Talk 8: Presentable ∞-categories

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Presentable categories

We will start of by quickly sketching this definition for ordinary categories, to get an intuition about the concepts which we will later extend to ∞-categories. There are two main ideas in the background:

- Passing from a small category to a *cocomplete* category (has all small colimits) in a universal way, is realized by categories of presheaves - can think of this a "Free generating"
- All locally presentable categories can be obtained as certain localizations of presheaf categories

Recall the Yoneda embedding for a small category A:

$$\begin{aligned} y : A &\to \mathrm{Fun}(A^{op}, \mathrm{Set}) \\ a &\mapsto \hom_A(-, a). \end{aligned}$$

For $X : A^{op} \to \mathrm{Set}$ we can define the *comma category* (y/X) by:

- Objects: (a, α) with $a \in A$ and $\alpha : y(a) \to X$ a natural transformation
- Morphism $f : (a, \alpha) \to (a', \alpha')$: Morphism $f : a \to a'$ in A which satisfies

$$\begin{array}{ccc} y(a) & \longrightarrow & y(a') \\ & \searrow \quad \swarrow & \\ & X. & \end{array}$$

We then have a projection functor

$$\begin{aligned} y/X &\xrightarrow{p} A \\ (a, \alpha) &\mapsto a. \end{aligned}$$

Further consider the composition

$$(y/X) \xrightarrow{p} A \xrightarrow{y} \mathrm{Fun}(A^{op}, \mathrm{Set}).$$

Proposition 0.1. *Let A be a small category and $X \in Fun(A^{op}, Set)$. Then there is a canonical isomorphism*

$$colim_{(y/X)} y \circ p \cong X.$$

We have that $\mathrm{Fun}(A^{op}, \mathrm{Set})$ is cocomplete for A a small category. This proposition tells us in a vague sense that the presheaf category $\mathrm{Fun}(A^{op}, \mathrm{Set})$ is the universal approximation of A by a cocomplete category, since every presheaf can be canonically written as a colimit of things of the form $\mathrm{Hom}_A(-, a)$ for $a \in A$. So we are interested in describing categories which are obtained by such presheaves. This will be done when we show that a presentable category is obtained from a localization of $\mathrm{Fun}(A^{op}, \mathrm{Set})$ for some small category A.

First we need some sort of smallness notion.

Definition 0.2. Let κ be a regular cardinal. A category $\mathcal{C}$ is *κ-accessible* of $\mathcal{C}$ admits κ-filtered colimits if there is a small subcategory $\mathcal{D} \subseteq \mathcal{C}$ such that

1. Every object in $\mathcal{C}$ can canonically be written as a κ-filtered colimit of objects in $\mathcal{D}$
2. $\mathrm{hom}_{\mathcal{C}}(d, -) : \mathcal{C} \to \mathrm{Set}$, $d \in \mathcal{D}$, preserves κ-filtered colimits.

We say $\mathcal{C}$ is *accessible* if it is κ-accessible for some regular cardinal κ.

A functor $F : \mathcal{C} \to \mathcal{D}$ is *κ-accessible* if $\mathcal{C}$ and $\mathcal{D}$ both admits κ-filtered colimits, and these are preserves by F. Again, we say F is *accessible* if it is κ-accessible for some κ.

The idea is that an accessible category admits certain filtered colimits and is formally determined by a small subcategory consisting of small objecs.

Definition 0.3. A category $\mathcal{C}$ is called *locally presentable* if it is cocomplete and accessible.

Example 0.4.

- Set is locally presentable
- $\mathrm{Fun}(A^{op}, \mathrm{Set})$ is locally presentable when A is small - in particular is sSet locally presentable
- $\mathrm{Mod}(R)$ and $\mathrm{Ch}(R)$ is both locally presentable for R a ring
- Cat of small categories is locally presentable.

One of the main reasons why we care about presentable categories is the so called adjoint functor theorem.

