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CASTELNUOVO-MUMFORD REGULARITY FOR PROJECTIVE CURVES

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Introduction.

"This is not Mathematics; this is Theology!"
(Gordan in reply to a Hilbert's paper)

The aim of this essay is to discuss a cohomological method for algebraic projective curves, that is grounded on a important numeric invariant, called *Castelnuovo-Mumford regularity.* In particular, we will provide a detailed proof for a regularity upper bound, providing then some examples that explain its geometric significance.

Castelnuovo-Mumford regularity is a fundamental invariant in commutative algebra and algebraic geometry. As a matter of fact, first outlines of its existence have appeared since late XIXth century, very long time before the proper and formal definition.

The starting argument in which regularity makes its appearance, either if not clearly stated, can be found in the works of Guido Castelnuovo (1865 – 1952), precisely in the celebrated paper [**5**], in which he deals with the study of certain linear series over the projective space $\mathbf{P}^3$, dedicating much of the discussion to their dimension. In particular, Castelnuovo directed his interests towards the linear series over an algebraic curve X of given degree d that are cut out by surfaces of degree f; that is, they are obtained intersecting X with degree f algebraic surfaces: such intersection clearly contains df points, but computing the dimension of the generated linear series is not an immediate task. In geometric terms, finding the dimension equals to ask how many linear conditions a curve X imposes to a surface S that contains X (*postulation* problem). Italian geometers were indeed aware of the fact that the desired dimension r_f, *for sufficiently large* f, must satisfy

$$r_f = df - g(X)$$

if X is an algebraic smooth curve in the projective space $\mathbf{P}^3$; this result was proved using techniques specialized by German geometry school. The relation tells, in other words, that the linear series cut out on X by degree f surfaces is complete and non special. Castelnuovo work aims, in this context, to derive a lower bound φ such that the previous formula holds for every $f \geq \varphi$ and, in the same time, an upper bound for the defect $fd - g(X) - r_f$, namely the error we make using the formula for some $f < \varphi$. The concept of regularity arose from the discussion of the first problem: indeed, completeness and non speciality of linear series can be translated in modern language with homological conditions concerning the surjectivity of certain arrows between cohomology modules. Moreover, as we shall see further, the fact that the formula holds *definitively* has a corresponding analogy with the modern definition, which involves algebraic sheaves.

Another early, yet muted, glimpse of regularity can be found in the revolutionary paper [**16**] by David Hilbert (1862 – 1943). The paper lays the foundation of free ideal resolutions theory, of which the Syzygy Theorem represents a remarkable example: it shows the finiteness of the free minimal resolution associated to an ideal generated by a finite number of homogeneous polynomials. Hilbert's work changed drastically the mathematics environment, promoting the gradual abandonment of outdated "constructive" methods, sometimes giving rise to heated debates. For instance, an earlier Hilbert's paper was the object of a curious anecdote. The paper solves, in a purely abstract way, a long-standing problem of invariant theory, a very diffused branch of mathematics in the second half of the XIXth century; professor Paul Gordan of Erlangen, a distinguished scholar of the time, read the article and, having found his entire scientific research summarized in a statement, wrote to Hilbert his complain, saying "this is not mathematics; it's theology!" ([**21**]).

Nevertheless, from the modern point of view, Hilbert's work is highly constructive: in fact, the Syzygy Theorem gives not only the finiteness of free resolution, but also an explicit way to compute it (see the fifth chapter of this document for more details). There was, however, a subtler question that gave rise, in the following years, to various controversies: *is it possible to bound from above the number of steps required to build up the free minimal resolution of an arbitrary homogeneous ideal?* In particular, it was interesting to know if this number could be bound with numeric characters associated to the only ideal, excluding, instead, that it could not be arbitrarily big. The affirmative answer came in 1926, thanks to the work of Grete Hermann (1901 – 1984), who proved in [**15**] that the free minimal resolution of a finitely generated homogeneous ideal can be *computed* in a (finite) number of operations, that is bounded only by characters derived from the number of indeterminates in the ambient ring and from the maximum degree of generators. It is quite surprising that Castelnuovo-Mumford regularity provides exactly that upper bound; more precisely, the regularity of a module M rules the behaviour of its Hilbert function, marking the integer from which the function equals the Hilbert polynomial. Hermann's work has been forgotten for decades, mostly because the lack of the technology necessary to carry over computations heavy enough to require a significant estimate of the necessary steps; the advent of the calculator and the development of more and more efficient computational systems have made this sort of argument a fundamental aspect, since it provides a true estimate of algorithmic complexity.

It was only in 1966 that David Mumford gave the first formally correct definition of *regularity*; inspired by the works of Castelnuovo, he proposed the concept of *m-regular sheaf in the sense of Castelnuovo*, using coherent ideal sheaves in the projective space: such a sheaf $\mathscr{I}$ over $\mathbf{P}^r_k$ is called m-regular if $H^i(\mathbf{P}^r_k, \mathscr{I}(m-i)) = 0$ for every $i > 0$ and the regularity of $\mathscr{I}$ is the minimum, if it exists, among the integers m that make $\mathscr{I}$ be a m-regular sheaf. Mumford managed to prove a first upper bound for the regularity of coherent sheaves, opening the road to a new "algorithmic approach" in the study of classical topics in algebraic geometry. Indeed, even if the original definitions rely on sheaf cohomology, there is a rather easy interpretation that uses syzygies of an ideal; for this reason, regularity plays an important role in the "classic" algebraic geometry, in which varieties are still defined explicitly by their equations.

Regularity is a fundamental concept also in commutative algebra. In 1982, Akira Ooishi defined the regularity of a graded finitely generated module by means of local cohomology, extensively introduced by Alexander Grothendieck (1928 – 2014) in the Sixties. A few years later, David Eisenbud and Shiro Goto carried out a major result, showing that the algebraic definition of regularity for a graded module over a polynomial ring is closely related to characters coming from the free minimal resolution of the module itself *(Betti numbers)*.

In the same paper, Eisenbud and Goto expressed the following conjecture: *what are the conditions on X such that the inequality* $\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim}(X) + 1$ *holds for a projective variety X?* Castelnuovo, in its 1893 paper, proved precisely that the inequality holds if X is a smooth non degenerate projective curve in $\mathbf{P}^3$; nearly a century after, in 1983, Gruson, Lazarsfeld and Peskine proved in [**17**] that the same result holds for projective irreducible curves that are not degenerate over an algebraically closed field. The result has been extended to smooth projective surfaces over a characteristic 0 field by Lazarsfeld in 1987. Nevertheless, the conjecture is still open in the other cases and makes object of a lively research.

We shall now review point to point the contents of the thesis; the work was divided in two separate parts, the first one containing all the algebraic and geometric preliminary tools required to understand the second one, in which the crucial arguments are concentrated.

First chapter is a brief *excursus* of some notable topics of homological algebra and dimension theory; depth, length of modules and Cohen-Macaulay property are the most important subjects included.

The second chapter introduces the language of algebraic sheaves, with particular respect to locally free and invertible sheaves, Weil and Cartier divisors and their expression by invertible sheaves. Some outlines of linear systems theory are given too. In the last section we introduce the concept of vector bundle, with the essential purpose to show its equivalence with the locally free sheaves language.

The following third chapter focuses deeper in the theory of algebraic sheaves, analyzing the details of *differential calculus* over algebraic varieties.

The discussion ranges from purely algebraic arguments, like Kähler relative differentials over modules and their specialization in the case of field extensions, to the application of sheaf theory in the introduction of the relative differentials sheaf. Much emphasis is given presenting the homological properties of these sheaves, in particular deriving the important *Euler sequence* 3.2. The last section has a geometric flavour and introduces the concepts of *canonical sheaf* and *genus* using the tools presented in the preceding two chapters.

In the fourth chapter we study briefly some basic properties of algebraic curves, with particular regard to the most important results in classical theory, like *Riemann-Roch theorem* for smooth curves and some properties of *degree*.

Finally, the last chapter of first part introduces us the language of commutative algebra which we will need further. The prime sections are devoted to a broad exposition of syzygies and free resolutions theory, with a pointed interest towards the graded case; the third section contains, instead, some outlines of the vast theory of *determinantal ideals*, sealed at last by *Hilbert-Burch Theorem*. The last two sections contain some simple geometric applications of the ideas exposed, showing a method to determine the free resolutions of arbitrary sets of point in the projective plane.

The second part begins with two technical chapters. The first one is dedicated to a systematic study of local cohomology and its multiple links with other cohomology theories. Over the sections we introduce many different definitions and computation methods, using extension functors, Koszul complexes or Čech cohomology. The last section collects other notable results, amongst which a vanishing theorem involving depth and dimension. The seventh chapter deals briefly with the construction of Eagon-Northcott complex, together with a quick summary of symmetric algebra properties.

The eighth chapter is one of the central parts of the essay and contains a detailed exposition of the modern theory of *Castelnuovo-Mumford regularity* for graded finitely generated modules. In particular, the first section show a characterization of regularity using local cohomology modules; the following section interpolate the previous section's result with some algebraic tools and strengthens hypotheses over the modules in order to obtain a simpler definition of regularity. The third section offers a first bound of regularity in a geometrically significant case, the arithmetically Cohen-Macaulay varieties; in the following chapter we will study the same inequality, but with coarser hypothesis. The last section brings the major notions of the chapter to the context of algebraic sheaves, bridging them with the traditional Mumford's definition of regularity. We present also a link theorem between regularity of coherent sheaves and finitely generated modules.

The last chapter is completely devoted to the full proof of *Gruson-Lazarsfeld-Peskine Theorem*, the fundamental result that proves Eisenbud-Goto conjecture in the case of projective non degenerate curves. Our proof treat only the smooth case, and a final summary of the proof is presented in the sixth section, while the previous sections compete to the reduction of the problem, using powerful tools often borrowed from other chapters; amongst them, let us name Fitting ideals, Koszul complex and Eagon-Northcott complex. Finally, in the last section some computations over significant geometric objects are carried out, in order to show the power and the precision of the above theorem.

Part 1

PRELIMINARY OUTLINES

CHAPTER 1

Homological algebra.

This chapter is exclusively devoted to recall some important homological algebra concepts and theorems that will be used through the subsequent parts of this essay.

1.1. Depth.

Let A be a noetherian ring and let M be a finitely generated A-module. A sequence of elements $a_1, \dots, a_n \in A$ is called *M-regular* if

- a_1 is not a zero-divisor of M;
- for every $i > 1$, a_i is not a zero-divisor in $M/(a_1, \dots, a_{i-1})M$.

Amongst regular sequences, those who are contained in an ideal $I \subseteq A$ such that $IM \neq M$ deserve a particular regard. Note that the noetherian condition on A is essential to guarantee the finiteness of regular sequences. We show in the next theorem how it is possible to characterize regular sequences using extension functors.

Let us recall that the *support* of a finitely generated A-module M is the set

$$\mathrm{supp}(M) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\} = V(\mathrm{Ann}_A(M))$$

Recall also that a prime ideal $\mathfrak{p}$ is an *associated prime* to M if there exist an immersion $A/\mathfrak{p} \hookrightarrow M$. The set of associated primes to M is usually called $\mathrm{Ass}_A(M)$.

THEOREM 1.1. (GROTHENDIECK) *Let A be a noetherian ring and M a finitely generated A-module. Let $I \subseteq A$ be an ideal such that $IM \neq M$ and let $n > 0$ be an integer. Then the following statements are equivalent:*

1. $\mathrm{Ext}^k_A(N, M) = 0$ *for every $k < n$ and for every finitely generated A-module N such that* $\mathrm{supp}(N) \subseteq V(I)$*;*
2. $\mathrm{Ext}^k_A(A/I, M) = 0$ *for every $k < n$;*
3. *there exists a finitely generated A-module N such that* $\mathrm{supp}(N) = V(I)$ *and* $\mathrm{Ext}^k_A(N, M) = 0$ *for every $k < n$;*
4. *there exists a finite M-regular sequence $a_1, \dots, a_n$ in I.*

PROOF. See [**9, 20**]. □

In particular, the Theorem assures that the following definition is well posed.

