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Presentable Categories

Yoneda embedding: A small cat

$$\begin{aligned} y: A &\longrightarrow \mathrm{Fun}(A^{op}, \mathrm{Set}) =: P(A) \\ a &\longmapsto \hom_A(-,a) \end{aligned}$$

$X \in P(A) \rightsquigarrow$ comma category (y/X):

- ob: (a, α), $a \in A$, $\alpha: y(a) \to X$
- mor $f: (a,\alpha) \to (a', \alpha')$ is a morp $f: a \to a'$ in A s.t.

$$\begin{tikzcd} y(a) \arrow[rr] \arrow[dr, "\alpha"'] & & y(a') \arrow[dl, "\alpha'"] \\ & X & \end{tikzcd}$$

Projection:

$$\begin{aligned} (y/X) &\xrightarrow{P} A \\ (a,\alpha) &\longmapsto a \end{aligned}$$

$$\rightsquigarrow \quad \begin{aligned} (y/X) &\xrightarrow{P} A \xrightarrow{y} P(A) \\ (a,\alpha) &\longmapsto a \longmapsto \hom_A(-,a) \end{aligned}$$

Prop: A small cat., $X \in P(A)$. Then there is a canonical iso

$$\mathrm{colim}_{(y/X)}\, y \circ P \cong X$$

Def: κ regular cardinal. A cat. $\mathcal{C}$ is κ-accessible if $\mathcal{C}$ admits κ-filtered colimits and there exists small subcat. $\mathcal{D} \subseteq \mathcal{C}$ s.t.

1) Every object in $\mathcal{C}$ can be written as a κ-filtered colimit of objects in $\mathcal{D}$

2) $\hom_{\mathcal{C}}(d,-): \mathcal{C} \to \mathrm{Set}$, $d \in \mathcal{D}$, preserves κ-filtered colimits

- Say $\mathcal{C}$ accessible if κ-accessible for some κ
- A functor $F: \mathcal{C} \to \mathcal{D}$ is κ-accessible if both $\mathcal{C}$ & $\mathcal{D}$ admits κ-filtered colimits and F preserves these

• A functor $F: \mathcal{C} \to \mathcal{D}$ is [illegible] if $\mathcal{C}$ [illegible] admits κ-filtered colimits and F preserves these.

Def: A cat. $\mathcal{C}$ is <u>locally presentable</u> if it is cocomplete & accessible.

Ex:
- Set $\mathcal{D}$ = FinSet
- $\mathrm{Fun}(A^{op}, \mathrm{Set})$ when A is small ($\mathcal{D}$ = representable functors)
- $\mathrm{Mod}(R)$, $\mathrm{Ch}(R)$ for R a ring $\rightarrow \mathcal{D} = \mathrm{Ch}^b(\text{Fin. gen. proj. } R)$
  $\mathcal{D}$ = fin. gen. proj.
- Cat of small categories

<u>Adjoint Functor Theorem</u> $F: \mathcal{C} \to \mathcal{D}$ between locally presentable categories.

1) F admits a right adjoint $\Leftrightarrow$ F preserves all colimits
2) F admits a left adjoint $\Leftrightarrow$ F preserves all limits and is accessible

Def: A <u>reflective localization</u> is an adjunction
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd}$$
w. R fully faithful.

Def: Let $\mathcal{C}$ be a cat. and $S \subset \mathrm{mor}\,\mathcal{C}$.

- $c \in \mathcal{C}$ is <u>S-local</u> if for all $f: c_1 \to c_2$ in S the induced map
$$f^*: \hom_{\mathcal{C}}(c_2, c) \longrightarrow \hom_{\mathcal{C}}(c_1, c)$$
is a bijection.
- A morphism $f: c_1 \to c_2$ in $\mathcal{C}$ is an <u>S-local equivalence</u> if for all S-local objects $c \in \mathcal{C}$:
$$f^*: \hom_{\mathcal{C}}(c_2, c) \to \hom_{\mathcal{C}}(c_2, c)$$
bijection.

Prop:
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd}$$
reflective localization, and $S \subset \mathrm{mor}\,\mathcal{C}$ which are inverted by L. Then
1) The essential image of R is precisely the S-local obj.

1) The essential image of R is precisely the S-local obj.

2) The S-local eq. are precisely the maps in S

Def: A reflective localization (L, R) is called <u>accessible</u> if R is accessible.

Thm: A category is locally presentable

$$\Leftrightarrow$$

If it is equivalent to ~~the~~ an accessible reflective localization of $P(A)$ for some small category A.