Theorem 0.5 (Adjoint functor theorem)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between locally presentable categories. Then*

1. *F admits a right adjoint if and only if it preserves all colimits*

2. *F admits a left adjoint if and only if it preserves all limits and is accessible.*

Up to equivalence, all locally presentable categories can be obtained as certain localizations of presheaf categories, therefor we will now look at the localizations.

Definition 0.6. A *reflective localization* is an adjunction $(L, R) : \mathcal{C} \to \mathcal{D}$ with the right adjoint R fully faithful.

In the case where we have such a reflective localization, it can be shown that $\mathcal{D}$ is equivalent to the localization $\mathcal{C}[S^{-1}]$, where S is the collection of morphisms in $\mathcal{C}$ which are mapped to isomorphisms by L.

Definition 0.7. Let $\mathcal{C}$ be a category and S a collection of morphisms in $\mathcal{C}$.

- $c \in \mathcal{C}$ is said to be *S-local* if
$$f^* : \hom_{\mathcal{C}}(c_2, c) \to \hom_{\mathcal{C}}(c_1, c)$$
is a bijection for all morphisms $f : c_1 \to c_2$ in $\mathcal{C}$.

- A morphism $f : c_1 \to c_2$ in $\mathcal{C}$ is an *S-local equivalence* if for all S-local objects $c \in \mathcal{C}$ the map
$$f^* : \hom_{\mathcal{C}}(c_2, c) \to \hom_{\mathcal{C}}(c_1, c)$$
is a bijection.

Proposition 0.8. *Let $(L, R) : \mathcal{C} \to \mathcal{D}$ be a reflective localization, and S the collection of morphisms in $\mathcal{C}$ which are inverted by L. Then*

1. *The essential image of R consists precisely of the S-local objects*

2. *The S-local equivalences are precisely the maps in S.*

So we see that a reflective localization is determined by exactly the collection of morphisms S. We say that a reflective localization $(L, R) : \mathcal{C} \to \mathcal{D}$ is an *accessible reflective localization* if the right adjoint R is accessible as well as being fully faithful.

Theorem 0.9. *A category is locally presentable if and only if it is equivalent to an accessible reflective localization of $Fun(A^{op}, Set)$ for some small category A.*

Presentable ∞-categories

The point of this presentation is in theory to give an introduction to presentable ∞-categories, which I will do. But since the proofs are rather complicated, I have decided to use this time to give a feeling for the definition, why we care about it and an interesting equivalent way of describing this definition. This will let me also talk a bit about other very interesting ideas in ∞-category land, which is often very helpful, namely construction of the Yoneda embedding, representable functors and localization. Ther references in this section is for *Higher Topos Theory* by Lurie.

Before we will be able to define presentable ∞-categories, we need to define *accessible* in this ∞-categorical setting, which will be describe how an ∞-category can be described using only a "small" amount of data. But this is not for ordinary small ∞-categories, we need to consider ∞-categories where the cardinality of the collection of objects is less than κ, where κ is some regular cardinal.

Definition 0.10. A simplicial set K is called *κ-small* if the cardinality of K_n is less than κ for all n.

So for an ∞-category this means that the cardinality of the collection of objects and of morphisms is less than κ and of higher morphisms a swell. Note that the issue is most often the size of the collection of objects, not morphisms.

Definition 0.11. An simplicial set $\mathcal{C}$ is called *κ-filtered* if for all κ-small $K \in \mathrm{sSet}$ and every morphism $f : K \to \mathcal{C}$ there is a morphism

$$\hat{f} : K^{\triangleright}(= K * \Delta^0) \to \mathcal{C}$$

extending f:

$$\begin{array}{ccc} K^{\triangleright} & \xrightarrow{\hat{f}} & \mathcal{C}. \\ \uparrow & \nearrow_{f} & \\ K & & \end{array}$$

If $\mathcal{C}$ is ω-filtered, where ω is the smallest infinite cardinal, we call it *filtered.*

Recall that a colimit of $p : K \to \mathcal{C}$ is an inital object i $\mathcal{C}_{p/}$, and we say it is *κ-filtered* if K is κ-filtered.