DEFINITION 1.1. The *depth* $\operatorname{depth}(I, M)$ of an A-module M respect to an ideal $I \subseteq A$ is defined as following:

- if $IM \neq M$, then $\operatorname{depth}(I, M)$ is the maximal length of a M-regular sequence contained in I;
- if $IM = M$, then $\operatorname{depth}(I, M) = \infty$.

If A is a local ring, we will write simply $\operatorname{depth}(M)$ to denote the number $\operatorname{depth}(\mathfrak{m}, M)$, being $\mathfrak{m}$ the maximal ideal in A. Here we show some notable properties of depth.

PROPOSITION 1.1. *Let A be a noetherian ring:*

(1) *if $I \subseteq A$ is an ideal and $M = 0$, then* $\operatorname{depth}(I, M) = 0$*;*
(2) *if A is a local ring,* $\operatorname{depth}(M) = 0$ *if and only if* $\mathfrak{m} \in \operatorname{Ass}_A(M)$*;*
(3) *if* $\mathfrak{p} \in \operatorname{Spec}(A)$*, then* $\operatorname{depth}(M_\mathfrak{p}) = 0$ *if and only if* $\mathfrak{p} \in \operatorname{Ass}_A(M)$*;*
(4) *if* $\mathfrak{p} \in \operatorname{Ass}_A(M)$*, then* $\operatorname{depth}(\mathfrak{p}, M) = 0$*;*
(5) $\operatorname{depth}(M_\mathfrak{p}) \geq \operatorname{depth}(\mathfrak{p}, M)$ *for every prime ideal* $\mathfrak{p} \subseteq A$*.*

PROOF. (Omitted) □

There exist remarkable relations that link depth, projective dimension and Krull dimension.

PROPOSITION 1.2. *Let A be a local ring and M a finitely generated A-module. Then*

$$\operatorname{depth}(M) \leq \dim(A/\mathfrak{p})$$

for every associated prime $\mathfrak{p} \in \operatorname{Ass}_A(M)$*.*

PROOF. See [**20**]. □

The most important formula, however, is stated in the following theorem.

THEOREM 1.2. (AUSLANDER-BUCHSBAUM FORMULA) *Let A be a local ring and M a finitely generated A-module having finite projective dimension. Then*

$$\operatorname{depth}(A) = \operatorname{depth}(M) + \dim^{(\mathrm{proj})}(M)$$

PROOF. See [**2, 9, 20**]. □

The greatest part of the arguments presented for local rings can be rearranged to hold in the context of graded modules over graded rings. Let $S = \bigoplus S_j$ be a graded ring such that S_0 is a field and such that S acts as a finitely generated S_0-algebra. We set

$$\mathfrak{m} := \bigoplus_{j \geq 1} S_j$$

the *irrelevant ideal.* It is a maximal ideal in S, and many of its properties can be paired with the properties of the maximal ideal in local rings. A notable result is the following.

COROLLARY 1.1. *Let M be a graded S-module, with the above notations. If M has finite projective dimension, therefore*

$$\mathrm{depth}(\mathfrak{m}, S) = \mathrm{depth}(\mathfrak{m}, M) + \dim^{(\mathrm{proj})}(M)$$

A detailed exposition about homological methods for graded modules and rings can be found in [**13**].

DEFINITION 1.2. A local noetherian ring $(A, \mathfrak{m})$ is a *regular local ring* if $\mathfrak{m}$ can be generated by exactly $\dim(A)$ elements.

An equivalent definition can be arranged considering the vector space $\mathfrak{m}/\mathfrak{m}^2$ over the residue field $k = A/\mathfrak{m}$. Nakayama's Lemma implies that A is a regular local ring if $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(A)$.

1.2. Cohen-Macaulay rings.

Let A be a ring and $I \subseteq A$ an ideal. The *dimension* of I is defined as

$$\dim(I) := \dim(A/I)$$

If M is an A-module, we set also

$$\dim_A(M) := \dim(A/\mathrm{Ann}_A(M))$$

where $\mathrm{Ann}_A(M) := \{a \in A \mid am = 0 \text{ for some } m \in M\}$ is the annihilator ideal.

REMARK 1.1. If $M = I$, that is we think I endowed with its A-module structure, we must state clearly the category in which we take dimension. If indeed A is an integral domain, therefore $\mathrm{Ann}_A(I) = (0)$, so $\dim_A(I) = \dim(A)$ as A-module, but $\dim(I) = \dim(A/I)$ as ideal, and this in general differs from $\dim(A)$. For such reasons, to avoid any ambiguity, we will always write $\dim(I)$ for the *ideal dimension* of I and $\dim_A(I)$ for the *A-module dimension* of I.

In the special case A is an integral domain acting as finitely generated k-algebra over a field k, then for every ideal $I \subseteq A$ the formula $\dim(R/I) = \dim(R) - \dim(I)$ holds.

In the previous section, we remarked that, if A is a local ring, then we have

$$\mathrm{depth}(M) \leq \dim(A_\mathfrak{m})$$

In the case A is a regular local ring, we know that any set of generators for the maximal ideal $\mathfrak{m}$ defines a $\mathfrak{m}$-regular sequence of maximal length, namely $\mathrm{depth}(A) = \dim(A_\mathfrak{m})$. Indeed, this property holds in a more general context.

LEMMA 1.1. *Let A be a ring such that* $\mathrm{depth}(A) = \dim(A_\mathfrak{m})$ *for every maximal ideal $\mathfrak{m} \subseteq A$. Therefore, for every proper ideal $I \subseteq A$, we have* $\mathrm{depth}(I, A) = \min\{\dim(A_\mathfrak{p}) : \mathfrak{p} \in \mathrm{Spec}(A/I)\}$.

PROOF. See [**9**]. □

DEFINITION 1.3. A ring A such that $\operatorname{depth}(\mathfrak{m}, A) = \dim(A_\mathfrak{m})$ holds for every maximal ideal $\mathfrak{m} \subseteq A$ is called *Cohen-Macaulay ring.*

Amongst Cohen-Macaulay rings, a significant position is owned by regular local rings. Here we show some of the most important properties of Cohen-Macaulay rings.

PROPOSITION 1.3. *Let A be a ring. The following statements are equivalent:*

(1) *A is a Coehn-Macaulay ring;*
(2) *$A_\mathfrak{p}$ is a Cohen-Macaulay ring for every prime ideal $\mathfrak{p} \subseteq A$;*
(3) *$A_\mathfrak{m}$ is a Cohen-Macaulay ring for every maximal ideal $\mathfrak{m} \subseteq A$..*

PROOF. Let A be a Cohen-Macaulay ring and let be $\mathfrak{p} \subseteq A$ a prime ideal. Then, calling $m_\mathfrak{p}$ the maximal ideal in $A_\mathfrak{p}$,

$$\dim((A_\mathfrak{p})_{m_\mathfrak{p}}) = \dim(A_\mathfrak{p}) = \operatorname{depth}(\mathfrak{p}, A) \leq \operatorname{depth}(m_\mathfrak{p}, A_\mathfrak{p}) \leq \dim((A_\mathfrak{p})_{m_\mathfrak{p}})$$

Therefore $A_\mathfrak{p}$ is a Cohen-Macaulay ring. Property (3) follows straightforwardly. Let us finally assume that every localization $A_\mathfrak{m}$ is a Cohen-Macaulay ring for every maximal ideal $\mathfrak{m} \subseteq A$. Then

$$\operatorname{depth}(m_\mathfrak{m}, A_\mathfrak{m}) = \operatorname{depth}(\mathfrak{m}, A)$$

Since we have $\dim((A_\mathfrak{m})_{m_\mathfrak{m}}) = \dim(A_\mathfrak{m})$, hence we prove that A is a Cohen-Macaulay ring. □

The next result shows a peculiar characterization of Cohen-Macaulay rings that uses the associated polynomial rings; the non trivial proof can be found [**9**], together with more detailed aspects of the question.

PROPOSITION 1.4. *A ring A is Cohen-Macaulay if and only if $A[x]$ is Cohen-Macaulay.*

1.3. Length of modules.

Let A be a ring and M an A-module. A chain of submodules in M, namely

$$N_0 \subsetneq N_1 \subsetneq \ldots \subsetneq N_r$$

is said to have *length* r.

DEFINITION 1.4. Let A be a ring and M an A-module. The *length* of M is the supremum $\operatorname{length}(M)$ amongst the lengths of every chain of submodules in M.

Length measures the "size" of M in the same way Krull dimension does for rings. Nevertheless, length and Krull dimension of modules usually do not coincide. For example, let $M = k^n$ be a vector space, so that

$$\dim_k(M) = \dim(k/\operatorname{Ann}_k(M)) = \dim(k) = 0$$

and this generally differs from $\operatorname{length}(M) = n$.

The following theorems characterize rings and modules of finite length. Let us recall that every artinian ring is noetherian, but noetherianity does not imply artinianity.

PROPOSITION 1.5. *Let A be a ring. The following statements are equivalent:*

(1) *A is a noetherian ring and every prime ideal in A is maximal;*
(2) *A is an A-module of finite length;*
(3) *A is an artinian ring.*

If, moreover, any on the above three conditions holds, A is a semi-local ring (namely, has a finite number of maximal ideals).

This proposition leads also to an interesting geometric argument, allowing us to characterize zero-dimensional varieties.

COROLLARY 1.2. *Let X be an algebraic set over an algebraically closed field k. The following statements are equivalent:*

(1) *X is a finite set;*
(2) *the coordinate ring $A(X)$ is a vector k-space having finite dimension that equals exactly the cardinality of X;*
(3) *$A(X)$ is an artinian ring.*

The following result deals with modules of finite length and offers their characterization together with an useful relation between length and Krull dimension.

THEOREM 1.3. *Let A be a noetherian ring and let M be a finitely generated A-module. The following statements are equivalent:*

(1) *M has finite length;*
(2) $\dim_A(M) = 0$*;*
(3) *every prime ideal containing* $\mathrm{Ann}_A(M)$ *is maximal;*
(4) $A/\mathrm{Ann}_A(M)$ *is an artinian ring.*

More details of the facts exposed above can be found in [**9**], supplied with the according proofs.

CHAPTER 2

Sheaves, divisors and vector bundles.

In this chapter some basic facts of algebraic geometry are proposed, with the main purpose of fixing notations and terminology for the next chapters.

Let us recall that a scheme X is called *noetherian* if there exists an affine finite open covering $\{\mathrm{Spec}(A_i)\}_{i=1}^n$ such that every A_i is a noetherian ring. Moreover, in order to avoid pathological behaviours, unless we state differently, we will always assume that every scheme X is *separated*, namely a scheme such that the diagonal map $X \longrightarrow X \times X$ is a closed immersion. More detailed information and facts about scheme theory can be retrieved in [**11, 12, 18**].

2.1. Locally free sheaves.

Let $X = (X, \mathscr{O}_X)$ be a scheme and let be $\mathscr{F}$ an algebraic sheaf over X. We recall the following definitions.

(S1) $\mathscr{F}$ is a *quasi-coherent sheaf* if there exists an open covering $\{U_i\}_{i\in I}$ of X such that every U_i is an affine open set in X and $\mathscr{F}|_{U_i} \simeq \widetilde{M}_i$ holds for some $\mathscr{O}_X(U_i)$-module M_i;

(S2) if X is a noetherian scheme, $\mathscr{F}$ is a *coherent sheaf* if it is quasi-coherent and M_i are finitely generated modules.

Definitions **(S1)** and **(S2)** could be also expressed in a more general form without the noetherian hypothesis on X; since we will take into account only sheaves over noetherian schemes, no such degree of generality is required.

In the theory of algebraic sheaves the following theorem has great significance.