Presentable ∞-Categories

κ : regular cardinal, ω : smallest infinite cardinal

Def: $K \in sSet$ is <u>κ-small</u> if K_n has cardinality less than κ

Def: $\mathcal{C} \in sSet$ is <u>κ-filtered</u> if for all κ-small $K \in sSet$ and every $f: K \to \mathcal{C}$ the exists a morphism

$$\bar{f}: \underset{\substack{\| \\ K \star \Delta^0}}{K^{\triangleright}} \to \mathcal{C}$$

extending f:

$$\begin{tikzcd} K^{\triangleright} \arrow[r, "\bar{f}"] & \mathcal{C} \\ K \arrow[u, hook] \arrow[ur, "f"'] \arrow[u, phantom, shift left=3, "\circlearrowleft"] & \end{tikzcd}$$

If $\mathcal{C}$ is ω-filtered then it is called <u>filtered</u>.

Ex: $\mathbb{N} \in Cat$ ob: $0, 1, 2, \ldots$

mor: $0 \to 1 \to 2 \to 3$

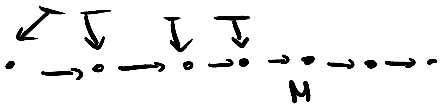
K ω-small $\rightsquigarrow$ K finite

$$f: K \to N(\mathbb{N})$$

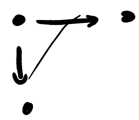
choose $M \in \mathbb{N}$ s.t. $M > f(x)$ for $\forall x \in K_0$

$$\hat{f}: K^{\triangleright} \to N(\mathbb{N})$$
$$K \longmapsto f(K)$$

$$\begin{aligned} f : K &\longrightarrow N(\mathbb{N}) \\ K &\longmapsto f(K) \\ \text{terminal} &\longmapsto M \end{aligned}$$



Ex: non-example:



Rem: $\mathcal{C}^{op} \to \mathcal{S}$, $\mathcal{C}$ small $\leftrightarrow$ right fibrations $\tilde{\mathcal{C}} \to \mathcal{C}$

Def: $\mathcal{C}$ small ∞-cat.

$$\mathrm{Ind}_{\kappa}(\mathcal{C}) \subseteq P(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$$

full subcat. spanned by those $F: \mathcal{C}^{op} \to \mathcal{S}$ for which the associated right fibration $\tilde{\mathcal{C}} \to \mathcal{C}$ has κ-filtered domain.

- $\kappa = \omega$: $\mathrm{Ind}(\mathcal{C})$ ∞-category of Ind-objects

Prop: (5.3.4.18) $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_\infty$, $\mathcal{C}$ small and $\mathcal{D}$ admits κ-filtered colimits. Then comp. w. Yoneda embedding induces an eq. of ∞-cat.

$$\underbrace{\mathrm{Map}_{\kappa}}_{\text{preserves } \kappa\text{-filtered colimits}}(\mathrm{Ind}_{\kappa}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

Def: $p: K \to \mathcal{C}$ initial in $\mathcal{C}_{p/}$ a colimit is called a <u>κ-filtered colimit</u> if K is κ-filtered.

Def: $\mathcal{C} \in \mathrm{Cat}_\infty$ is called κ-<u>accessible</u> if $\exists$ small $\mathcal{C}^0 \in \mathrm{Cat}_\infty$ together

Def: $\mathcal{C} \in \mathrm{Cat}_\infty$ is called κ-__accessible__ if $\exists$ small $\mathcal{C}^\circ \in \mathrm{Cat}_\infty$ together with an eq.

$$\mathrm{Ind}(\mathcal{C}^\circ) \xrightarrow{\sim} \mathcal{C}$$

$\mathcal{C}$ is called __accessible__ if κ-accessible for some κ

Def: $\mathcal{C}$ κ-accessible. $F: \mathcal{C} \to \mathcal{C}'$ between ∞-cats is called __κ-accessible__ if it preserves κ-filtered colimits.

Ex: (5.4.2.7) $\mathcal{S}$ is accessible

- Sp is accessible

Rem: $\mathcal{C}$ an accessible ∞-cat.

5.4.4.3 • $K \in \mathrm{sSet}$ small $\rightsquigarrow$ $\mathrm{Fun}(K, \mathcal{C})$ is accessible

$\hookrightarrow$ $\mathcal{P}(\mathcal{C})$ is accessible for $\mathcal{C}$ small

5.4.5.16 • $p: K \to \mathcal{C}$, $K \in \mathrm{sSet}$ small $\rightsquigarrow$ $\mathcal{C}_{p/}$, $\mathcal{C}_{/p}$ accessible
5.4.6.7

5.4.6.6 • Pull-back Cat_∞:

$$\begin{tikzcd} X' \arrow[r, "q'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ Y' \arrow[r, "q"'] & Y \end{tikzcd}$$

Then if X, Y', Y, p, q are accessible $\Rightarrow X', p', q'$ are accessible.