Example 0.12. The easiest example of a filtered ∞-category, is $N(\mathbb{N})$, with $\mathbb{N}$ is the category which has the natural numbers as objects and morphisms $n \to m$ if and only if $m > n$, which are unique. First note that if K is ω-small simplicial set, then it is finite. So if we have a functor $f : K \to N(\mathbb{N})$ then we can let $M \in \mathbb{N}$ such that $M > f(X)$ for all $X \in K_0$. Then there exists a natural extension

$$\hat{f} : K^{\triangleright} \to N(\mathbb{N}),$$

such that $\hat{f}(*) = M$, where $*$ is the new terminal object which we have assigned to K.

Example 0.13. Something which is not filtered is the category

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

If $f(K)$ lands in the two non-inital points, then there is no way to extend this to $K^{\triangleright}$, since there is no point to send the new terminal object to, since they don't both have arrows to another object.

It can be shown that presheaves $\mathcal{C}^{op} \to \mathscr{S}$ from a small ∞-category $\mathcal{C}$, is in one-to-one correspondence to right fibrations $\tilde{\mathcal{C}} \to \mathcal{C}$.

Definition 0.14. Let $\mathcal{C}$ be a small ∞-category. Then we define

$$\mathrm{Ind}_\kappa(\mathcal{C}) \subseteq P(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathscr{S})$$

as the full subcategory of those presheaves $F : \mathcal{C}^{op} \to \mathscr{S}$ for which the associated right fibration $\tilde{\mathcal{C}} \to \mathcal{C}$ has κ-filtred domain.

In the case where $\kappa = \omega$, we write $\mathrm{Ind}(\mathcal{C}) := \mathrm{Ind}_\kappa(\mathcal{C})$ and call it the ∞-*category of Ind-categories.*

The idea is that we formally add κ-filtered colimits to $\mathcal{C}$. It is classified by the following universal property

Proposition 0.15 (Proposition 5.3.4.18)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Assume that $\mathcal{C}$ is small and $\mathcal{D}$ admits κ-filtered colimits. Then composition with the Yoneda embedding induces an equivalence of ∞-categories*

$$Map_\kappa(Ind_\kappa(\mathcal{C}), \mathcal{D}) \to Fun(\mathcal{C}, \mathcal{D}),$$

where the left hand side denotes the ∞-category of all functors from $Ind_\kappa(\mathcal{C})$ to $\mathcal{D}$ which preserves all κ-filtered colimits.

Example 0.16. $\mathrm{Ind}(N(\mathrm{FinSet})) \simeq N(\mathrm{Set})$, where FinSet is the category of finite sets.

Definition 0.17. An ∞-category $\mathcal{C}$ is said to be κ-*accessible* if there exists a small ∞-category $\mathcal{C}^0$ together with an equivalence

$$\mathrm{Ind}_\kappa(\mathcal{C}^0) \xrightarrow{\simeq} \mathcal{C}.$$

We say that $\mathcal{C}$ is *accessible* if it is κ-accessible for some regular cardinal κ.

Definition 0.18. Let $\mathcal{C}$ be an accessible ∞-category. Then a functor $F : \mathcal{C} \to \mathcal{C}'$ between ∞-categories is called *accessible* if it preserves κ-filtered colimits (called being κ-*continuous*) for some κ.

In general accessible ∞-categories are not small, but they will still work as our "smallness" condition, since they are determined by only a "small" amount of data. The same is true for an accessible functor between accessible ∞-categories.

Example 0.19.

- (5.4.2.7) The ∞-category of spaces is accessible $\mathscr{S}$
- The ∞-category of spectra Sp

In general: Most non-small ∞-categories which you care about are accessible

Remark 0.20. The notion of an accessible ∞-category is closed under many constructions

5.4.4.3 If K is a small simplicial set and $\mathcal{C}$ an accessible ∞-category, then so is $\mathrm{Fun}(K, \mathcal{C})$

  – In particular, since $\mathscr{S}$ is accessible, we get that for any small ∞-category $P(\mathcal{C})$ is again accessible (Note, we only need $\mathcal{C}$ to be a small simplicial set)

5.4.5.16 Let $\mathcal{C}$ be an accessible ∞-category and $p : K \to \mathcal{C}$ be a digram indexed by a small simplicial set K. Then $\mathcal{C}_{p/}$ is again accessible.

