\times K) \arrow[dr, "f\,\square\, g"', "{\color{magenta}(f\times \mathrm{id}_L,\ \mathrm{id}_B\times g)}"] & \\
 & & B\times L
\end{tikzcd}
$$

Thm: (R. 15.7) $\overline{\text{InnHorn}} \,\square\, \overline{\text{Cell}} \subset \overline{\text{InnHorn}}$

Pf: ($\mathcal{C}\in \mathrm{QCat}$, $K\in \mathrm{sSet} \Rightarrow \mathrm{Fun}(K,\mathcal{C})\in \mathrm{QCat}$)

Let $K\in$ sSet and note $\emptyset \in$ sSet. We have

$(i: \Lambda^n_\ell \hookrightarrow \Delta^n) \in \text{InnHorn} \subset \overline{\text{InnHorn}}$, $\quad 0<\ell<n$

$(j: \emptyset \hookrightarrow K) \in \overline{\text{Cell}}$ $:= \{ {\color{magenta} K \hookrightarrow L \mid K, L \in \mathrm{sSet}} \}$

Hence

$$i\,\square\, j : (\Lambda^n_\ell \times K) \coprod_{\Lambda^n_\ell\times\emptyset} (\Delta^n\times \emptyset) \cong \Lambda^n_\ell \times K \longrightarrow \Delta^n\times K$$

is in $\overline{\text{InnHorn}}\,\square\,\overline{\text{Cell}} \subset \overline{\text{InnHorn}}$.

$i\,\square\, j$ inner anodyne $\Rightarrow$ For any $\mathcal{C}\in\mathrm{QCat}$, $f: \Lambda^n_k\times K \to \mathcal{C}$, there exists a lift

$$
\begin{tikzcd}
\Lambda^n_\ell \times K \arrow[r] \arrow[d, "i\square j"'] & \mathcal{C} \\
\Delta^n\times K \arrow[ur, magenta, "\exists"']
\end{tikzcd}
$$

Using the adjunction

$$
\begin{tikzcd}
\mathrm{sSet} \arrow[r, bend right, "\mathrm{Fun}(K,-)"'] \arrow[r, phantom, "\perp"] & \mathrm{sSet} \arrow[l, bend right, "-\times K"']
\end{tikzcd}
$$

we get that this is equivalent to

$$
\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & \mathrm{Fun}(K,\mathcal{C}) \\
\Delta^n \arrow[ur, magenta, "\exists"']
\end{tikzcd}
\qquad \Rightarrow \mathrm{Fun}(K,\mathcal{C})\in\mathrm{QCat}
$$

$\square$

We also want it to be an "invariant" notion:

$K\simeq K' \in \mathrm{sSet} \Rightarrow \mathrm{Fun}(K,\mathcal{C}) \simeq \mathrm{Fun}(K',\mathcal{C})$

$\mathcal{C}\simeq\mathcal{C}'\in \mathrm{QCat} \Rightarrow \mathrm{Fun}(K,\mathcal{C}) \simeq \mathrm{Fun}(K,\mathcal{C}')$

Equivalences of ∞-categories & simplicial sets

(Following Rezk - equivalent to Groth's)

Def: Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories

- A categorical inverse to F consist of

${\color{magenta} \mathrm{id}_{\mathcal{C}} \quad G \quad \Uparrow\!\sim \quad F}$

Def: Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend right, "F"'] & \mathcal{D} \arrow[l, bend right, "G"'] \arrow[r, bend right, "\mathcal{C}"'] & \mathcal{C} \arrow[l, bend right, "F"'] \arrow[ll, bend right=70, "id_{\mathcal{C}}"'] \arrow[ll, phantom, bend right=35, "\Uparrow\simeq"] \arrow[r] & {} \end{tikzcd}$$

- A <u>categorical inverse</u> to F consist of
  - A functor $G: \mathcal{D} \to \mathcal{C}$
  - Two natural equivalences: $GF \simeq id_{\mathcal{C}}$ & $FG \simeq id_{\mathcal{D}}$
- F is called a <u>categorical equivalence</u> if it admits a categorical inverse.
- $\mathcal{C}$ and $\mathcal{D}$ are called <u>equivalent</u> if there exist a categorical equivalence between them.

