Descent & completion

K_G^* $\qquad K_G^0(X) = G\text{-}K(\mathbb{C}\text{-}G\text{-vector bundles}/X)$

$K_G^0(pt) = R(G)$

$$\begin{array}{ccccc} K_G^* & = & [S^0, K]^G & = & [S^0, K^G] \\ \downarrow & & \downarrow & & \downarrow \\ K_*^{hG} & = & [EG_+, K]^G & = & [S^0, K^{hG}] \end{array}$$

$$\begin{array}{c} K^G \\ \| \\ F(S^0, K)^G \\ \downarrow \\ F(EG_+, K)^G \\ \| \\ K^{hG} \end{array}$$

Calculate this

$$\begin{array}{ccccccc} * \subseteq & EG_+^{(0)} & \subseteq & EG_+^{(1)} & \subseteq & EG_+^{(2)} & \subseteq \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ & G_+ & & EG_+^{(1)}/EG_+^{(0)} & & EG_+^{(2)}/EG_+^{(1)} & \\ & & & \| & & \| & \\ & & & \bigvee_{i \in I_1} \Sigma G_+ & & \bigvee_{i \in I_2} \Sigma^2 G_+ & \end{array} \qquad (+)$$

Take the SS of this filtered complex.

Hence $E_1^{s,t} = [EG_+^{(s)}/EG_+^{(s-1)}, K]^G_{-t} \Rightarrow [EG_+, K]^G_{t-s}$

$$= \Big[\bigvee_{i \in I_s} \Sigma^s G_+, K\Big]^G_{-t}$$

$$= \mathrm{Hom}(H_s(\bigvee \Sigma^s G_+), K)_{s+t}$$

To understand E_2

$$\bigvee_{i\in I_s} \Sigma^s G_+ = EG_+^{(s)}/EG_+^{(s-1)} \xrightarrow[\text{d.}]{\text{induce}} \Sigma EG_+^{(s-1)}/EG_+^{(s-2)} = \bigvee_{i\in I_{s-1}} \Sigma^s G_+$$

$$\searrow \Sigma EG_+^{(s-1)}$$

& $[G_+, G_+]_0^G \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}G}(H_0G_+, H_0G_+)$

So just need to recognise SS of (†) in $H_*(\ )$

where it has E_1

$$\boxed{H_0(\bigvee_{i\in I_0} G_+) \leftarrow H_1(\bigvee_{i\in I_1} \Sigma G_+) \leftarrow H_2(\bigvee_{i\in I_2} \Sigma^2 G_+) \leftarrow \cdots}$$

& it collapses at E_2

free res^n of $\mathbb{Z}$

Converges to $H_*(EG_+) \cong H_0(EG_+) = \mathbb{Z}$.

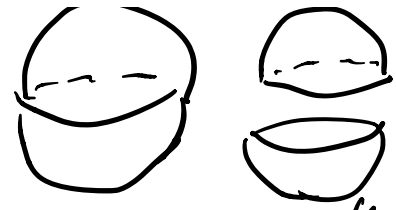
Hence $E_2^{s,t} = H^s(G; K^t) \Rightarrow \pi_{-t-s}(K^{hG})$

DESCENT SS.

<u>Example 1</u>: K = Atiyah-Segal $\mathbb{C}$-equiv K theory.

$$K_* = \mathbb{Z}[v^{\textcircled{2}}, v^{-1}]$$

& G acts trivially.



Splice two triv bundles
$S^1 \to GL_1(\mathbb{C})$

Hence

$$H^*(G;\mathbb{Z})[v,v^{-1}] \Rightarrow \pi_*(K^{hG}) = \pi_*^G(EG_+, K)$$
$$= \pi_*(BG_+, K)$$
$$= K^*(BG_+)$$

$K_G^*(pt)$

<u>Example 2</u>: K = Atiyah-Segal $\mathbb{C}$-equiv K-theory
$G = Q = Gal(\mathbb{C}|\mathbb{R})$
acting by complex conj^n.

$$H^*(Q;K^*) \Rightarrow \pi_*(K^{hQ})$$

$H^*(Q;K^*) = \mathbb{Z}[v,v^{-1}]$

$\pi_*(K^{hQ}) = \pi_*(K^Q)$ ($\mathbb{Z}/2$ in odd degrees)