Theorem 2.1. (Serre) *Let k be a field, X a projective k-scheme and $\mathscr{F}$ an algebraic coherent sheaf over X. Therefore*

(1) *$H^p(X, \mathscr{F})$ is a finitely generated vector k-space for every $p \geq 0$;*
(2) *there exists an integer $n_0 > 0$ such that $H^p(X, \mathscr{F}(n)) = 0$ for every $p > 0$ and for every $n \geq n_0$.*

Proof. See [**11, 12, 18**]. □

Let us recall the following definition.

Definition 2.1. Let X be a (noetherian) scheme and $\mathscr{F}$ an algebraic sheaf over X. We say that $\mathscr{F}$ is a *locally free sheaf* if there exists an open covering $\{U_i\}_{i\in\Lambda}$ of X such that

$$\mathscr{F}|_{U_i} \simeq \bigoplus_{i\in\Lambda} \mathscr{O}_{U_i}$$

REMARK 2.1. It is worthy to remark that locally free sheaves are also quasi-coherent; if moreover Λ is a finite set, they are also coherent. If X is a connected scheme and Λ is a finite set, the concept of *rank of a sheaf* can be defined: it is indeed the integer $r > 0$ such that $\mathscr{F}|_{U_i} \simeq \mathscr{O}^r_{U_i}$. So, in the coherent case, locally free sheaves have a well defined rank.

PROPOSITION 2.1. *A coherent algebraic sheaf $\mathscr{F}$ over a scheme X is locally free if and only if the stalk $\mathscr{F}_p$ is a free $\mathscr{O}_{X,p}$-module for every $p \in X$.*

PROOF. Let $\mathscr{F}$ be locally free; then there exists a suitable open covering $\{U_i\}_{i\in I}$ of X such that isomorphisms $\mathscr{F}|_{U_i} \simeq \mathscr{O}^r_{U_i}$ are induced. Hence, the isomorphisms $\mathscr{F}_p \simeq (\mathscr{O}_{X,p})^r$ are induced in a natural way.

Conversely, let us assume $\mathscr{F}$ has free stalks and let us reduce to consider an affine open set $U = \operatorname{Spec}(A) \subseteq X$, where A is a noetherian ring, such that $\mathscr{F}|_U \simeq \widetilde{M}$. The sheaf $\widetilde{M}$ is locally free if and only if M is a projective A-module, namely if and only if the localization $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module for every $\mathfrak{p} \in \operatorname{Spec}(A)$. Calling $p \in \operatorname{Spec}(A)$ the schematic point corresponding to the prime ideal $\mathfrak{p}$, therefore $\mathscr{O}_{X,p} \simeq A_\mathfrak{p}$ holds, forcing $\mathscr{F}_p \simeq M_\mathfrak{p}$ and this concludes the proof. □

Amongst the locally free sheaves with finite rank, those having unitary rank deserve major attention.

DEFINITION 2.2. An algebraic sheaf $\mathscr{F}$ over a scheme X is said an *invertible sheaf* if it is a locally free sheaf of rank 1.

It is moreover practicable to arrange isomorphism classes of invertible sheaves in an abelian group $\operatorname{Pic}(X)$, commonly called *Picard group*, with the operation of tensor product. When we deal with integral k-schemes of finite type (essentially, they are algebraic varieties over a field k), there is an alternative description of the Picard group that involves Cartier divisor; the latter has the advantage of having a geometric meaning.

2.2. Divisors and linear systems.

Let X be a noetherian integral scheme, such that it is regular in codimension 1.

DEFINITION 2.3. A *prime divisor* in X is an integral, closed subscheme of codimension 1 in X. A *Weil divisor* is an element of the free abelian group $\mathbb{Z}$-generated over the set of prime divisors in X. We will call $\operatorname{Div}(X)$ the group of Weil divisors.

More explicitly, a Weil divisor $D \in \mathrm{Div}(X)$ owns a representation of the following form:

$$D := \sum_j n_j Y_j$$

where $n_j \in \mathbb{Z}$, the Y_j are prime divisors in X and the sum is finite. When also $n_j \geq 0$ for every j, the divisor D is called *effective*.

Let now $K := \mathrm{Quot}(\mathscr{O}_{X,\eta})$ be the field of rational functions over X, where η is the generic point. It can be proved that, for every rational non-zero function $f \in K^* := K \backslash \{0\}$, the following divisor is well defined:

$$(f) := \sum_Y \nu_Y(f) Y$$

where $\nu_Y : K^* \longrightarrow \mathbb{Z}$ is the discrete valuation associated to the ring $\mathscr{O}_{Y,\eta}$ (note that the codimension 1 regularity implies that $\mathscr{O}_{Y,\eta}$ is a discrete valuation ring ring), and Y varies amongst all the prime divisors in X. Divisors of the above form are called *principal divisors* and they form a subgroup $\mathrm{Prin}(X)$.

DEFINITION 2.4. Two divisors D, D' are said to be *linearly equivalent* if $D - D' \in \mathrm{Prin}(X)$. The factor group $\mathrm{Cl(X)} := \mathrm{Div}(X)/\mathrm{Prin}(X)$ is then called *class group* of X.

The following result introduces the notion of *degree* in divisors theory.

THEOREM 2.2. *Let k be a field and $\mathbf{P}^r := \mathbf{P}^r_k$ a (schematic) projective space. Let also $D = \sum n_j Y_j \in \mathrm{Div}(\mathbf{P}^r)$ be a generic divisor, where $Y_j = V(f_j^{m_j})$ is a projective hypersurface of degree m_j. Let us define the degree*

$$\deg(D) := \sum_j n_j m_j$$

and let be H an hyperplane in $\mathbf{P}^r$.

(1) *If D has degree d, then D is linearly equivalent to $d \cdot H$.*
(2) *For every $f \in K^*$ we have* $\deg(f) = 0$.
(3) *The induced morphism* $\deg : \mathrm{Div}(\mathbf{P}^r) \longrightarrow \mathbb{Z}$ *quotients to an isomorphism* $\mathrm{Cl}(\mathbf{P}^r) \simeq \mathbb{Z}$.

PROOF. [**18, 24**]. □

The main concern about Weil divisors is that the major part of the interesting results holds only under very restrictive hypotheses regarding the regularity of the underlying scheme. In future, we would like to use divisors on arbitrary schemes.

Let us recall the construction of the *rational functions sheaf.* Let X be a scheme and $U = \mathrm{Spec}(A_U)$ an open affine set. Let also S_U be the set of elements in A_U that are not zero divisors; since S_U forms a multiplicative subset, we can define $K_U := S_U^{-1} A_U$, namely the total quotient ring of A_U. Thus, for every $U = \mathrm{Spec}(A_U)$, setting

$$U \mapsto S_U^{-1} A_U$$

gives a presheaf of rings over X, whose associated sheaf $\mathscr{K}$ is the desired sheaf of rational functions over X. Let us remark that, when X is an integral scheme, the sheaf $\mathscr{K}$ is just the constant sheaf that for every open U gives the rational function field $K = \mathrm{Quot}(\mathscr{O}_{X,\eta})$, being $\eta \in X$ the generic point.

REMARK 2.2. Clearly, $\mathscr{K}^*$ e $\mathscr{O}_X^*$ will respectively indicate the sheaves of non-vanishing rational and regular functions over X; moreover, there is a trivial sheaf immersion of $\mathscr{O}_X^*$ into $\mathscr{K}^*$.

DEFINITION 2.5. A *Cartier divisor* over X is a global section of the quotient sheaf $\mathscr{K}^*/\mathscr{O}_X^*$.

To derive an explicit description of Cartier divisors, let us recall the following exact sequence:

$$0 \longrightarrow \mathscr{O}_X^* \longrightarrow \mathscr{K}^* \longrightarrow \mathscr{K}^*/\mathscr{O}_X^* \longrightarrow 0 \tag{2.2.1}$$

A global section $D \in \Gamma(X, \mathscr{K}^*/\mathscr{O}_X^*)$ is thus represented by an open covering $\{U_i\}_{i\in I}$ of X together with a collection of rational functions $f_i \in \mathscr{K}^*(U_i)$ for every $i \in I$; these must be chosen such that the exactness is respected, namely $f_i f_j^{-1} \in \mathscr{O}_X^*(U_i \cap U_j)$ for every pair of indices $i, j \in I$. For the sake of brevity, we will write $\{(U_i, f_i)\}_{i\in I}$ to mean such a representation of a Cartier divisor over X.

DEFINITION 2.6. A Cartier divisor D is called *principal* if it belongs to the image of the natural map $\mathscr{K}^*(X) \longrightarrow \mathscr{K}^*/\mathscr{O}_X^*(X)$, namely if there exists a global section $f \in \mathscr{K}^*(X)$ such that the system $\{(X, f)\} =: (X, f)$ represents D.

We write $\mathrm{Cart}(X)$ to denote the Cartier divisors group: one could note that, even if $\Gamma(X, \mathscr{K}^*/\mathscr{O}_X^*)$ is a multiplicative group, Cartier divisors are usually written in additive notations, echoing the language of Weil divisors. Principal divisors form themselves a subgroup of $\mathrm{Cart}(X)$ that will be written as $\mathrm{Pr}(X)$.

DEFINITION 2.7. Two divisors D_1, D_2 are said to be *linearly equivalent* if $D_1 - D_2 \in \mathrm{Pr}(X)$.

The following theorem establishes the link between Cartier and Weil divisors.

THEOREM 2.3. *Let X be a noetherian, separated, integral and locally factorial scheme (namely, such that every local ring of X is an unique factorization domain). Then* $\mathrm{Div}(X) \simeq \mathrm{Cart}(X)$*. Moreover, this isomorphism sends principal divisors in principal divisors.*

PROOF. See [**18**]. □

With the hypotheses of Theorem 2.3, the isomorphism descend to the quotient and induces thus $\mathrm{Cl}(X) \simeq \mathrm{Cart}(X)/\mathrm{Pr}(X)$. The latter factor group contains formally Cartier divisors modulo linear equivalence, but it can also be understood as group of isomorphism classes of invertible sheaves. One could prove, in fact, that

$$\mathrm{Pic}(X) \simeq H^1(X, \mathscr{O}_X)$$

so from exact sequence (2.2.1) it follows $\mathrm{Pic}(X) \simeq \mathrm{Cart}(X)/\mathrm{Pr}(X)$. The isomorphism can be defined explicitly too, at least in the case X is an integral scheme. Let $D = \{(U_i, f_i)\}_{i \in I}$ be a Cartier divisor over an integral scheme X. It is immediate to see that D defines a system of transition functions, defined as

$$\varphi_{ij} = f_i f_j^{-1}$$

(namely, the multiplication for the element $f_i f_j^{-1} \in \Gamma(U_i \cap U_j, \mathscr{O}_X^*)$). In fact, we have that $\varphi_{ii} = 1$ and

$$\varphi_{ik} = f_i f_k^{-1} = f_i f_j^{-1} f_j f_k^{-1} = \varphi_{ij} \circ \varphi_{jk}$$

on the open set $U_i \cap U_j \cap U_k$. Gluing these data, there exists an unique sheaf $\mathscr{O}_X(D)$ that is an invertible sheaf: for every $i \in I$ the isomorphism φ_i is the multiplication by the non-zero function f_i and explicitly

$$\Gamma(U_i, \mathscr{O}_X(D)) = f_i^{-1} \cdot \Gamma(U_i, \mathscr{O}_X)$$

In fact, if $\sigma_i \in \Gamma(U_i, \mathscr{O}_X(D))$ and $\sigma_j \in \Gamma(U_j, \mathscr{O}_X(D))$, therefore $\varphi_i(\sigma_i) = f_i \sigma_i$ and $\varphi_j(\sigma_j) = f_j \sigma_j$. In particular $\sigma_j = f_i f_j^{-1} \sigma_i = \varphi_{ij}(\sigma_i)$. For these reasons, we will often use the language of divisors and invertible sheaves in interchangeable way, when the context request one of them.