Prop: (R 18.5). Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then F is a categorical equivalence iff the induced functor

$$F^*: \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$
$$(\mathcal{D} \xrightarrow{H} \mathcal{E}) \longmapsto (\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{H} \mathcal{E})$$
precomposing

is a categorical equivalence for all $\mathcal{E} \in \mathrm{QCat}$.

Pf: "$\Rightarrow$" Let $G: \mathcal{D} \to \mathcal{C}$ be the categorical inverse to F.

$\rightsquigarrow$ $G^*: \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{D}, \mathcal{E})$, $\mathcal{E} \in \mathrm{QCat}$
$$(\mathcal{C} \xrightarrow{H} \mathcal{E}) \xmapsto[\text{precomposing}]{} (\mathcal{D} \xrightarrow{G} \mathcal{C} \xrightarrow{H} \mathcal{E})$$

G^* is categorical inverse to F^*:

$$\eta: GF \overset{\simeq}{\Rightarrow} id_{\mathcal{C}} \rightsquigarrow \eta^*: G^*F^* \overset{\simeq}{\Rightarrow} id_{\mathrm{Fun}(\mathcal{C},\mathcal{E})}$$
$$\gamma: FG \overset{\simeq}{\Rightarrow} id_{\mathcal{D}} \rightsquigarrow \gamma^*: F^*G^* \overset{\simeq}{\Rightarrow} id_{\mathrm{Fun}(\mathcal{D},\mathcal{E})}$$

Using composition is functorial

"$\Leftarrow$" Assume

$$F^*: \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

categorical equivalence for any $\mathcal{E} \in \mathrm{QCat}$, with H categorical inverse

<u>Note</u>: $H \Rightarrow H': \mathcal{C} \to \mathcal{D} \rightsquigarrow hH \Rightarrow hH': h\mathcal{C} \to h\mathcal{D}$

We get that

$$hF^*: h\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\sim} h\mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

is an equivalence of ordinary categories.

Write $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$ for the subcategory with $\mathrm{ob}\,\mathcal{C}^{\simeq} = \mathrm{ob}\,\mathcal{C}$ and only the isomorphisms.

Then $\pi_0 \mathcal{C}^{\simeq}$ is the isomorphism classes, so since

$$(f: X \to Y) \in \mathcal{C}^{\simeq} \iff (hf: X \to Y) \in h\mathcal{C}^{\simeq}$$

we get

$$\pi_0 \mathcal{C}^{\simeq} \simeq \pi_0 h\mathcal{C}^{\simeq}$$

So F^* induces bijection

$$\pi_0 hF^*: \pi_0 h\mathrm{Fun}(\mathcal{D}, \mathcal{E})^{\simeq} \xrightarrow{\sim} \pi_0 h\mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}$$
$$\Rightarrow \pi_0 F^*: \pi_0 \mathrm{Fun}(\mathcal{D}, \mathcal{E})^{\simeq} \xrightarrow{\sim} \pi_0 \mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}$$

Put $\mathcal{E} = \mathcal{C}$: $\pi_0 F^*: \pi_0 \mathrm{Fun}(\mathcal{D}, \overset{\mathcal{C}}{\cancel{\mathcal{E}}})^{\simeq} \xrightarrow{\sim} \pi_0 \mathrm{Fun}(\mathcal{C}, \mathcal{C})^{\simeq}$

Put $\mathcal{E}=\mathcal{C}$: $\quad \pi_0 F^*: \pi_0\mathrm{Fun}(\mathcal{D},\overset{\mathcal{C}}{\cancel{\mathcal{E}}})^{\simeq} \xrightarrow{\sim} \pi_0\mathrm{Fun}(\mathcal{C},\mathcal{C})^{\simeq}$

$$G \longmapsto \mathrm{id}_{\mathcal{C}}$$

$\rightsquigarrow$ $\exists G \in \pi_0\mathrm{Fun}(\mathcal{D},\mathcal{E})^{\simeq}$ s.t.

$$F^*(G) \simeq GF \simeq \mathrm{id}_{\mathcal{C}}.$$

Put $\mathcal{E}=\mathcal{D}$: $\quad \pi_0 F^*: \pi_0\mathrm{Fun}(\mathcal{D},\mathcal{D})^{\simeq} \xrightarrow{\sim} \pi_0\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\simeq}$

$$F^*(\mathrm{id}_{\mathcal{D}}) \simeq \mathrm{id}_{\mathcal{D}}F \simeq F \simeq F\mathrm{id}_{\mathcal{C}} \simeq FGF \simeq F^*(FG)$$