5.4.6.7 Let $\mathcal{C}$ be an accessible ∞-category and $p : K \to \mathcal{C}$ be a digram indexed by a small simplicial set K. Then $\mathcal{C}_{/p}$ is again accessible.

5.4.6.6 Let

$$\begin{array}{ccc} X' & \xrightarrow{q'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{q} & Y \end{array}$$

be a pullback in Cat_∞. Suppose X, Y, Y' are accessible and that both p and q are accessible functors. Then X', p' and q' are all accessible.

Now that we know that most things we would care about is accessible, we will now define the main topic in this presentation.

Definition 0.21. An ∞-category $\mathcal{C}$ is *presentable* if it is accessible and admits small colimits.

Remark 0.22. Presentable is sometimes called locally presentable. The idea is that we can write everything in an presentable ∞-category as colimits of a small amount of objects from a small ∞-category.

Example 0.23.

- (5.5.1.8) The ∞-category of spaces $\mathscr{S}$ is presentable
- If $\mathcal{C}$ is a small ∞-category, then $P(\mathcal{C})$ is presentable
- Sp is presentable

So why do we care? One of the main reasons is the following version of the adjoint functor theorem:

Theorem 0.24 (Adjoint Functor Theorem (5.5.2.9))**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories.*

1) The functor F has a right adjoint if and only if is preserves small colimits

2) The functor F has a left adjoint if and only if it is accessible and preserves small limits.

Intuition: We broke the symmetry when we defined accessible and presentable, so it makes sense that there is a difference. We made presentable categories such that left adjoints were well-behaved, so makes sense that the conditions for such is less than for right adjoints.

First we see that the "only if" ($\Rightarrow$) follows directly by the following two propositions:

Proposition 0.25 (Proposition 5.2.3.5)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories and assume it admits a right adjoint G. Then F preserves all colimits which exist in $\mathcal{C}$ and G preserves all limits which exist in $\mathcal{D}$.*

Proposition 0.26 (Proposition 5.4.7.7)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between accessible ∞-categories. If F admits a left or right adjoint, then F is accessible.*

To even state the main propositions which goes into proving the other direction, we will need the notion of a *representable object,* for which we need an ∞-categorical version of the Yoneda embedding. This is done by following 5.1.3.

First recall the adjunction $N_\Delta \vdash C[-] : \text{sSet} \to \text{sCat}$, where N_Δ is the coherent nerve. Let K be a simplicial set, then we have a simplicial functor

$$\begin{aligned} C[K] \times C[K]^{op} &\to \text{Kan} \\ (X, Y) &\mapsto \text{Sing}|\text{Hom}_{C[K]}(X, Y)|, \end{aligned}$$

where $|-| : \text{sSet} \to \text{Top}$ is an ordinary functor called *geometric realization.* $C[-]$ does not in general commute with products, but there is a natural map $C[K \times K^{op}] \to C[K] \times C[K]^{op}$, so we have a simplicial functor

$$C[K \times K^{op}] \to C[K] \times C[K]^{op} \to \text{Kan}.$$

Passing to the adjoint and recalling that $\mathscr{S} = N_\Delta(\text{Kan})$ we get a map

$$K \times K^{op} \to \mathscr{S}.$$

By further using the adjunction $- \times K^{op} \dashv \text{Fun}(K^{op}, -) : \text{sSet} \to \text{sSet}$ we get the so called *Yoneda embedding*

$$y : K \to \text{Fun}(K^{op}, \mathscr{S}) = P(K),$$

which can be shown to be fully faithful (5.1.3.1). If we consider the Yoneda embedding for an ∞-category $\mathcal{C}$ we have that it is given by

$$\begin{aligned} j : \mathcal{C} &\to \text{Fun}(\mathcal{C}^{op}, \mathscr{S}) \\ X &\mapsto \text{Map}_{\mathcal{C}}(-, X). \end{aligned}$$

By 5.1.3.2 we get that j preserves all small limits in $\mathcal{C}$ if $\mathcal{C}$ is a small ∞-category.