The concepts of *effective divisor* and *linear system* (or, formerly, *linear series*) are heritage of classical algebraic geometry.

Definition 2.8. A Cartier divisor D over a k-scheme X is *effective* if there exists a representing family $\{(U_i, f_i)\}_{i \in I}$ such that $f_i \in \mathscr{O}_X^*(U_i)$.

One can show that the set of all effective divisors that belong to the same linear equivalence class $[\mathscr{L}]$ is in one-to-one correspondence with the set $\Gamma(X, \mathscr{L})^*/k^*$ so own the structure of a projective space. This motivates the following definition.

Definition 2.9. A *complete linear system* (also called *linear series*) $\mathfrak{L}$ over X is the set of divisors associated to global sections $\Gamma(X, \mathscr{L}) \backslash \{0\}$, where $\mathscr{L}$ is an invertible sheaf over X. By *linear system* $|V|$ in general we mean a linear subspace of a complete linear system.

If $D = \{(U_i, f_i)\}_{i \in I}$ is an effective divisor over X, it is possible to associate a closed subscheme of X to D, and it is called *subscheme associated to* D. This is built from its own ideal sheaf $\mathscr{I}_D$: it is in fact enough to give

$$\Gamma(U_i, \mathscr{I}_D) := f_i \cdot \Gamma(U_i, \mathscr{O}_X)$$

In other words, one can establish an one-to-one correspondence between Cartier effective divisors and ideal sheaves over X that are locally principal (namely, locally generated by a single element). In a more geometric flavour, an effective divisor is set in correspondence with a 1-dimensional closed subscheme that is locally principal, namely locally defined by a single equation.

Definition 2.10. A sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is called *generated* in a point $x \in X$ if there exists a family of sections $\{s_i\}_{i\in I} \subseteq \Gamma(X, \mathscr{F})$ such that the corresponding germs $(s_i)_x$ are generators for the stalk $\mathscr{F}_x$. One says that $\mathscr{F}$ is *globally generated* if it is generated in every point of X.

In the special case $\mathscr{F}$ is an invertible sheaf, the definition can be also stated asking that there exists $\sigma \in \Gamma(X, \mathscr{F})$ such that $\sigma(x) \neq 0$. In general, every quasi-coherent sheaf is globally generated.

A remarkable example of globally generated sheaf is the twisted sheaf $\mathscr{O}_{\mathbf{P}^r_k}(1)$, called sheaf of *linear forms* over $\mathbf{P}^r_k$: one can consider homogeneous coordinates $X_0, \dots, X_r$ as global sections in $\Gamma(X, \mathscr{O}_{\mathbf{P}^r}(1)) = k[X_0, \dots, X_r]_{\mathrm{h},1}$. Clearly, for every point $x \in \mathbf{P}^r_k$ there exists an index $j \in \{0, \dots, r\}$ such that $X_j(x) \neq 0$.

This peculiar generation property of the linear form sheaf allows us to show a convenient way to define morphisms of the form $f : X \longrightarrow \mathbf{P}^r_k$, where X is a k-variety over an algebraic closed field. Let $\mathscr{L} := f^*\mathscr{O}_{\mathbf{P}^n}(1)$ be the inverse image sheaf and let $\sigma_i := f^*X_i$ be for every $i = 0, \dots, r$. Therefore $\mathscr{L}$ is globally generated by sections $\sigma_0, \dots, \sigma_r$: for every $x \in X$ there exists $j \in \{0, \dots, r\}$ such that $X_j(f(x)) \neq 0$, that is $\sigma_j(x) = f^*X_j(x) \neq 0$. We state, moreover, that the morphism f is uniquely determined by the pair $(\mathscr{L}, \{\sigma_0, \dots, \sigma_r\})$. If, indeed, we fix an invertible sheaf $\mathscr{L}$ such that it is globally generated by its section $s_0, \dots, s_r$ over X, thus setting

$$f(x) := [s_0(x) : \dots : s_r(x)]$$

defines a morphism $f : X \longrightarrow \mathbf{P}^r_k$. It is to be remarked that f is *non degenerate* (namely. $f(X)$ not contained in any hyperplane) if and only if the sections s_j are linearly independent.

The language of linear systems allows to express intrinsically the previous concepts, getting rid of projective coordinates. Let $f : X \longrightarrow \mathbf{P}^r$ be a non degenerate morphism; we know so that f is uniquely determined by the invertible sheaf $\mathscr{L} = f^*\mathscr{O}_{\mathbf{P}^r}(1)$ and by the global sections $\sigma_i = f^*X_i$ for $i = 0, \dots, r$. The linear map f^* acts over the global section of $\mathscr{L}$ taking every homogeneous polynomial $H = a_0X_0 + \dots + a_rX_r$ in the section $f^*H = a_0\sigma_0 + \dots + a_r\sigma_r$; in other words, f^* induces a map from the linear system $|\mathscr{O}_{\mathbf{P}^r}(1)|$ to $|V|$, being $V = \mathrm{im}(f^*)$. The subspace V is then identified to $\Gamma(\mathbf{P}^r, \mathscr{O}_{\mathbf{P}^r}(1))$, hence $\mathbf{P}^r$ identifies with the dual projective space $\mathbf{P}(V^\vee) = \Gamma(\mathbf{P}^r, \mathscr{O}_{\mathbf{P}^r}(1))^\vee$. One then expects to translate f as a morphism to $\mathbf{P}(V^\vee)$: in fact, it suffices to set

$$f(x) := H_x = \{\sigma \in V \mid \sigma(x) = 0\}$$

and it is clear that H_x is a hyperplane, an element in $\mathbf{P}(V^\vee)$. To see this, it is enough to understand that the natural morphism $V \longrightarrow \Gamma(X, \mathscr{L} \otimes \mathscr{O}_{X,x}/\mathfrak{m}_x)$ acting as $\sigma \mapsto \sigma \otimes \sigma(x)$ has kernel H_x. The significance of this formulation resides in having removed every coordinate reference in the definition of f.

2.3. Vector bundles.

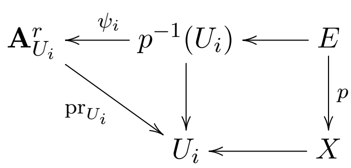
The powerful language of locally free sheaves has a corresponding geometric counterpart in the notion of *vector bundle*. Let X be a k-scheme, where k is a fixed field. In the following, we will omit specifying that fiber product is taken over $\mathrm{Spec}(k)$. We set, therefore

$$\mathbf{A}^r_X := X \times \mathbf{A}^r_k$$

and the following projection maps are defined: $\mathrm{pr}_X : \mathbf{A}^r_X \longrightarrow X$ and $\mathrm{pr}_{\mathbf{A}^r_k} : \mathbf{A}^r_X \longrightarrow \mathbf{A}^r_k$.

DEFINITION 2.11. A pair (E, p) where E is a k-scheme and $p : E \longrightarrow X$ is a morphism is called *vector bundle* of rank r if there exists an open covering $\mathfrak{U} = \{U_i\}_{i\in I}$ of X such that every U_i is an affine open set and

(1) for every $U_i \in \mathfrak{U}$ there exists an isomorphism $\psi_i : p^{-1}(U_i) \longrightarrow \mathbf{A}^r_{U_i}$ such that the following diagram commutes:



namely $\mathrm{pr}_{U_i} \circ \psi_i = p|_{p^{-1}(U_i)}$;

(2) for every $i, j \in I$, the map $\psi_{ij} = \psi_i \circ \psi_j^{-1} : \mathbf{A}^r_{U_i\cap U_j} \longrightarrow \mathbf{A}^r_{U_i\cap U_j}$ acts linearly on the fibers. That is to say, writing $\mathbf{A}^r_{U_i\cap U_j} = \mathrm{Spec}(A[x_1, \ldots, x_n])$, the morphism ψ_{ij} descends from an A-linear automorphism of $A[x_1, \ldots, x_n]$.

The open sets which belongs to $\mathfrak{U}$ are called *trivializations*.

DEFINITION 2.12. Let (E, p) and (F, q) be two vector bundles. An *isomorphism* of vector bundles is given by a scheme isomorphism $g : E \longrightarrow F$ such that $p = q \circ g$.

For every scheme morphism $f : X \longrightarrow Y$, a *section* of f over an open set $U \subseteq Y$ is a morphism $s : U \longrightarrow X$ such that $f \circ s = 1_U$. One can easily argue that, assigning every open set $U \subseteq Y$ the set $\mathscr{S}_f(U)$ made by sections of f over U, defines a presheaf of sets over Y. Moreover, if $\{U_j\}_{j\in I}$ is an open covering of Y and if $s_j \in \mathscr{S}_f(U_j)$ are sections such that, for every $i, j \in I$, the following property holds:

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$$

Hence we can define a glued section $s : Y \longrightarrow X$ setting $s(p) = s_i(p)$ if $p \in U_i$. This morphism is well defined thanks to the gluing properties and it is also a section of f over Y: in fact, if $p \in U_i$ for some $i \in I$, we have

$$f(s(p)) = f(s_i(p)) = p$$

Therefore $\mathscr{S}_f$ is a sheaf of sets over Y.

In particular, if (E, p) is a rank n vector bundle over X, let us call $\mathscr{S}_E$ the sheaf of sections of p over the open sets of X. One can then prove that $\mathscr{S}_E$ inherits a natural structure of $\mathscr{O}_X$-module and, further, it is a rank n locally free sheaf. Let us recall the following general statement:

THEOREM 2.4. *Let $(X, \mathscr{O}_X)$ be a scheme and $Y = \operatorname{Spec}(A)$ an affine scheme. Therefore, the natural map*

$$\begin{aligned} \Phi : \operatorname{Hom}_{\mathbf{Sch}}(X, Y) &\longrightarrow \operatorname{Hom}_{\mathbf{CommR}}(A, \mathscr{O}_X(X)) \\ (f, f^\sharp) &\mapsto f^\sharp_Y \end{aligned}$$

is a bijection.

PROOF. See [**12**]. $\square$

Tu set in an $\mathscr{O}_X$-module structure over the sheaf $\mathscr{S}_E$ it suffices to define such a structure in a trivialization $U \subseteq X$, that is, an open set such that $p^{-1}(U) \simeq \mathbf{A}^r_U$. So let us assume that, without loss of generality, $X = \operatorname{Spec}(A)$ and $E = \mathbf{A}^r_Y$. Thus, using Theorem 2.4, a section $E \longrightarrow A$ corresponds to an A-algebra morphism $A[x_1, \dots, x_n] \longrightarrow \mathscr{O}_X(X) = A$. In other words,

$$\mathscr{S}_E(\operatorname{Spec}(A)) = \operatorname{Hom}_A(A[x_1, \dots, x_1], A)$$

and this has a natural A-module structure. In the general situation, the $\mathscr{O}_X$-module structure of $\mathscr{S}_E$ can be retrieved from a suitable affine open covering made of trivializations. Moreover, let us remark that $\mathscr{S}_E$ is a rank n locally free sheaf: taking the same affine open cover made of trivializations $\{U_j\}_{j \in I}$, let $j \in I$ be a fixed index and let us consider the following sections:

$$\begin{aligned} \varepsilon_i : U_j &\longrightarrow \mathbf{A}^n_{U_j} \\ p &\mapsto (p, e_i) \end{aligned}$$

where e_i is the i-th coordinate corresponding point. Rather clearly, every other section $s : U_j \longrightarrow \mathbf{A}^n_{U_j}$ decomposes as the sum

$$s = a_1\varepsilon_1 + \ldots + a_n\varepsilon_n$$

Finally, we can define an isomorphism $\mathscr{S}_E(U_j) \simeq \mathscr{O}_X(U_j)^n$ simply setting $s \mapsto (a_1, \dots, a_n)$.