$\pi_*(KO)$

Nilpotence of η

$H^*(Q;\mathbb{Z}) = \mathbb{Z}[x]/(2x)$, $\deg x = 2$

$H^*(Q;\tilde{\mathbb{Z}}) = \Sigma^{-1}(x)$

$$\boxed{0 \to \mathbb{Z} \to \mathbb{Z}Q \to \tilde{\mathbb{Z}} \to 0}$$

Here $Q = \langle g \rangle$, $gv = -v$

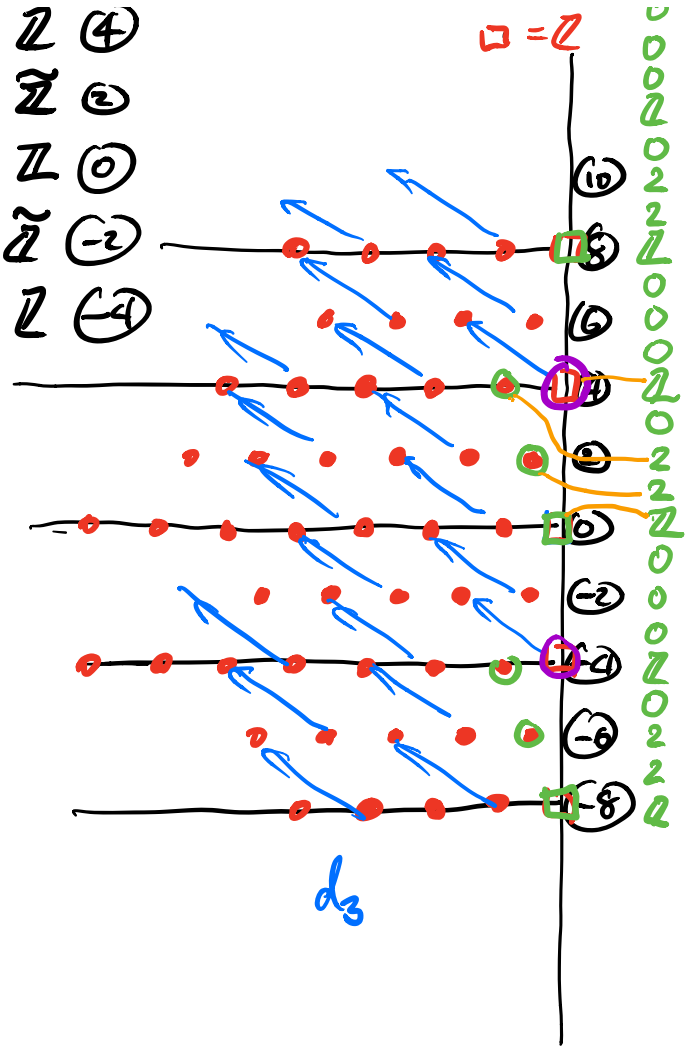
ie $\mathbb{Z}$ (8)
$\tilde{\mathbb{Z}}$ (6)

KO_*

$\bullet = \mathbb{Z}/2$

$$\mathbb{Z}[v,v^{-1}] = \begin{array}{l} \mathbb{Z}\ (4) \\ \tilde{\mathbb{Z}}\ (2) \\ \mathbb{Z}\ (0) \\ \tilde{\mathbb{Z}}\ (-2) \\ \mathbb{Z}\ (-4) \end{array}$$

$E_2^{**} =$



<u>Example 1</u> implies

$$\begin{array}{ccc} K^G & \longrightarrow & K^{hG} \\ \pi_*(K) & \longrightarrow & \pi_*(K^{hG}) \\ \| & & \| \\ [S^0, K]^G & \longrightarrow & [EG_+, K]^G \\ \| & & \| \\ K_G^*(S^0) & & K_G^*(EG_+) \end{array}$$

<u>want</u>

accessible via descent

<u>Atiyah Segal completion thm</u>:

The map induces J-adic completion

$$\begin{array}{ccc} K_G^*(S^0) & \longrightarrow & K_G^*(EG_+) \\ \| & & \wr\| \\ R(G)[v,v^{-1}] & & R(G)^\wedge_J[v,v^{-1}] \end{array}$$

where $J = \ker(R(G) \longrightarrow \mathbb{Z})$ is

<u>Example</u>: $G = C_n$. ~~[illegible]~~ $\subseteq \mathbb{C}$

acts freely on $S(\mathbb{C})$ (unit sphere)

$$\begin{array}{ccc} & & \cap\| \\ \text{—}\|\text{—} & & S(2\mathbb{C}) \\ & & \cap\| \\ \text{—}\|\text{—} & & S(3\mathbb{C}) \\ & & \cap\| \end{array}$$