Definition 0.27. Let $\mathcal{C}$ be an ∞-category. An object F in $P(\mathcal{C})$ (i.e. a functor $F : \mathcal{C}^{op} \to \mathscr{S}$) is called *representable* if it's in the essential image of the Yoneda embedding, i.e. if $F(-) \simeq \text{Map}_{\mathcal{C}}(-, X)$ for some object $X \in \mathcal{C}$. In this case we say it is *represented* by X.

There is also a dual version, where we say that $F \in P(\mathcal{C})$ is *corepresentable* if it's in the essential image of the "dual" Yoneda embedding, i.e. if $F(-) \simeq \mathrm{Map}_{\mathcal{C}}(X, -)$.

We are now ready to state the main ingredients for the proof of the "if" part of the adjoint functor theorem.

Proposition 0.28 (Proposition 5.5.2.7). *Let $\mathcal{C}$ be a presentable ∞-category and $F : \mathcal{C} \to \mathscr{S}$ a functor. Then F is corepresented by an object of $\mathcal{C}$ if and only if F is accessible and preserves small limits.*

Proposition 0.29 (Proposition 5.2.4.2). *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then the following is equivalent:*

1. *The functor F has a left adjoint*
2. *For every pullback diagram*

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{D}' \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

if p is a corepresentable left fibration, then p' is also a corepresentable left fibration.

When we talk about a corepresentable left fibration, we mean that the presheaf $\mathcal{C}^{op} \to \mathscr{S}$ it corresponds to, is corepresentable.

Proof of Adjoint Fucntor Theorem "if". We will only show part two, since part one i similar but easier. So let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories which is accessible and preserves small limits. Further let $F' : \mathcal{D} \to \mathscr{S}$ be a corepresentable functor. We then know by 5.5.2.7 that F' is accessible and preserves small limits, hence the same holds for the composition

$$F' \circ F : \mathcal{C} \to \mathcal{D} \to \mathscr{S},$$

and note that it is again a presheaf. By again invoking 5.5.2.7 we get that $F' \circ F$ is corepresentable. If we denote the corepresentable left fibrations associated to F' and $F' \circ F$ by $F' : \tilde{\mathcal{D}} \to \mathcal{D}$ and $F' \circ F : \tilde{\mathcal{C}} \to \mathcal{C}$ respectively, we have that they fit into the following pullback

$$\begin{array}{ccc} \widetilde{\mathcal{C}} & \longrightarrow & \widetilde{\mathcal{D}} \\ {\scriptstyle F' \circ F}\big\downarrow & & {\scriptstyle F'}\big\downarrow \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D}. \end{array}$$

Since we have just showed that this pullback diagram satisfies the conditions in 5.2.4.2 (2) we get that F has a left adjoint as desired. ∎

Localization

Now that we have the definition of presentable ∞-category and we know why we care about it, we will take a look at an equivalent description, for which we will need the following definition:

Definition 0.30. A functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is a *localization* if F admits a fully faithful right adjoint. We further say it is an *accessible localization* if the right adjoint is accessible.

Note: This is the terminology from Lurie, but in Groth it is called "*reflective localization*". Most often it is called *Bousfield localization* to distinguish it from other kinds of localizations, e.g. Dwyer Kan localization.