THEOREM 2.5. *There exists a one-to-one correspondence between vector bundles and locally free sheaves.*

PROOF. Let (E, p) be a vector bundle; then there exists an open cover $\{U_i\}_{i \in I}$ of X such that $p^{-1}(U_i) \simeq U_i \times \mathbf{A}^r_k \simeq U_i \times k^n$ and such that the following diagram commutes

$$\begin{array}{ccc} p^{-1}(U_{ij}) & \xrightarrow{\ \simeq\ } & U_j \times k^n \\ \big\downarrow{=} & & \big\uparrow{\psi} \\ p^{-1}(U_{ij}) & \xrightarrow{\ \simeq\ } & U_j \times k^n \end{array}$$

The diagram is completed by an isomorphism ψ acting linearly on the fiber (namely k^n). Hence defining a vector bundle (E, p) is the same thing as giving a system $\{(U_i)_{i\in I}, (\psi_{ij})\}$ made of an open cover of X and linear isomorphisms ψ_{ij}. But it is clear that these data defines a rank n locally free sheaf. The converse correspondence is clear. □

The previous theorem allows us to cease any distinctions between the notions of vector bundle and locally free sheaf; even if we will use the term "vector bundle", only historical importance of the expression is remarked, since the methods employed will take inspiration mostly from the language of sheaves.

More details about theory of vector bundles can be found in [**23**], together with an extensive essay on their topological properties.

CHAPTER 3

Differentials.

In this chapter we introduce some tools commonly used for differential calculus, formalized so that they work in the context of rings, modules and algebraic sheaves. A more exhaustive treatment of the subject can be found in [**18, 19**]; the proofs of the greatest part of algebraic results appear in [**20**].

3.1. Kähler differentials.

Let A be a (commutative unital) ring and let B be an A-algebra and M a B-module.

Definition 3.1. An *A-derivation* over B in M is an additive map $d : B \longrightarrow M$ such that

$$d(\beta_1\beta_2) = \beta_1 d(\beta_2) + \beta_2 d(\beta_1), \quad d(a\beta) = ad(\beta)$$

for every $\beta, \beta_1, \beta_2 \in B$ and $a \in A$.

A-derivations are collected into a B-module $\mathrm{Der}_A(B, M)$.

Definition 3.2. We define the *relative differential forms module* of B over A as the B-module $\Omega_{B/A}$ endowed with an A-derivation $d : B \longrightarrow \Omega_{B/A}$, such that the following universal property holds: for every B-module M and for every A-derivation $\delta : B \longrightarrow M$ there exists a B-modules morphism $f : \Omega_{B/A} \longrightarrow M$ such that $\delta = f \circ d$.

In category terms, the universal property states that

$$\mathrm{Der}_A(B, M) \simeq \hom_A(\Omega_{B/A}, M)$$

namely, the functor $\mathrm{Der}_A(B, -) : \mathbf{Mod}_B \longrightarrow \mathbf{Mod}_B$ can be represented by the object $\Omega_{B/A}$.

An explicit construction of the module $\Omega_{B/A}$ can be obtained taking the free B-module F generated over the set of symbols $\{\mathrm{d}\beta \mid \beta \in B\}$ and quotienting it with the submodule generated by elements

$$\mathrm{d}(\beta_1 + \beta_2) - \mathrm{d}\beta_1 - \mathrm{d}\beta_2,$$
$$\mathrm{d}(\beta_1\beta_2) - \beta_1\mathrm{d}(\beta_2) - \beta_1\mathrm{d}(\beta_1),$$
$$\mathrm{d}(a\beta) - a\,\mathrm{d}(\beta)$$

for every $\beta, \beta_1, \beta_2 \in B$ and $a \in A$. In such a way, the universal derivation $d : B \longrightarrow \Omega_{B/A}$ is obtained setting $b \mapsto \mathrm{d}b$ for every $b \in B$. Moreover, in this description $\Omega_{B/A}$ is generated by the set $\{\mathrm{d}\beta \mid \beta \in B\}$ as a B-module.

PROPOSITION 3.1. *Let B be an A-algebra, $f : B \otimes_A B \longrightarrow B$ the diagonal morphism $\beta \otimes \beta' \mapsto \beta\beta'$ and $I = \ker f$. Let us consider $B \otimes_A B$ endowed with the B-module structure inherited by left multiplication. Therefore I/I^2 inherits a natural B-module structure. Let us define the map $D : B \longrightarrow I/I^2$ setting*

$$D\beta := (\beta \otimes 1 - 1 \otimes \beta) + I^2$$

Therefore, the couple $(I/I^2, D)$ defines a relative differential forms module of B over A.

PROPOSITION 3.2. *Let A' and B be two A-algebras, and $C = B \otimes_A A'$. Then $\Omega_{C/A'} \simeq \Omega_{B/A} \otimes_B C$. Moreover, if $S \subseteq B$ is a multiplicative subset, then $\Omega_{S^{-1}B/A} \simeq S^{-1}\Omega_{B/A}$.*

EXAMPLE 3.1. Let be $B = A[X_1, \dots, X_n]$ a polynomial ring. Then we can see that $\Omega_{B/A}$ is the rank n free module having the set $\{\mathrm{d}X_1, \dots, \mathrm{d}X_n\}$ as a basis. In fact, let $P_1, \dots, P_n \in B$ be such that

$$\sum_{i=1}^{n} P_i \,\mathrm{d}X_i = 0$$

and let $\partial_j \in \mathrm{Der}_A(B, B)$ be the canonical formal derivative respect to an arbitrary index j. Therefore, since $\mathrm{Der}_A(B, B) \simeq \hom_A(\Omega_{B/A}, B)$, there exists a B-modules morphism $f : \Omega_{B/A} \longrightarrow B$ such that $f(\mathrm{d}X_j) = \partial_j$. Then

$$0 = f\left(\sum_{i=1}^{n} P_i \mathrm{d}X_i\right) = P_j$$

Since j has been chosen arbitrarily, it follows that $\mathrm{d}X_i$ form a basis for $\Omega_{B/A}$.

PROPOSITION 3.3. (FIRST EXACT SEQUENCE) *Let $A \longrightarrow B$ and $B \longrightarrow C$ be two ring morphisms. Therefore there exists a natural C-modules exact sequence*

$$\Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0$$

PROPOSITION 3.4. (SECOND EXACT SEQUENCE) *Let B be an A-algebra, I an ideal in B and $C = B/I$. Therefore there exists a natural C-modules exact sequence*

$$I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow 0$$

where for every $b \in I$ we have set $\delta(b + I^2) = (\mathrm{d}b) \otimes 1$.

COROLLARY 3.1. *If B is a finitely generated A-algebra, or if it obtained localizing a finitely generated A-algebra, then $\Omega_{B/A}$ is also a finitely generated B-module.*

PROOF. Let us assume that $B = A[X_1, \dots, X_n]/I$, without loss of generality. Using the second exact sequence, if we call $P = A[X_1, \dots, X_n]$, it holds that

$$I/I^2 \xrightarrow{\delta} \Omega_{P/A} \otimes_P B \longrightarrow \Omega_{B/A} \longrightarrow 0$$

Now, $\Omega_{P/A}$ is finitely generated, as B is, so every quotient taken over $\Omega_{P/A} \otimes_P B$ is finitely B-generated too. In particular, $\Omega_{B/A} \simeq \Omega_{P/A} \otimes_P B/\mathrm{im}(\delta)$ is finitely B-generated. □

3.2. Differentials and field extensions.

Let us now restrict to consider differential modules defined over field extensions or local rings. Recall that a field extension K/k is called *separably generated* if there exists a transcendence basis $\{x_i\}$ for K over k such that K is a separable extension of $k(\{x_i\})$. Recall also that a field extension K/k has a transcendence basis $\{x_i\}$ if and only if x_i are algebraically independent over k and K acts as an algebraic extension of $k(\{x_i\})$. One could prove that every field extension admits a transcendence basis, and that every two transcendence basis have the same cardinality $\mathrm{trdeg}(K/k)$, which is called *transcendence degree.*

THEOREM 3.1. *Let K/k be a finitely generated field extension. Therefore $\Omega_{:K/k}$ is a finitely generated vector K-space and*

$$\dim_K \Omega_{K/k} \geq \mathrm{trdeg}(K/k)$$

Equality holds if and only if K/k is a separably generated field extension.

PROOF. See [**20**]. In particular, note that if K/k is a finite algebraic field extension, then $\Omega_{K/k} = 0$ if and only if K/k is separable (namely, every polynomial over k has distinct roots over K). □

LEMMA 3.1. *Let A be a noetherian local ring, k its residue field and K its quotient field. If M is a finitely generated A-module and if $\dim_k(M \otimes_A k) = \dim_K(M \otimes_A K) = r$, therefore M is a rank r free module.*

PROPOSITION 3.5. *Let B be a local ring containing a field k, such that k is isomorphic to the residue field $B/\mathfrak{m}$. Therefore the map $\delta : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{B/k} \otimes_B k$ defined setting $\beta + \mathfrak{m}^2 \mapsto (\mathrm{d}\beta) \otimes 1$ is an isomorphism.*

PROOF. Using the second exact sequence, the cokernel of δ is given by $\Omega_{k/k} = 0$, so δ is a surjection. To show that δ is injective, it is enough to show that the dual map

$$\delta^\vee : \hom_k(\Omega_{B/k} \otimes_B k, k) \longrightarrow \hom_k(\mathfrak{m}/\mathfrak{m}^2, k)$$

is surjective. Since

$$\hom_k(\Omega_{B/k} \otimes_B k, k) \simeq \hom_B(\Omega_{B/k}, k) \simeq \mathrm{Der}_k(B, k)$$

if $d : B \longrightarrow k$ is a k-derivation, $\delta^\vee(d)$ is thus the morphism obtained restricting d to $\mathfrak{m}$, noting that $d(\mathfrak{m}^2) = 0$: in fact, take $x, y \in \mathfrak{m}$, so $d(xy) = xd(y) + yd(x) \in \mathfrak{m}$; this means that $d(xy) = 0 \in k \simeq B/\mathfrak{m}$. Let us now prove that $\delta^\vee$ is surjective. Take $h \in \hom_k(\mathfrak{m}/\mathfrak{m}^2, k)$ and, for every $b \in B$, consider the decomposition $b = \lambda + c$ with $\lambda \in k$ and $c \in \mathfrak{m}$. Let us define $d : B \longrightarrow k$ setting $d(b) := h(c + \mathfrak{m}^2)$. Therefore d is a k-derivation in B and $\delta^\vee(d) = h$. □

PROPOSITION 3.6. *Let B be a local ring containing a field k, such that k is isomorphic to the residue field $B/\mathfrak{m}$. Moreover, let k be a perfect field and B the localization of a finitely generated k-algebra. Therefore $\Omega_{B/k}$ is a* $\dim(B)$ *rank free B-module if and only if B is a regular local ring.*

PROOF. Let us assume that $\Omega_{B/k} \simeq B^{\dim(B)}$. Then $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(B)$ by the previous Proposition. Using the definition, this means straightforwardly that B is a regular local ring. Let us assume, conversely, that B is a r-dimensional regular local ring. Using Proposition 3.1 we prove that $\Omega_{B/k}$ is finitely generated. Moreover, the regularity of B implies that $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = r$ and the previous Proposition implies that $\dim_k(\Omega_{B/k} \otimes_B k) = r$. On the other hand, let K be the quotient field of B. Then using the formula from the Proposition 3.2 it follows that

$$\Omega_{B/k} \otimes_B K \simeq \Omega_{K/k}$$

Since k is a perfect field, K/k is a separably generated extension ([**9**]) and using Theorem 3.1 we find that $\dim_K(\Omega_{K/k}) = \text{trdeg}(K/k) = r$ (this holds because the dimension of a finitely generated algebra equals the transcendence degree of its own quotient field over the ground field). It follows that, since $\dim_k(\Omega_{B/k} \otimes_B k) = r = \dim_K(\Omega_{K/k} \otimes_B k)$, we can use Lemma 3.1 to conclude that $\Omega_{B/k}$ is free and has rank r. □

3.3. Differentials sheaves.

Now, let X, Y be two separated schemes and let $f : X \longrightarrow Y$ be a schemes morphism. Let us assume $\text{Spec}(A) = U \subseteq Y$ and $\text{Spec}(B) = V \subseteq X$ are open affine sets such that $f(V) \subseteq U$. We therefore define the *sheaf of relative differentials* for V over U setting

$$\Omega_{V/U} := \widetilde{\Omega_{B/A}}$$

that is, the associated sheaf to the relative differential forms module of B over A. Calling I the kernel of the diagonal morphism $B \otimes_A B \longrightarrow B$, it is clear that the ideal sheaf $\mathscr{I}$ associated to I is the ideal sheaf associated to the diagonal subscheme $\Delta(X)$ on $X \otimes_Y X$. In other words, $\Omega_{B/A} \simeq I/I^2$ and $\Omega_{V/U}$ is nothing more than the inverse image sheaf of the quotient $\mathscr{I}/\mathscr{I}^2$. One then can give the following definition.