Def: An ∞-category $\mathcal{C}$ is __presentable__ if it is accessible and and admits small colimits

Ex:
- $\mathcal{S}$
- $\mathcal{P}(\mathcal{C})$ for $\mathcal{C}$ small
- Sp

$\Big\}$ presentable

__Adjoint functor Theorem__ $F: \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories. Then

1) F has a right adjoint $\Leftrightarrow$ F preserves all small colimits

1) F has a right adjoint <=> F preserves all small colimits

2) F has a left adjoint <=> F preserves all small limits & is accessible

only if =>

5.2.3.5 Prop: $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories and assume it admits a right adjoint G. Then F preserves all ~~small~~ limits in $\mathcal{C}$ and G preserves all limits in $\mathcal{D}$.

Prop 5.4.7.7 $F : \mathcal{C} \to \mathcal{D}$ between accessible ∞-categories. Then if F admits an adjoint, then F is accessible

To show "<=" we need representable functors

Yoneda embedding - (§5.3.1)

$$\begin{tikzcd} sSet \arrow[r, bend left, "C[-]"] \arrow[r, phantom, "\perp"] & sCat \arrow[l, bend left, "N_\Delta"] \end{tikzcd}$$

$K \in sSet$, then we have a simplicial functor

$$C[K \times K^{op}] \to C[K] \times C[K^{op}] \longrightarrow \mathcal{K}an$$
$$(X, Y) \longmapsto \mathrm{Sing}\,|\mathrm{Hom}_{C[K]}(X, Y)|$$

$$\overset{C[-] \dashv N_\Delta}{\rightsquigarrow} \quad K \times K^{op} \to N_\Delta(\mathcal{K}an) = \mathcal{S}$$

$$\overset{- \times K^{op} \dashv \mathrm{Fun}(K^{op}, -)}{\rightsquigarrow} \quad \text{Yoneda embedding } y : K \to \mathrm{Fun}(K^{op}, \mathcal{S}) = \mathcal{P}(K)$$

fully faitful by 5.1.3.1

In case $\mathcal{C} \in \mathcal{C}at_\infty$:

$$j : \mathcal{C} \longrightarrow \mathcal{P}(\mathcal{C})$$
$$x \longmapsto \mathrm{Map}_{\mathcal{C}}(-, x) \qquad \mathrm{hom}_{\mathcal{C}}(-, x)$$

and if $\mathcal{C}$ is small j preserves all small limits

$x \mapsto \mathrm{Map}_{\mathcal{C}}(-,x)$

and if $\mathcal{C}$ is small j preserves all small limits.

<u>Def:</u> $\mathcal{C} \in \mathrm{Cat}_\infty$. $F \in \mathcal{P}(\mathcal{C})$ is called <u>representable</u> if it is in the essential image of the Yoneda embedding ($F(-) \simeq \mathrm{hom}_{\mathcal{C}}(-, C)$, say F is <u>represented</u> by $C \in \mathcal{C}$)

<u>Dual</u> $F \in \mathcal{P}(\mathcal{C}^{op})$ is <u>corepresentable</u> if $F(-) \simeq \mathrm{hom}_{\mathcal{C}}(C, -)$.

<u>Prop 5.5.2.7</u> $\mathcal{C}$ presentable ∞-category and $F : \mathcal{C} \to \mathcal{S}$. Then F is corep. by an ob. in $\mathcal{C}$ iff F is accessible and preserves small limits

<u>Prop: 5.2.4.2</u> : Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then TFAE:

1) F admits a left adjoint
2) For every pullback diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d, "p'"'] & \mathcal{D}' \arrow[d, "p"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

if p is a corep. left fibration, p' is also a corep. left fibration.

<u>Pf of AFT if (2):</u> $F : \mathcal{C} \to \mathcal{D}$ functor between ~~∞-catop~~ presentable ∞-categories which is accessible and preserves small limits

Let $F' : \mathcal{D} \to \mathcal{S}$ corepresentable functor

$\overset{5.5.2.7}{\leadsto}$ F' is accessible and preserves small limits

... is accessible and preserves small

5.5.2.? $\rightsquigarrow$ F' is accessible and preserves small limits

$\rightsquigarrow$ $F'\circ F: \mathcal{C} \to \mathcal{D} \to \mathcal{S}$ is accessible and preserves small limits

5.5.2.7
$\rightsquigarrow$ $F'\circ F$ is corepresentable

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r] \arrow[d, "F'\circ F"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{D}} \arrow[d, "F'"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

∎

Localization

Def: $F: \mathcal{C} \to \mathcal{D}$ between ∞-categories is a <u>localization</u> if it admits a fully faithful right adj. R.

- It is an <u>accessible localization</u> if R is accessible.