Theorem 0.31 (Theorem 5.5.1.1 (1)+(5)). *Let $\mathcal{C}$ be an ∞-category. Then the following is equivalent:*

1. *$\mathcal{C}$ is presentable*
2. *There exists a small ∞-category $\mathcal{D}$ such that $\mathcal{C}$ is an accessible localization of $P(\mathcal{D})$.*

Due to this theorem we see that we want to understand localizations to understand presentable ∞-categories. If $L : \mathcal{C} \to \mathcal{D}$ is a localization, we denote the composition of L with the right adjoint by $L : \mathcal{C} \to \mathcal{C}$, which is called the *localization functor*. The idea of localization is that a localization functor is determined, up to equivalence, by the collection S of all morphisms f in $\mathcal{C}$ which satisfies that Lf is an equivalence. First we note the following equivalent ways of recognizing localization functors

Proposition 0.32 (Proposition 5.2.7.4). *Let $\mathcal{C}$ be an ∞-category and $L : \mathcal{C} \to \mathcal{C}$ an endofunctor with essential image $L\mathcal{C} \subseteq \mathcal{C}$. Then the following conditions are equivalent*

1. *There exists a functor $F : \mathcal{C} \to \mathcal{D}$ with a fully faithful right adjoint $G : \mathcal{D} \to \mathcal{C}$ and an equivalence between $G \circ F$ and L,*
2. *When L is considered as a functor $\mathcal{C} \to L\mathcal{C}$, it is left adjoint to the inclusion $L\mathcal{C} \hookrightarrow \mathcal{C}$,*
3. *There exists a natural transformation $\eta : id_{\mathcal{C}} \Rightarrow L$, such that for every object $c \in \mathcal{C}$, the morphisms*
$$L(\eta(c)), \eta(Lc) : Lc \to LLc,$$
in $\mathcal{C}$, are equivalences.

In part two we see that $L : \mathcal{C} \to L\mathcal{C}$ is clearly a localization. If we conversely have a localization $L : \mathcal{C} \to \mathcal{D}$, we can identify $\mathcal{D}$ with a full subcategory of $\mathcal{C}$, by using the right adjoint from part one, hence we are in the case of part two again. This means that this proposition gives us equivalent ways of describing localization functors. In particular, we get that a localization functor $L : \mathcal{C} \to \mathcal{C}$ is left adjoint to the inclusion $L\mathcal{C} \subseteq \mathcal{C}$, and an endofunctor is a localization if it can be equipped with a natural transformation η as in part three.

We will now consider how accessible localizations of an arbitrary presentable ∞-category.

Definition 0.33. Let $\mathcal{C}$ be an ∞-category and S a collection of morphisms of $\mathcal{C}$. We say that $x \in \mathcal{C}$ is *S-local* if for every morphism $f : z \to y$ in $\mathcal{C}$ the induced map

$$f^* : \mathrm{Map}_{\mathcal{C}}(z, x) \to \mathrm{Map}_{\mathcal{C}}(y, x)$$

is a weak equivalence. We write $S^{-1}\mathcal{C}$ for the full subcategory of S-local objects in $\mathcal{C}$.

Further, a morphisms $f : z \to y$ in $\mathcal{C}$ is called an *S-local equivalence* if, for every S-local object x, the induced map

$$f^* : \mathrm{Map}_{\mathcal{C}}(z, x) \to \mathrm{Map}_{\mathcal{C}}(y, x)$$

is a weak equivalence.

Proposition 0.34 (Proposition 5.5.4.1)**.** *Let $\mathcal{C}$ be an ∞-category and $L : \mathcal{C} \to \mathcal{C}$ a localization functor. Further, let S denote the collection of all morphisms f in $\mathcal{C}$ such that Lf is an equivalence. Then*

1. *An object c of $\mathcal{C}$ is S-local if and only if it belongs to $L\mathcal{C}$.*

2. *Every S-local equivalence in $\mathcal{C}$ belongs to S.*

3. *Suppose that $\mathcal{C}$ is accessible. Then the following conditions is equivalent:*

   (a) The ∞-category $L\mathcal{C}$ is accessible

   (b) The functor $L : \mathcal{C} \to \mathcal{C}$ is accessible

   (c) there exists a (small) subset $S_0 \subseteq S$ such that every S_0-local object is S-local.

