DERIVED COMPLETION IN ALGEBRA

Plan: ① Recap + motivation
② L_0^I-completion
③ Derived completion

§1: RECAP + MOTIVATION

$E^*(BG_+)$ is hard to compute as BG is an infinite complex.
Instead we can view it as $E_G^*(EG_+) = E^*(BG_+)$ in the equivariant world and we have a comparison map

$$E_G^* \longrightarrow E_G^*(EG_+)$$

Assume E_G is a module over R_G; and write

$$I = \ker\Big(\underset{\substack{\| \\ E_G^*(S^0)}}{E_G^*} \longrightarrow \underset{\substack{\| \\ E_G^*(G_+)}}{E^*}\Big)$$

We build EG_+ out of free cells G_+ and so by definition we see that $E_G^*(EG_+)$ is a colimit of I-torsion things.
The "best" colimit of I-torsion things is the I-adic completion $(-)_I^\wedge$ so it is natural to ask whether

$$E_G^*(EG_+) = (E_G^*)_I^\wedge \ ?$$

or more generally whether

$$\underbrace{E_G^*(EG_+ \wedge X)}_{\text{exact in } X} = \underbrace{E_G^*(X)_I^\wedge}_{\text{rarely exact in } X} \ ?$$

So we see that such a statement cannot be true in general.
To rectify this we can derive $(-)_I^\wedge$ to make it exact.

It turns out that the best answer one can hope for more generally is a convergent spectral sequence

$$\underbrace{H_*^I(E_G^*(X))}_{\text{GOAL: Explain this side}} \Longrightarrow E_G^*(EG_+ \wedge X).$$

§2: L_0^I - COMPLETION

Fix a commutative Noetherian ring R and an ideal $I = (x_1, \dots, x_n)$. ← not strictly necessary

Problems with $(-)_I^\wedge$:

① $(-)_I^\wedge$ is neither left nor right exact in general

② Collection of I-adically complete modules is not abelian

So we replace $(-)_I^\wedge$ with a "better" version.

Defn: We write L_n^I for the nth left derived functor of $(-)_I^\wedge$. Of know particular note is $L_0^I M := \operatorname{coker}\left((P_1)_I^\wedge \longrightarrow (P_0)_I^\wedge\right)$ for $P_\bullet$ a projective resolution of M.
We say that M is <u>L_0^I-complete</u> iff $M \xrightarrow{\cong} L_0^I M$.

Note: (i) Can take left derived functors of non-right exact F (e.g. $(-)_I^\wedge$) and all properties remain except that $L_0 F = F$.

(ii) By construction L_0^I is right exact.

Key properties [Greenlees-May]:

(1) L_0^I is idempotent (i.e. $L_0^I M \xrightarrow{\cong} L_0^I L_0^I M$)

(2) The map $M \longrightarrow M_I^\wedge$ factors as

$$M \longrightarrow L_0^I M \twoheadrightarrow M_I^\wedge$$

(3) If M is I-adically complete then M is L_0^I-complete (more generally if $M = L_0^I N$ then M is L_0^I-complete).

(4) By (2) one sees that

$$\begin{array}{c} M \text{ } I\text{-adically} \\ \text{complete} \end{array} \iff \underbrace{\begin{array}{c} M \text{ } L_0^I\text{-complete} \\ \text{and} \\ I\text{-adically separated} \end{array}}_{\substack{M \longrightarrow M_I^\wedge \\ \text{is injective}}}$$

The category $\mathrm{Mod}_R^{L_0^I}$ has a universal property:

<u>Prop</u> [Salch]: $\mathrm{Mod}_R^{L_0^I}$ is the smallest full subcategory of Mod_R which:

* contains all I-adically complete modules
* is abelian
* the inclusion is exact.

This shows that L_0^I-completion is significantly better behaved than I-adic completion, and it appears more naturally from topology.

<u>Remark:</u> M is L_0^I-complete $\iff$ $\underbrace{M \text{ is an } I\text{-contramodule}}_{\substack{\mathrm{Hom}_R(R[\frac{1}{x_i}], M) \\ \| \\ 0 \\ \| \\ \mathrm{Ext}_R^1(R[\frac{1}{x_i}], M)}}$ $\forall x_i$ (recall $I = (x_1, \dots, x_n)$)

§ DERIVED COMPLETION

It is natural to consider the replacement of $(-)_I^\wedge$ on $D(R)$, i.e, its total left derived functor $\mathbb{L}(-)_I^\wedge$ defined by

$$\mathbb{L}(M)_I^\wedge = P_I^\wedge$$

where $P \xrightarrow{\sim} M$ is a dg-projective replacement of $M \in D(R)$.

This isn't very computable or easy to work with though but we can give an alternative description.

If $x \in R$ is regular we get a diagram

$$\begin{tikzcd}
K(x) & R \arrow[r, "\cdot x"] \arrow[d, "id"'] & R \arrow[r] \arrow[d, "\cdot x"] & R/x \arrow[d] \\
& R \arrow[r, "\cdot x^2"] \arrow[d, "id"'] & R \arrow[r] \arrow[d, "\cdot x"] & R/x^2 \arrow[d] \\
& R \arrow[r, "\cdot x^3"] \arrow[d] & R \arrow[r] \arrow[d] & R/x^3 \arrow[d] \\
& \vdots & \vdots & \vdots \\
K_\infty(x) & R \arrow[r] & R\left[\frac{1}{x}\right] &
\end{tikzcd}$$

limit of this is $\hat{R}_{(x)}$

INFACT: $\hat{R}_{(x)} = \lim_i \left(R/x^i \to R/x^{i-1} \right)$

<u>DEFN</u>: (i) The unstable Koszul complex $K(I)$ is

$$K(I) = \hat{\bigotimes_{i=1}} \left(\underset{0}{R} \xrightarrow{\cdot x_i} \underset{-1}{R} \right).$$

(ii) The stable Koszul complex $K_\infty(I)$ is

$$K_\infty(I) = \hat{\bigotimes_{i=1}} \left(R \longrightarrow R\left[\frac{1}{x}\right] \right)$$

<u>Note</u>: ① Should think of $K(I)$ as a "nice" replacement for R/I: If

* if I is regular then $K(I)$ is a free resolution of R/I
* if I isn't regular, then $K(I)$ is perfect but R/I isn't and $\mathrm{Loc}(K(I)) = \mathrm{Loc}(R/I)$.