<u>Thm</u> (5.5.1.1) $\mathcal{C} \in Cat_\infty$. Then TFAE:

1) $\mathcal{C}$ is presentable
2) $\exists$ small $\mathcal{D} \in Cat_\infty$ s.t. $\mathcal{C}$ is an accessible localization of $\mathcal{P}(\mathcal{D})$

$$\mathcal{P}(\mathcal{D}) \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{C} \quad (\perp)$$

$L: \mathcal{C} \to \mathcal{D}$ a localization $\rightsquigarrow$ $L: \mathcal{C} \to \mathcal{C}$ localization functor

<u>Prop</u> 5.2.7.4: $\mathcal{C} \in Cat_\infty$, $L: \mathcal{C} \to \mathcal{C}$ an endofunctor with essential image $L\mathcal{C} \subseteq \mathcal{C}$. Then TFAE:

1) $\exists$ functor $F: \mathcal{C} \to \mathcal{D}$ w. a fully faithful right adjoint R and an eq. between $G\circ F$ and L.

2) $L: \mathcal{C} \to L\mathcal{C}$ is left adjoint to $L\mathcal{C} \subseteq \mathcal{C}$

eq. between ...

2) $L : \mathcal{C} \to L\mathcal{C}$ is left adjoint to $L\mathcal{C} \subseteq \mathcal{C}$

3) There exists a nat. trans. $\eta : id_{\mathcal{C}} \Rightarrow L$ s.t. $\forall c \in \mathcal{C}$ the morphisms
$$L(\eta(c)),\ \eta(Lc) : Lc \to LLc$$
in $\mathcal{C}$, are equivalences

<u>Def:</u> $\mathcal{C} \in Cat_\infty$, $S \subset mor(\mathcal{C})$.

- $x \in \mathcal{C}$ is <u>S-local</u> if for every $f: x \to y$ in S the induced map
$$f^* : Map_{\mathcal{C}}(z,x) \longrightarrow Map_{\mathcal{C}}(y,x)$$
is a w.eq.
- A morphism $f: z \to y$ in $\mathcal{C}$ is called an <u>S-local equivalence</u> if for S-local objects $c \in \mathcal{C}$ the induced map
$$f^* : Map_{\mathcal{C}}(z,x) \longrightarrow Map_{\mathcal{C}}(y,x)$$
is a w.eq.

$S^{-1}\mathcal{C} \subseteq \mathcal{C}$ full subcat. of S-local objects.

<u>Prop 5.5.4.1</u> $\mathcal{C} \in Cat_\infty$ and $L : \mathcal{C} \to \mathcal{C}$ a localization functor. Further $S \subset Mor(\mathcal{C})$ for which Lf is an ~~eq~~ eq. Then

1) $c \in \mathcal{C}$ is S-local $\Leftrightarrow$ $c \in L\mathcal{C}$
2) Every S-local eq. belongs to S
3) If $\mathcal{C}$ is accessible, then TFAE:
   i) $L\mathcal{C}$ is accessible
   ii) $L : \mathcal{C} \to \mathcal{C}$ is accessible
   iii) $\exists$ small subset $S_0 \subseteq S$ s.t every S_0-local ob. is S-local.

S as above satisfies the following (5.~~5~~5.4.10)

- closed under formation of colimits in S
- stable under formation of retracts
- contains all eq. of $\mathcal{C}$
- Stable under cobase change
- Given a 2-simplex of $\mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, "h"] \\ & Z \end{tikzcd}$$

then if any two of f, g, h are in S, then so is the third (2-out-of-3 prop)

$\Big\}$ Strongly saturated (SS)

Rem: $\{S_\alpha\}_{\alpha \in A}$ collection of SS classes of morphisms in $\mathcal{C}$, then $\bigcap_{\alpha \in A} S_\alpha$ again SS.

Therefor, any class of morphisms $S_0 \subseteq \mathrm{mor}(\mathcal{C})$ is contained in a minimal SS class of mor. in $\mathcal{C}$ S.

- Write $S = \overline{S_0}$, S is SS class generated by S_0
- S_0 is small, $S = \overline{S_0}$: S is of small generation

Prop 5.5.4.15: $\mathcal{C}$ presentable ∞-category and $S \subseteq \mathrm{mor}(\mathcal{C})$ small. Then

1) $\forall c \in \mathcal{C}$ $\exists$ $s: c \longrightarrow c'$ in $\mathcal{C}$ s.t. $c' \in S^{-1}\mathcal{C}$ and $s \in \overline{S}$
2) $S^{-1}\mathcal{C}$ is presentable
3) $S^{-1}\mathcal{C} \subseteq \mathcal{C}$ has a left adjoint L $\quad L: \mathcal{C} \to S^{-1}\mathcal{C} \to \mathcal{C}$
4) For every morphism f in $\mathcal{C}$ TFAE:
   i) f is an S-local eq.
   ii) $f \in \overline{S}$
   iii) Lf is an eq.