This gives us that the essential image of a localization functor $L : \mathcal{C} \to \mathcal{C}$ is exactly the S-local objects, where S is as in the proposition above. So a localization functor is completely determined by S. So next step must be to try and understand this class of morphisms a bit better. It can be shown that it satisfies the following properties(5.5.4.10):

- S is closed under formation of colimits in S

- S is stable under the formation of retracts

- Contains all equivalences in $\mathcal{C}$

- Stable under cobase change

- Suppose we are given a 2-simplex of $\mathcal{C}$ corresponding to the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle g}\searrow \qquad \swarrow{\scriptstyle h} & \\ & Z. & \end{array}$$

if any two if f, g and h belongs to S, then so does the third. This is called the 2-*out-of*-3 *property* with respect to 2-simplices.

A class of morphisms which satisfies these properties are called *strongly saturated*.

Remark 0.35 (Remark 5.5.4.7)**.** It can be shown that if $\{S_\alpha\}_{\alpha \in A}$ is a collection of strongly saturated classes of morphisms of an ∞-category $\mathcal{C}$, which admits colimits, then the intersection $\cap_{\alpha \in A} S_\alpha$ is also strongly saturated. So any collection S_0 of morphisms in $\mathcal{C}$ is contained in some minimal strongly saturated class of morphisms S. In this case we write $S = \bar{S_0}$, and we refer to S as the strongly saturated class of morphisms *generated* by S_0. We will say that S is *of small generation* if $S = \bar{S_0}$ where $S_0 \subseteq S$ is small.

Proposition 0.36 (5.5.4.15)**.** *Let $\mathcal{C}$ be a presentable ∞-category and let S be a small collection of morphisms of $\mathcal{C}$. Let $\bar{S}$ denote the strongly saturated class of morphisms generated by S. Then*

1. *For each $c \in \mathcal{C}$ there exists a morphism $s : c \to c'$ such that c' is S-local and s belongs to $\bar{S}$*
2. *The ∞-category $S^{-1}\mathcal{C}$ is presentable*
3. *The inclusion $S^{-1}\mathcal{C} \subseteq \mathcal{C}$ has a left adjoint L*
4. *For every morphism f in $\mathcal{C}$, the following are equivalent:*
   - *(a) The morphism f is an S-equivalence*
   - *(b) The morphism f belongs to $\bar{S}$*
   - *(c) The induced morphism Lf is an equivalence.*

So for any small set of morphism S of a presentable ∞-category $\mathcal{C}$, then the inclusion $S^{-1}\mathcal{C} \subseteq \mathcal{C}$ is a localization, and every accessible localization arises in this way. It can further be shown that a strongly saturated class S of morphisms in a presentable ∞-category $\mathcal{C}$ is of small generation, if and only if there is an accessible localization functor $L : \mathcal{C} \to \mathcal{C}$ such that S is exactly those morphisms of $\mathcal{C}$ which are mapped to equivalences by L.

Finally we have the following universal property characterizing the localization $S^{-1}\mathcal{C}$. Note that $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the full subcategory spanned by the left adjoint functors $\mathcal{C} \to \mathcal{D}$.

Proposition 0.37 (Proposition 5.5.4.20)**.** *Let $\mathcal{C}$ be a presentable ∞-category and $\mathcal{D}$ an arbitrary ∞-category. Further let S be a small set of morphisms of $\mathcal{C}$, and $L : \mathcal{C} \to$*

$\S^{-1}\mathcal{C} \subseteq \mathcal{C}$ an associated accessible localization functor. Composition with L induces a functor

$$\eta : Fun^L(S^{-1}\mathcal{C}, \mathcal{D}) \to Fun^L(\mathcal{C}, \mathcal{D}).$$

The functor η is fully faithful, and the essential image of η consists of those functors $f : \mathcal{C} \to \mathcal{D}$ such that $f(s)$ is an equivalence in $\mathcal{D}$ for each $s \in S$.