Since F^* is a bijection: $\mathrm{id}_{\mathcal{D}} \simeq FG$. $\square$

Def: A map $f: X \to Y$ between simplicial sets is called a categorical equivalence if for every $\mathcal{C} \in \mathrm{QCat}$, the induced map

$$f^*: \mathrm{Fun}(Y,\mathcal{C}) \longrightarrow \mathrm{Fun}(X,\mathcal{C})$$
$$(Y \xrightarrow{h} \mathcal{C}) \longmapsto (X \xrightarrow{f} Y \xrightarrow{h} \mathcal{C})$$

is a categorical equivalence (of ∞-categories).

This is Groth prop 2.5 (iii)

$f: X \to Y \in sSet$ eq. is $\mathfrak{C}f: \mathfrak{C}X \to \mathfrak{C}Y$ is an eqx.

Prop: (G 2.5 (ii)) Let $F: \mathcal{C} \to \mathcal{D}$ be a categorical equivalence of ∞-categories. Then

$$F_*: \mathrm{Fun}(K,\mathcal{C}) \longrightarrow \mathrm{Fun}(K,\mathcal{D})$$
$$(K \xrightarrow{H} \mathcal{C}) \overset{\text{postcomp}}{\longmapsto} (K \xrightarrow{H} \mathcal{C} \xrightarrow{F} \mathcal{D})$$

is a categorical equivalence of ∞-categories for all $K \in sSet$.

Pf: Same idea as above. Let G be categorical inverse to F. Then G_* is categorical inverse to F_*. $\square$

Rem: Is an if and only if statement: Assume F_* is a categorical equivalence, and put $K = \Delta^0$:

$$\mathcal{C} \simeq \mathrm{Fun}(\Delta^0,\mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(\Delta^0,\mathcal{D}) \simeq \mathcal{D}.$$

Def: Let $\mathrm{Cat}_\infty^\Delta$ be the simplicial category defined by

- objects: ∞-categories
- $\mathrm{Map}(\mathcal{C},\mathcal{D}) = \mathrm{Fun}(\mathcal{C},\mathcal{D})^{\simeq}$

Applying the coherent nerve we get an ∞-category

$$\mathrm{Cat}_\infty = N_\Delta(\mathrm{Cat}_\infty^\Delta)$$

which is the <u>∞-category of ∞-categories</u>.

§2 Joins

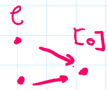
Def: Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$. The <u>join</u> of $\mathcal{C}$ and $\mathcal{D}$ is the category $\mathcal{C} \star \mathcal{D}$ with

- $\mathrm{Ob}(\mathcal{C} \star \mathcal{D}) \simeq \mathrm{ob}(\mathcal{C}) \amalg \mathrm{ob}(\mathcal{D})$

- $\mathrm{Ob}(\mathcal{C} \star \mathcal{D}) \simeq \mathrm{ob}(\mathcal{C}) \amalg \mathrm{ob}(\mathcal{D})$
- $$\mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(x,y) = \begin{cases} \hom_{\mathcal{C}}(x,y), & x,y \in \mathcal{C} \\ \hom_{\mathcal{D}}(x,y), & x,y \in \mathcal{D} \\ *, & x \in \mathcal{C},\ y \in \mathcal{D} \\ \emptyset, & x \in \mathcal{D},\ y \in \mathcal{C} \end{cases}$$
- Composition such that the inclusions

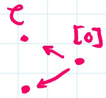
$$\mathcal{C} \hookrightarrow \mathcal{C} \star \mathcal{D} \hookleftarrow \mathcal{D}$$

are functors.



Ex: • Right cone: $\mathcal{C}^{\triangleright} := \mathcal{C} \star [0]$ attaching new terminal object

Left cone: $\mathcal{C}^{\triangleleft} := [0] \star \mathcal{C}$ attaching new initial object.