DEFINITION 3.3. Let X, Y be two schemes and $f : X \longrightarrow Y$ a schemes morphism. Let us assume $\{V_i\}_{i\in I}$ and $\{U_{j(i)}\}_{i\in I}$ are two open covers of X and Y, respectively, such that $f(V_i) \subseteq U_{j(i)}$ for every $i \in I$. Define then the *sheaf of relative differentials* of X over Y as the sheaf $\Omega_{X/Y}$ over X, obtained gluing the sheaves $\Omega_{V_i/U_{j(i)}}$, for $i \in I$, along

$$\Omega_{V_{i_1}\cap V_{j_2}/U_{j(i_1)}} \simeq \Omega_{V_{i_1}\cap V_{i_2}/U_{j(i_2)}}$$

REMARK 3.1. The formal definition is slightly more complicated: let $\Delta : X \longrightarrow X \times_Y X$ be the diagonal; in our hypotheses, Δ is a closed immersion. If we call $\mathscr{I}$ its ideal sheaf, we can thus define

$$\Omega_{X/Y} := \Delta^*(\mathscr{I}/\mathscr{I}^2)$$

Note that the quotient $\mathscr{I}/\mathscr{I}^2$ has a clear $\mathscr{O}_{\Delta(X)}$-module structure. Since Δ induces an isomorphism over X, one can see that $\Omega_{X/Y}$ inherits a natural $\mathscr{O}_X$-module structure too. Moreover, $\Omega_{X/Y}$ is a quasi-coherent by construction and, if Y is a noetherian scheme along with f a finite-type morphism, therefore $\Omega_{X/Y}$ is coherent too.

The following two results are the corresponding sheaf-theoretic sequences for the two modules exact sequence previously stated.

Proposition 3.7. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two schemes morphisms. Then there exists an exact sequence of sheaves over X*

$$f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

Proposition 3.8. *Let $f : X \longrightarrow Y$ be a scheme morphism and Z a closed subscheme of X defined by an ideal sheaf $\mathscr{I}$. Then there exists an exact sequence of sheaves over Z*

$$\mathscr{I}/\mathscr{I}^2 \longrightarrow \Omega_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{O}_Z \longrightarrow \Omega_{Z/Y} \longrightarrow 0$$

Example 3.2. Let be $X = \mathbf{A}^n_Y := \mathbf{A}^n_k \times Y$. Then $\Omega_{X/Y}$ is simply the free sheaf $\mathscr{O}^n_X$, globally generated by sections $\mathrm{d}X_1, \ldots, \mathrm{d}X_n$, where $X_1, \ldots, X_n$ are affine coordinates for $\mathbf{A}^n_Y$.

The following algebraic result has a remarkable relevance.

Theorem 3.2. (Euler sequence) *Let A be a ring, $Y = \mathrm{Spec}(A)$ and $X = \mathbf{P}^r_A$. Then there exists an exact sequence of sheaves*

$$0 \longrightarrow \Omega_{X/Y} \longrightarrow \mathscr{O}_X(-1)^{r+1} \longrightarrow \mathscr{O}_X \longrightarrow 0$$

Proof. Let $S = A[X_0, \ldots, X_n]$ be the homogeneous coordinate ring of X and $E = S(-1)^{n+1}$, the graded S-module with basis $e_0, \ldots, e_n$ in degree 1. Let us define a morphism $E \longrightarrow S$ setting $e_i \mapsto X_i$ and let M be its kernel. Therefore, the sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow S$$

is exact. Sheafifying the sequence, one obtains

$$0 \longrightarrow \widetilde{M} \longrightarrow \mathscr{O}_X(-1)^{n+1} \longrightarrow \mathscr{O}_X \longrightarrow 0$$

Now, note that $E \longrightarrow S$ is not globally surjective, but it is surjective in positive degree; so it gives rise to a surjective sheaves morphism. It remains to show that $\widetilde{M} \simeq \Omega_{X/Y}$. Indeed, localizing at X_i we obtain the S-module M_{X_i} which surjects onto S_{X_i} by means of the former map; hence M is a rank n free module and it is generated by the set $\{e_j - (X_j/X_i)e_i \mid j \neq i\}$. This implies, furthermore, that over the standard open cover $\{U_0, \ldots, U_n\}$ of X the sheaf $\widetilde{M}(U_i)$ is a $\mathscr{O}_X(U_i)$-module generated by the family of sections $\{e_j/X_i - (X_j/X_i^2)e_i \mid j \neq i\}$.

Let us now recall that $U_i \simeq \operatorname{Spec}(A[X_0/X_i, \ldots, X_n/X_i])$, hence

$$\Omega_{X/Y}(U_i) = \langle \mathrm{d}(X_0/X_i), \ldots, \mathrm{d}(X_n/X_i) \rangle_{\mathscr{O}_X(U_i)}$$

because X is smooth. Define a morphism $\varphi_i : \Omega_{X/Y}(U_i) \longrightarrow M^{\sim}(U_i)$ setting

$$\varphi_i(\mathrm{d}(X_j/X_i)) = \frac{X_i e_j - X_j e_i}{X_i^2}$$

We see φ_i is an isomorphism (it brings basis in basis); moreover, one can show that, amongst the indices i, the previous morphisms φ_i do glue, giving rise to a global sheaf isomorphism $\varphi : \Omega_{X/Y} \longrightarrow \widetilde{M}$. In fact, for every pair of indices i, j such that $U_i \cap U_j \neq \varnothing$ we have $X_k/X_i = (X_k/X_j)(X_j/X_i)$ for every k. So, over $\Omega_{X/Y}(U_i \cap U_j)$ we shall have

$$\mathrm{d}(X_k/X_i) = (X_k/X_j)\,\mathrm{d}(X_j/X_i) + (X_j/X_i)\,\mathrm{d}(X_k/X_j)$$

Hence

$$\varphi_i\left(\mathrm{d}(X_k/X_i) - (X_k/X_j)\,\mathrm{d}(X_j/X_i)\right) = \frac{X_j e_k - X_k e_j}{X_i X_j}$$

and furthermore

$$\varphi_j((X_j/X_i)\mathrm{d}(X_k/X_j)) = \frac{X_j}{X_i} \cdot \frac{X_j e_k - X_k e_j}{X_j^2} = \frac{X_j e_k - X_k e_j}{X_i X_j}$$

proving that $\varphi_j = \varphi_i$ over $U_i \cap U_j$. This proves that isomorphisms glue together, finishing the proof. $\square$

Let us remark that in the case X is a k-variety, it is naturally given a scheme morphism $X \longrightarrow \operatorname{Spec}(k)$. The relative differentials sheaf of X is then defined as $\Omega_{X/k} := \Omega_{X/\operatorname{Spec}(k)}$.

Proposition 3.9. *Let X be a k-variety over an algebraically closed field. Therefore the sheaf $\Omega_{X/k}$ is locally free and has $n = \dim X$ if and only if X is smooth.*

Proof. Let $x \in X$ be a closed point. Then $B = \mathscr{O}_{X,x}$ has dimension n and can be viewed as localization of a reduce k-algebra of finite type. Moreover, $(\Omega_{X/k})_x = \Omega_{B/k}$. Hence, using Proposition 3.6, $\Omega_{B/k}$ is a rank n free module if and only if B is a regular local ring, namely if and only if X is smooth in x. Then, by Proposition 2.1 we know that $\Omega_{X/k}$ is a rank n locally free sheaf if and only if its all stalks is a rank n free sheaf. Thesis follow immediately. $\square$

In general, the characterization does not hold with coarser hypothesis. There is however a remarkable related result.

Corollary 3.2. *If X is a k-variety, then there exists a dense open set $U \subseteq X$ that is also a smooth k-subvariety.*

Proof. If $\eta \in X$ is the generic point, then $K = \mathrm{Quot}(\mathscr{O}_{X,\eta})$ is a field having transcendence degree $n = \dim X$ over k and it acts as a finitely generated field extension of k. Then K/k is separably generated. Hence, by Proposition 3.6 again, it follows that $\Omega_{K/k}$ is a n-dimensional vector K-space: but $\Omega_{K/k} = (\Omega_{X/k})_\eta$ so there exists an open neighborhood U of η where $\Omega_{X/k}|_U$ is free of rank n by 3.9, namely U is a non singular k-(sub)variety. It is clear, also, that every neighborhood of η is dense in X. □

Theorem 3.3. *Let X be a smooth k-variety and let be Y an irreducible closed subscheme of X, defined by an ideal sheaf $\mathscr{I}$. Then Y is smooth if and only if*

(1) *$\Omega_{Y/k}$ is locally free;*
(2) *the sequence*

(3.3.1) $$0 \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow \Omega_{X/k} \otimes_{\mathscr{O}_X} \mathscr{O}_Y \longrightarrow \Omega_{Y/k} \longrightarrow 0$$

is exact.

In this condition, $\mathscr{I}/\mathscr{I}^2$ is a locally free sheaf having rank $r = \dim X - \dim Y$.

Proof. See [**18**]. □

3.4. Canonical sheaf.

Let us recall that, given an A-module M, we can define the n-th *exterior power* of M in the following way. Let be $T^n(M) = M \otimes_A \ldots \otimes_A M$ (repeated n times) and J the ideal in $T(M)$ containing the elements $m_1 \otimes \ldots \otimes m_n$ such that $m_i = m_j$ for some $1 \leq i < j \leq n$. We define then the exterior power A-module

$$\bigwedge^n M := T^n(M)/J_n$$

Every coset represented by a pure tensor $m_1 \otimes \ldots \otimes m_n$ is written $m_1 \wedge \ldots \wedge m_n$. For the sake of completeness, one sets $\bigwedge^0 M = A$ and $\bigwedge^1 M = M$. It is easy to see that, if M is a rank m finitely generated module, then $\bigwedge^n M$ is also finitely generated and has rank $\binom{m}{n}$. In general, if $\mathscr{F}$ is a $\mathscr{O}_X$-modules sheaf, we set $\bigwedge^n \mathscr{F}$ to define the associated sheaf to the presheaf such that, for every U,

$$\left(\bigwedge^n \mathscr{F}\right)(U) := \bigwedge^n \mathscr{F}(U)$$

If n is the rank of $\mathscr{F}$, the maximum exterior power $\bigwedge^n \mathscr{F}$ is called *determinant sheaf* of $\mathscr{F}$.

Let us recall the following important property of right-exactness of exterior powers.