[Dwyer–Greenlees]

② $K_\infty(I)$ only depends on I up to radical.

<u>DEFN</u>: The derived completion $\Lambda_I : D(R) \longrightarrow D(R)$ is defined by $\Lambda_I M = \mathrm{Hom}_R(K_\infty(I), M)$. Say M is <u>derived complete</u> if $M \xrightarrow{\simeq} \Lambda_I M$

Two natural questions:

① How does Λ_I relate to $\mathbb{L}(-)^\wedge_I$?

② How does $\Lambda_I / \mathbb{L}(-)^\wedge_I$ relate to L^I_n (and L^I_0)?

Answer to ①:

THM [Greenlees-May, Scherzel, Porta-Shaul-Yekutieli]:
There is a natural quasi-isomorphism

$$\Lambda_I(M) \xrightarrow{\simeq} \mathbb{L}(M)^\wedge_I$$

for all $M \in D(R)$. If M is a module (i.e in a single degree) then in particular

$$H_n(\Lambda_I M) \cong L^I_n M.$$

$$\overset{\shortparallel}{H^I_n M} \longleftarrow n^{th} \text{ local homology of } M$$

Answer to ②:

THM [Greenlees-May]:
There is a strongly convergent spectral sequence

$$E^2_{p,q} = H^I_p(H_q M) \Longrightarrow H_{p+q}(\Lambda_I M).$$

Some consequences:

(i) M is derived complete $\Longleftrightarrow$ $H_n M$ is L^I_0-complete.

(ii) If M is a module, then M is L^I_0-complete $\Longleftrightarrow$ M is derived complete

(iii) M I-adically complete $\Longrightarrow$ M L^I_0-complete $\Longrightarrow$ M derived complete.

Note: This spectral sequence is really the algebraic incarnation of the "best answer" we claimed in the recap.

§ DERIVED COMPLETION VIA BOUSFIELD LOCALIZATION.

Recall that for $K \in D(R)$, the K-localization of $M \in D(R)$ is a map $M \xrightarrow{f} L_K M$ with:

① $L_K M$ K-local (i.e., if $Z \otimes K \simeq 0$ then $[Z, L_K M] = 0$)

② $M \xrightarrow{f} L_K M$ is a K-equivalence i.e. $K \otimes f$ is an equiv.

THM [Greenlees-May]:
The derived completion Λ_I is the $K(I)$-localization.

Proof: We have a map $M \longrightarrow \Lambda_I M$ induced by $K_\infty(I) \to R$ so it is enough to verify ① + ② from above.

① Suppose $Z \otimes K(I) \simeq 0$.
Then

$$[Z, \Lambda_I M] \underset{\otimes\text{-hom adjunction}}{\simeq} [K_\infty(I) \otimes_R Z, M] \underset{K_\infty(I) \text{ is a colimit of } K(I)}{\simeq} 0$$

② $$K(I) \otimes_R \mathrm{Hom}_R(K_\infty(I), M) \underset{K(I) \text{ perfect}}{\simeq} \mathrm{Hom}_R(K_\infty(I), K(I) \otimes M)$$

$$\underset{K(I) \text{ self dual}}{\simeq} \mathrm{Hom}_R(\Sigma^n K(I) \otimes_R K_\infty(I), M)$$

$$\underset{K(I) \text{ has torsion homology so } K(I) \otimes_R R[\frac{1}{x}] \simeq 0}{\simeq} \mathrm{Hom}_R(\Sigma^n K(I), M)$$

$$\underset{K(I) \text{ perfect and self-dual}}{\simeq} K(I) \otimes_R M.$$

~~Remark: Didn't actually need to use $K(I)$ self-dual. Instead can argue that~~

$$\cancel{\mathrm{Hom}_R(\check{C}_I R, K(I) \otimes_R M) \simeq *}$$

~~since $\check{C}_I R$ built out of $R[\frac{1}{x}]$ and $\mathrm{Hom}_R(R[\frac{1}{x}],$~~

Remark: Didn't actually need to use that $DK(I) \simeq \Sigma^n K(I)$. Instead one sees that

$$K_\infty(I) \otimes_R DK(I) \simeq \mathrm{Hom}_R(K(I), K_\infty(I)).$$

Now $\mathrm{Hom}_R(K(I), \check{C}_I R) \simeq 0$ where $K_\infty(I) \to R \to \check{C}_I R$ since $\check{C}_I R$ is built from $R[\frac{1}{x}]$ so result follows.

§ WHERE NEXT / WHAT IS THIS GOOD FOR?

1. Can make sense of Λ_I in other settings (e.g. Spectra, G-spectra)
2. There is a "dual" story about torsion and local cohomology which leads to Gorenstein phenomena in topology. [Dwyer-Greenlees-Iyengar]

   or even better in any triangulated category $\rightsquigarrow$ stratification á la BIK.