Hence $EG = S(\infty\mathbb{C})$

$$EG_+ = S(\infty\mathbb{C})_+ \to S^0 \longrightarrow S^{\infty\mathbb{C}}$$

$$\begin{array}{ccccc} \uparrow & & \uparrow & & \uparrow \\ S(3\mathbb{C})_+ & \longrightarrow & S^0 & \longrightarrow & S^{3\mathbb{C}} \\ \uparrow & & \uparrow & & \uparrow \\ S(2\mathbb{C})_+ & \longrightarrow & S^0 & \longrightarrow & S^{2\mathbb{C}} \\ \uparrow & & \uparrow{=} & & \uparrow \\ S(\mathbb{C})_+ & \longrightarrow & S^0 & \longrightarrow & S^{\mathbb{C}} \end{array}$$

$$
\begin{tikzcd}
K_G^*(EG_+) \arrow[d] & K_G^*(S^0) \arrow[l] \arrow[d] & K_G^*(S^{-\infty}) \arrow[l] \arrow[d] \\
R(G)/\!/e(G)^3 \arrow[d] & R(G) \arrow[l] \arrow[d] & R(G) \arrow[l, "e(G)^3"'] \arrow[d] \\
R(G)/\!/e(G)^2 \arrow[d] & R(G) \arrow[l] \arrow[d] & R(G) \arrow[l, "e(2G)"', "e(G)^2"] \arrow[d] \\
R(G)/\!/e(G) & R(G) \arrow[l] \arrow[d, equal] & R(G) \arrow[l, "e(G)"'] \arrow[d, "e(G)"] \\
 & R(G) & R(G) \arrow[l, "e(G)"']
\end{tikzcd}
$$

$\varprojlim^1 = R(G)^\wedge_{e(G)}$

(*)

$$S(V)_+ \longrightarrow S^0 \xrightarrow{i} S^V \leftarrow \text{ex rep}^n$$

$$\downarrow K_G^0$$

$$
\begin{tikzcd}
K_G^0(S(V)_+) \arrow[d, equal] & K_G^0(S^0) \arrow[l] \arrow[d, equal] & K_G^0(S^V) \arrow[l, "i^*"'] \arrow[d, "\cong"] \\
R(G)/\!/e(V) & R(G) \arrow[l] & R(G) \arrow[l, "e(V)"']
\end{tikzcd}
$$

$\leftarrow$ Thom iso

If V is 1-dim^l $e(V) = 1 - v \in R(G)$

$*$ is obtained from the following by applying $\mathrm{Hom}_R(\underline{\ \ }, M)$ $(M = R)$

$$\Sigma^{-1} R/\!/e^\infty \simeq \Sigma^{-1} \frac{R[\frac{1}{e}]}{R} \to \boxed{R \longrightarrow R[\tfrac{1}{e}]}$$

stable Koszul complex $K_\infty(e)$

$$
\begin{tikzcd}
\Sigma^{-1}R/\!/e^3 = K(e^3) \arrow[r] & R \arrow[r, "e^3"] & R \\
\Sigma^{-1}R/\!/e^2 = K(e^2) \arrow[r] \arrow[u] & R \arrow[r, "e^2"] \arrow[u, equal] & R \arrow[u, "e"] \\
\Sigma^{-1}R/\!/e = K(e) \arrow[r] \arrow[u] & R \arrow[r, "e"] \arrow[u, equal] & R \arrow[u, "e"]
\end{tikzcd}
$$

Hence

$$M^{\wedge h}_{(e)} = \mathrm{Hom}(K_\infty(e), M)$$

is derived completion

$$\overset{0}{(R} \xrightarrow{e^n} \overset{-1}{R)}$$ Koszul complexes

$$
\begin{tikzcd}
K(e^n) \arrow[r] & R \arrow[r, "e^n"] & R \arrow[d] \\
\mathrm{Hom}(K(e^n), M) & \mathrm{Hom}(R, M) \arrow[l] & \mathrm{Hom}_R(R, M) \arrow[l, "e^n"'] \\
 & M \arrow[u, equal] & M \arrow[u, equal]
\end{tikzcd}
$$