Proposition 3.10. *Let $0 \longrightarrow A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \longrightarrow 0$ be a finitely generated free modules exact sequence, with A having rank* 1. *Therefore there exists an exacts sequence*

$$0 \longrightarrow A \otimes \bigwedge^{p-1} C \longrightarrow \bigwedge^p B \longrightarrow \bigwedge^p C \longrightarrow 0$$

PROOF. Firstly, note that if we call r the rank of C, then B has rank $r+1$. Moreover, if $\{b_1, \ldots, b_{r+1}\}$ is a basis for B, we know that the elements $b_{i_1} \wedge \ldots \wedge b_{i_p}$ generate $\bigwedge^p B$; hence there exists a natural map $\bigwedge^p B \longrightarrow \bigwedge^p C$ defined by

$$b_{i_1} \wedge \ldots \wedge b_{i_p} \mapsto f(b_{i_1}) \wedge \ldots \wedge f(b_{i_p})$$

and it is clearly surjective. One could define, therefore, a map $A \otimes \bigwedge^{p-1} B \longrightarrow \bigwedge^p B$ by means of $a \otimes (b_{i_1} \wedge \ldots \wedge b_{i_{p-1}}) \mapsto a \wedge b_{i_1} \wedge \ldots \wedge b_{i_{p-1}}$, being a a generator of A. This latter map vanishes over the kernel of the morphism $A \otimes \bigwedge^{p-1} B \longrightarrow A \otimes \bigwedge^{p-1} C$ obtained setting

$$a \otimes (b_{i_1} \wedge \ldots \wedge b_{i_{p-1}}) \mapsto a \otimes (f(b_{i_1}) \wedge \ldots \wedge f(b_{i_{p-1}}))$$

This means that a map $A \otimes \bigwedge^{p-1} C \longrightarrow \bigwedge^p B$ is induced naturally. Furthermore, one sees that $A \otimes \bigwedge^{p-1} C$ belongs to the kernel of $\bigwedge^p B \longrightarrow \bigwedge^p C$, since $A = \ker(g)$. Switching to ranks and using Stiefel formula for binomial coefficients, we find that

$$\binom{r}{p-1} + \binom{r}{p} = \binom{r+1}{p}$$

Since every module considered is free, $A \otimes \bigwedge^{p-1} C$ is the desired kernel. $\square$

REMARK 3.2. The Proposition admits also a symmetric formulation, assuming to deal with a free modules exact sequence $0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$, with G having rank 1. There is an exact sequence

$$0 \longrightarrow \bigwedge^p E \longrightarrow \bigwedge^p F \longrightarrow G \otimes \bigwedge^{p-1} E \longrightarrow 0$$

The same results hold replacing free modules with vector bundles.

DEFINITION 3.4. Let X be a smooth k-variety. We define the *tangent sheaf* as the sheaf over X defined by

$$\mathscr{T}_X := \mathscr{H}om(\Omega_{X/k}, \mathscr{O}_X)$$

If $Y \subseteq X$ is a smooth subvariety, we define also the *normal sheaf* of Y in X as

$$\mathscr{N}_{Y/X} := \mathscr{H}om(\mathscr{I}_Y/\mathscr{I}_Y^2, \mathscr{O}_X)$$

Finally, we define the *canonical sheaf* of X setting

$$\omega_X := \bigwedge^{\dim X} \Omega_{X/k}$$

Note that $\mathscr{T}_X$ is a locally free sheaf and has rank $\dim X$. In fact, when X is a smooth variety, $\Omega_{X/k}$ is locally free and has rank $\dim X$, compelling $\mathscr{H}om(\Omega_{X/k}, \mathscr{O}_X)$ to be alike. Moreover, $\bigwedge^{\dim X} \Omega_{X/k}$ is invertible and in particular, ω_X is a coherent sheaf.

Let be $\mathbf{P}^r := \mathbf{P}^r_k$ the schematic projective space and let us consider the Euler sequence explained in Theorem 3.2, dualized accordingly to the current notations:

$$0 \longrightarrow \Omega^1_{\mathbf{P}^r} \longrightarrow \mathscr{O}_{\mathbf{P}^r}(-1)^{r+1} \longrightarrow \mathscr{O}_{\mathbf{P}^r} \longrightarrow 0$$

Tensoring by $\mathscr{O}_{\mathbf{P}^r}(1)$, we obtain the exact sequence

$$0 \longrightarrow \Omega^1_{\mathbf{P}^r}(1) \longrightarrow \mathscr{O}^{r+1}_{\mathbf{P}^r} \longrightarrow \mathscr{O}_{\mathbf{P}^r}(1) \longrightarrow 0$$

Let $\Omega^p_{\mathbf{P}^r} = \bigwedge^p \Omega_{\mathbf{P}^r/k}$ be the sheaf of differential p-forms over $\mathbf{P}^r$; let us recall that $\Omega^r_{\mathbf{P}^r} = \omega_{\mathbf{P}^r}$. Taking p-th exterior power and using Proposition 3.10, we find the sequence

$$0 \longrightarrow \bigwedge^p \Omega^1_{\mathbf{P}^r}(1) \longrightarrow \bigwedge^p \mathscr{O}^{r+1}_{\mathbf{P}^r} \longrightarrow \mathscr{O}_{\mathbf{P}^r}(1) \otimes \bigwedge^p \Omega^1_{\mathbf{P}^r}(1) \longrightarrow 0$$

Note that $\bigwedge^p \Omega^1_{\mathbf{P}^r}(1) = \Omega^p_{\mathbf{P}^r}(p)$. Hence setting $p = r+1$ forces $\Omega^{r+1}_{\mathbf{P}^r} = 0$ because $\Omega^1_{\mathbf{P}^r} \simeq (\mathscr{T}_{\mathbf{P}^r})^\vee$ has rank r, giving the isomorphism

$$\mathscr{O}_{\mathbf{P}^r} = \bigwedge^{r+1} \mathscr{O}^{r+1}_{\mathbf{P}^{r+1}} \simeq \mathscr{O}_{\mathbf{P}^r}(1) \otimes \omega_{\mathbf{P}^r}(r)$$

It follows then $\omega_{\mathbf{P}^r} \simeq \mathscr{O}_{\mathbf{P}^r}(-r-1)$.

Definition 3.5. The *geometric genus* of a projective variety X is the integer $p_g(X) := \dim_k(\Gamma(X, \omega_X))$.

The definition is well posed: in fact, if X is a projective variety, Serre's theorem shows that every module $H^p(X, \omega_X)$ is a finitely generated vector k-space. Let us recall also that, for a projective scheme X over a field k, the *arithmetic genus* is defined setting ([**18**])

$$\begin{aligned} p_a(X) : &= (-1)^{\dim X+1} (1 - \chi(X, \mathscr{O}_X)) = \\ &= (-1)^{\dim X+1} \left(1 - \sum_{j=0}^{\dim X} (-1)^j \dim_k H^j(X, \mathscr{O}_X)\right) \end{aligned}$$

At the conclusion of the chapter, we want to recall the following notable results.

Theorem 3.4. (Serre duality) *Let X be a n-dimensional k-scheme and let be $\mathscr{F}$ a vector bundle over X. Therefore*

$$H^p(X, \mathscr{F})^\vee \simeq H^{n-p}(X, \mathscr{F}^\vee \otimes \omega_X)$$

where $\mathscr{F}^\vee := \mathscr{H}om(\mathscr{F}, \mathscr{O}_X)$.

Proof. See [**18**]. □

Theorem 3.5. (Adjunction formula) *Let Y be a r-codimensional smooth subvariety in a smooth k-variety X. Then $\omega_Y \simeq \omega_X \otimes_{\mathscr{O}_X} \bigwedge^r \mathscr{N}_{Y/X}$. If $r = 1$, let us consider Y as a divisor over X and let be $\mathscr{L}$ the invertible sheaf over X that is associated to Y. Then $\omega_Y \simeq \omega_X \otimes_{\mathscr{O}_X} \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{O}_Y$.*

Proof. See [**18**]. □

Remark 3.3. Note that the differentials sheaf and the canonical sheaf are both *implicitly defined* over the variety X. This means that themselves, together with their numeric invariants, are invariant under isomorphism of varieties. In general, birational invariance does not hold, except for geometric genus. Useful references for these arguments are, amongst others, [**1, 11, 24**].

CHAPTER 4

Curves.

In these pages, a *curve* is a k-scheme X, where k is an assigned field, with the following properties:

(C1) X is integral (namely $\mathscr{O}_{X,p}$ is a reduced ring for every $p \in X$ and X is an irreducible topological space);
(C2) X has dimension 1 (namely, Krull dimension of each $\mathscr{O}_{X,p}$ is 1);
(C3) X is a projective variety (namely, there exists a closed immersion $X \subseteq \mathbf{P}^r_k$ for some $r > 0$).

We will often assume also

(C4) X is smooth (namely, $\mathscr{O}_{X,p}$ is a local regular ring for every $p \in X$).

Remark 4.1. With hypotheses **(C1)**−**(C4)**, if X is a curve we know that

$$p_a(X) = 1 - \dim_k H^0(X, \mathscr{O}_X) + \dim_k H^1(X, \mathscr{O}_X) = \dim_k H^1(X, \mathscr{O}_X)$$

Moreover $\omega_X = \Omega_{X/\kappa}$ is an invertible sheaf. Therefore $H^1(X, \mathscr{O}_X)$ and $H^0(X, \Omega_{X/\kappa})$ are one the dual of the other; and this means that for a projective smooth curve $p_a = p_g$ holds. Cleared that, in our usual hypotheses, geometric and arithmetic genus coincide, so we will talk only about "the genus" of a curve.

For a complete study about algebraic curves, one can see [**18**].

4.1. Riemann-Roch theorem.

In our hypotheses, Weil and Cartier divisors coincide over smooth curves. A divisor can thus be seen as a finite formal sum of integral multiples of points belonging to the curve. In particular, there is an isomorphism between the class group $\mathrm{Cl}(X)$ of divisor modulo linear equivalence and the Picard group $\mathrm{Pic}(X)$, containing isomorphism classes of invertible sheaves over X. We will write $\mathscr{O}_X(D)$ for an invertible sheaf associated to a divisor D over X (up to isomorphism).

The set of effective divisor linearly equivalent to a given divisor D forms the complete linear system $|\mathscr{O}_X(D)|$, which we will denote $|D|$. Its dimension is $l(D) = \dim_k \Gamma(X, \mathscr{O}_X(D))$.

LEMMA 4.1. *Let D be a divisor over a curve X. Then, if $l(D) \neq 0$, necessarily* $\deg(D) \geq 0$. *Moreover, if $l(D) \neq 0$ and* $\deg(D) = 0$ *then D is linearly equivalent to the zero divisor, that is $\mathscr{O}_X(D) \simeq \mathscr{O}_X$.*

PROOF. If $l(D) \neq 0$, the linear system $|D|$ is not empty, so D is linearly equivalent to some effective divisor having necessarily non negative degree. Since degree does not depend from the linear equivalence class, we can claim that $\deg(D) \geq 0$. Let us also suppose that $\deg(D) = 0$. Then D is linearly equivalent to an effective divisor of zero degree, which is necessarily the zero divisor. □

Let $\Omega_{X/k}$ be the relative differentials sheaf over X. Since X is 1-dimensional, $\Omega_{X/k}$ is invertible and is isomorphic to the canonical sheaf ω_X over X. Every divisor belonging to the linear equivalence class of ω_X (that is to say, every D such that $\mathscr{O}_X(D) \simeq \omega_X$) is called a *canonical divisor* over X and it is denoted K_X, if no ambiguity arises.

THEOREM 4.1. (RIEMANN, ROCH) Let D be a divisor over a curve X of genus g. Therefore

$$l(D) - l(K_X - D) = \deg(D) + 1 - g$$

PROOF. We know that the divisor $K_X - D$ corresponds to the invertible sheaf $\omega_X \otimes \mathscr{O}_X(D)^{-1}$. Using Serre's duality, we have

$$H^0(X, \omega_X \otimes \mathscr{O}_X(D)^{-1}) \simeq H^1(X, \mathscr{O}(D))^\vee$$

For this reason, we can compute

$$\begin{aligned} \chi(X, \mathscr{O}_X(D)) &= \dim_k H^0(X, \mathscr{O}_X(D)) - \dim_k H^1(X, \mathscr{O}_X(D)) = \\ &= \dim_k H^0(X, \mathscr{O}_X(D)) - \dim_k H^0(X, \omega_X \otimes_{\mathscr{O}_X} \mathscr{O}_X(D)^{-1}) = \\ &= l(D) - l(K_X - D) \end{aligned}$$

and it is sufficient to prove that

$$\chi(X, \mathscr{O}_X(D)) = \deg(D) + 1 - g$$

We may separate the proof in two cases.