Def Let $X, Y \in s\mathrm{Set}$. Then we define the join of X and Y as $X \star Y \in s\mathrm{Set}$ by

$$(X \star Y)_n := X_n \cup Y_n \cup \left(\bigcup_{i+j+1=n} X_i \times Y_j \right), \quad n \geq 0,\ i,j \geq 0$$

Note that we have canonical inclusions

$$\begin{array}{ccc} X \hookrightarrow X \star Y & , & Y \hookrightarrow X \star Y \\ X_n \longmapsto X_n & & Y_n \longmapsto Y_n \end{array}$$

compatible w. the nerve functor

Prop: (6.2.13) Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$. Then

Isomorphism of simplicial sets

$$N(\mathcal{C} \star \mathcal{D}) \cong N(\mathcal{C}) \star N(\mathcal{D})$$

Join of categories   Join of simplicial sets

Pf: Want to construct a map $N(\mathcal{C} \star \mathcal{D}) \longrightarrow N(\mathcal{C}) \star N(\mathcal{D})$.

Object in $N(\mathcal{C} \star \mathcal{D})_n$ has one of the following forms:

1) $s_0 \to \cdots \to s_n$, $s_i \in \mathcal{C}$
2) $\underbrace{s_0 \to \cdots \to s_k}_{\text{In } \mathcal{C}} \longrightarrow \underbrace{s_{k+1} \longrightarrow \cdots \to s_n}_{\text{in } \mathcal{D}}$
3) $s_0 \to \cdots \to s_n$, $s_i \in \mathcal{D}$

Since in $\mathcal{C} \star \mathcal{D}$ there is no maps from objects in $\mathcal{D}$ to objects in $\mathcal{C}$.

Consider case 2:

$$C: \underbrace{s_0 \longrightarrow \cdots \longrightarrow s_k}_{A \in N(\mathcal{C})_k} \to \underbrace{s_{k+1} \longrightarrow \cdots \longrightarrow s_n}_{B \in N(\mathcal{D})_{n-k-1}}$$

Then $(A,B) \in N(\mathcal{C})_n \star N(\mathcal{D})_n = N(\mathcal{C})_n \cup N(\mathcal{D})_n \cup \left(\bigcup_{i+j+1=n} N(\mathcal{C})_i \times N(\mathcal{D})_j \right)$ since $k + (n-k-1) + 1 = n$.

Case 1 & 3: $s_0 \to \cdots \to s_n$ is a n-simplex in $N(\mathcal{C})$. (or $N(\mathcal{D})$.)

Then $(\sigma,\tau) \in N(\mathcal{C})_n * N(\mathcal{D})_n = N(\mathcal{C})_n \cup N(\mathcal{D})_n \cup \left(\bigcup_{i+j+1=n} N(\mathcal{C})_i \times N(\mathcal{D})_j\right)$
since $k + (n-k-1) + 1 = n$.

Case 1 & 3: $s_0 \to \cdots \to s_n$ is a n-simplex in $N(\mathcal{C})_n$ (or $N(\mathcal{D})_n$) and is simply map to that.

This is a bijection. $\square$

As with $\mathrm{Fun}(-,-)$ we also want this to be 'stable' and 'invariant'

Thm: (6.2.15) If

i) If $\mathcal{C}, \mathcal{D} \in \mathrm{QCat} \Rightarrow \mathcal{C} * \mathcal{D} \in \mathrm{QCat}$

ii) If $F: \mathcal{C} \to \mathcal{C}'$ and $G: \mathcal{D} \to \mathcal{D}'$ are equivalences of ∞-categories then the same is true for

$$F * G : \mathcal{C} * \mathcal{D} \to \mathcal{C}' * \mathcal{D}'.$$

Join is a functor
$*: \mathrm{sSet} \times \mathrm{sSet} \to \mathrm{sSet}$

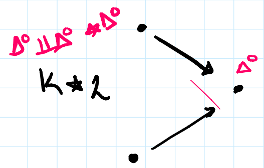
Join of categories

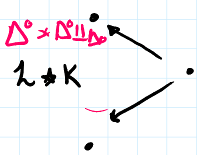
Ex: • $[n] * [m] \simeq [n+m+1]$

$$0_n \to \cdots \to n \to 0_m \to \cdots \to m$$

• $\Delta^{n+m+1} \simeq N([n+m+1]) \simeq N([n] * [m])$
$\simeq N([n]) * N([m]) \simeq \Delta^n * \Delta^m$

Ex: The join of simplicial sets is not symmetric.
Consider $K = \Delta^0 \amalg \Delta^0$, $L = \Delta^0$





Example of right cone left cone

Def. Let $K \in \mathrm{sSet}$. Then we define

- Right cone: $K^{\triangleright} := K * \Delta^0$
- Left cone: $K^{\triangleleft} := \Delta^0 * K$