(1) Let us assume $D = 0$. Then it is straightforward that

$$\chi(X, \mathscr{O}_X(D)) = \chi(X, \mathscr{O}_X) = \dim_k H^0(X, \mathscr{O}_X) - \dim_k H^1(X, \mathscr{O}_X) = 1 - g$$

as we wished, being $H^0(X, \mathscr{O}_X) \simeq k$ for every projective variety X.

(2) Let D be now an arbitrary divisor and let be p a point. Therefore p and $D + p$ are divisors over the curve. We will show that the theorem holds for D if and only if it holds for $D + p$. Proceeding this way, we reduce the proof to the previous point, since every divisor D can be obtained from 0 adding or subtracting a point step by step. Let us consider $P = \{p\}$ as a closed subscheme of X: its structure sheaf is the skyscraper sheaf concentrated in p, that is

$$k(P) := \mathscr{O}_P(U) = \begin{cases} \kappa & \text{if } p \in U \\ 0 & \text{otherwise} \end{cases}$$

Moreover, the ideal sheaf is $\mathscr{I} = \mathscr{O}_X(-p)$. The following exact sequence holds

$$0 \longrightarrow \mathscr{O}_X(-p) \longrightarrow \mathscr{O}_X \longrightarrow k(P) \longrightarrow 0$$

Tensoring with $\mathscr{O}_X(D+p)$, we keep the exactness without acting on $k(P)$, and we find

$$0 \longrightarrow \mathscr{O}_X(D) \longrightarrow \mathscr{O}_X(D+p) \longrightarrow \kappa(P) \longrightarrow 0$$

Switching to Euler-Poincaré characteristic, it follows that

$$\chi(X, \mathscr{O}_X(D+p)) = \chi(X, \mathscr{O}_X(D)) + \chi(\kappa(P)) = \chi(X, \mathscr{O}_X(D)) + 1$$

On the other hand, $\deg(D+p) = \deg(D)+1$, therefore the formula holds for D if and only if it holds for $D+p$.

□

REMARK 4.2. Let X be a curve of genus g. One can prove that the canonical divisor has genus $2g-2$. In fact, using the Riemann-Roch theorem for $D = K_X$, since

$$l(K_X) = \dim_k H^0(X, \omega_X) = p_g(X) = g$$

and $l(0) = 1$, we find

$$g - 1 = \deg K_X + 1 - g$$

and thus $\deg K_X = 2g - 2$.

4.2. Degree of projective varieties.

In this section we will introduce the concept of *degree* for an algebraic variety, as a natural generalization of degree for algebraic hypersurfaces.

Let $X \subseteq \mathbf{P}^r := \mathbf{P}^r_k$ be a projective variety, $I_X \subseteq k[X_0, \dots, X_r]$ its homogeneous ideal and $S_X = k[X_0, \dots, X_r]/I_X$ its homogeneous coordinate ring. Therefore, Hilbert's function and polynomial of S_X are, respectively, the *Hilbert's function* $\mathrm{Hilb}_X(t)$ of X and the *Hilbert's polynomial* $P_X(t)$ of X. The following results gives a characterization for the dimension of projective varieties.

THEOREM 4.2. *Hilbert's polynomial of a projective variety $X \subseteq \mathbf{P}^r$ has degree* $\dim(X)$.

PROOF. It is know that

$$P_{\mathbf{P}^r}(t) = \binom{t+r}{r}$$

which has degree r, so the thesis holds for $X = \mathbf{P}^r$. If X reduces to a point, then P_X is a constant and has zero degree. If X is a projective variety, there exists a chain of subvarieties

$$X_0 \subseteq X_1 \subseteq \ldots \subseteq X_{r-1} \subseteq X_r = \mathbf{P}^r$$

such that $\dim(X_i) = i$ and $X_d = X$ for some $d = \dim(X)$. Therefore, it is enough to prove that, for every strict inclusion $Y \subset X$ of projective varieties,

$\deg(P_Y) < \deg(P_X)$ holds. We can assume, without loss of generality, that $Y \subseteq X \cap H$, being H a degree s hypersurface, that is $H = V(h)$ for some homogeneous polynomial h of degree s. Let $R = S_X$ be the algebra of functions of X. Thus we have a graded modules exact sequence

$$0 \longrightarrow R(-s) \longrightarrow R \longrightarrow R/(h) \longrightarrow 0$$

where the first arrow represents multiplication by h. Let us assume that

$$P_R(t) = a_0\binom{t}{d} + a_1\binom{t}{d-1} + \ldots + a_d$$

Since $P_{R(-s)}(t) = P_R(t-s)$, we have

$$\begin{aligned} P_{R/(h)}(t) &= P_R(t) - P_R(t-s) = a_0\left[\binom{t}{d} - \binom{t-s}{d}\right] + \ldots = \\ &= sa_0\binom{t}{d} + \ldots \end{aligned}$$

where dots hide terms of degree $\leq d$. On the other hand, $\mathrm{Hilb}_{R/(h)}(t) \geq \mathrm{Hilb}_Y(t)$ since S_Y is a quotient of $R/(h)$; therefore

$$\deg(P_Y) \leq \deg(P_{R/(h)}) < \deg(P_X)$$

finishing the proof. □

Let $d = \dim(X)$. Then one could write P_X in the form

$$P_X(t) = \frac{\deg(X)}{d!}t^d + \ldots$$

where $\deg(X)$ is a integer, called *degree* of X. In other words, $\deg(X)$ is the leading coefficient of P_X multiplied by $\dim(X)!$

The following theorem shows some of the most relevant properties of degree.

THEOREM 4.3. *Degree of varieties has the following properties:*

1. $\deg(X) > 0$ *for every variety* $X \neq \varnothing$*;*
2. *a linear subspace in* $\mathbf{P}^r$ *has degree* 1*;*
3. *a degree* m *hypersurface in* $\mathbf{P}^r$ *has degree* m *as variety;*
4. *if* $X \subseteq \mathbf{P}^r$ *is a projective* d*-dimensional variety, with* $d \geq 2$*, and* H *is a hyperplane defined by an equation* $h = 0$*, such that* (h) *is a prime ideal in* S_X*, then* $Y = X \cap H$ *is a projective* $(d-1)$*-dimensional variety and* $\deg(Y) = \deg(X)$*.*

PROOF. Statement (1) is straightforward, since $P_X(n) > 0$ for every $n \gg 0$.

(2) If $L \subseteq \mathbf{P}^r$ is a linear subspace, then $S_L = k[X_0, \ldots, X_{\dim(L)}]$, so $P_L(t) = \binom{t+\dim(L)}{\dim(L)}$, thus $\mu(X) = 1$.

(3) Let us assume that X is a hypersurface $V(f)$, with $f \in k[X_0, \dots, X_r]$ a degree m homogeneous polynomial. If we set $S = k[X_0, \dots, X_r]$, we have an exact sequence

$$0 \longrightarrow S(-m) \longrightarrow S \longrightarrow S_X = S/(f) \longrightarrow 0$$

where the first arrow represents multiplication by f. Then we have

$$\begin{aligned} P_X(t) &= P_S(t) - P_S(t-m) = \binom{t+r}{r} - \binom{t+r-m}{r} = \\ &= m\binom{t+r-1}{r-1} + \dots \end{aligned}$$

hence

$$\deg(X) = \frac{m}{(r-1)!}\dim(X)! = m$$

(4) With the given hypotheses, $Y \subseteq H$ is a projective $(d-1)$-variety having $S_Y = S_X/(h)$ as coordinate ring. One has, then, the following exact sequence

$$0 \longrightarrow S_X(-1) \longrightarrow S_X \longrightarrow S_Y \longrightarrow 0$$

being h a homogeneous polynomial. It follows that

$$\begin{aligned} P_Y(t) &= P_X(t) - P_X(t-1) = \mu(X)\left[\binom{t+d}{d} - \binom{t+d-1}{d}\right] + \dots = \\ &= \deg(X)\binom{t+d-1}{d-1} + \dots \end{aligned}$$

hence thesis. □

Remark 4.3. A hyperplane H satisfying hypotheses stated in point (4) is often called *general hyperplane* and the intersection $X \cap H$ takes the name of *general hyperplane section.*

Moreover, we have the following notable inequality.

Corollary 4.1. *Let $X \subseteq \mathbf{P}^r$ be an irreducible non degenerate projective curve over an algebraically closed field. Then we have* $\deg(X) \geq r$.

Proof. Let us assume by contradiction that X has degree strictly lower than r. Then, for every choice of points $p_1, \dots, p_r \in X$, it should exists a hyperplane $H \simeq \mathbf{P}^{r-1}$ containing all of then; since, however, we assumed $\deg(X) < r$, the hyperplane section $X \cap H$ contains at least r points and can not be general, namely $\dim(H \cap X) = 1$; this would lead to say that $X \subseteq H$ and this is a contradiction. □

Lastly, we show an important result that we will need further in this document; a general form can be found in [**10**]. Let us call $h^i(X, -) = \dim(H^i(X, -))$ for every $i \geq 0$ and let us recall the following notation: for every invertible sheaf $\mathscr{L}$ over X and for every point $p \in X$ we shall denote with $\mathscr{L}(p)$ the invertible sheaf $\mathscr{L} \otimes \mathscr{O}_X(p)$, corresponding to the divisor

$\mathscr{L} + p$. In other words, tensoring an invertible sheaf with $\mathscr{O}_X(p)$ means adding the point p to the corresponding divisor.

DEFINITION 4.1. A line bundle $\mathscr{L}$ over X is called *general* if $\mathscr{L} = \mathscr{O}_X(p_1 + \ldots + p_m - q_1 - \ldots - q_n)$ for some $p_i, q_j \in X$ general points.

The useful properties of generic bundles are collected in the following couple of results.

LEMMA 4.2. *Let $\mathscr{L}$ be a line bundle such that $h^0(X, \mathscr{L}) \geq n$, for some $n \in \mathbb{N}$. Then, for every choice of general points $p_1, \ldots, p_n \in X$, we have $h^0(\mathscr{L}(-p_1 - \ldots - p_n)) = h^0(\mathscr{L}) - n$.*

PROOF. We proceed inducting on n. For $n = 0$, there is nothing to prove; let us suppose that the thesis holds for $n-1$ and let us prove it for n. Since $h^0(\mathscr{L}) > 0$, then $h^0(\mathscr{L}(-p_n)) = h^0(\mathscr{L}) - 1$. By inductive hypothesis

$$h^0(\mathscr{L}(-p_1 - \ldots - p_n)) = h^0(\mathscr{L}(-p_n)) - (n-1) = h^0(\mathscr{L}) - n$$

□

THEOREM 4.4. *Let X be a smooth curve of genus g over an algebraically closed field k. If $\mathscr{L}$ is a generic line bundle having degree $d \geq g-1$, then $\mathscr{L}$ is non special, that is $h^1(X, \mathscr{L}) = 0$.*

PROOF. Let us note that the thesis equals the fact that, for every general line bundle $\mathscr{L}$ having degree $d \geq g-1$, we have $h^0(\mathscr{L}) = d-g+1$. Moreover, if $d \geq 2g-1$, then $\deg(\omega_X \otimes \mathscr{L}^{-1}) = 2g - 2 - 2g + 1 = -1 < 0$, hence

$$h^1(\mathscr{L}) = h^0(\omega_X \otimes \mathscr{L}^{-1}) = 0$$

and the result holds. Let us now fix $g - 1 \leq d < 2g - 1$ and consider a general line bundle $\mathscr{M}$ having degree $2g - 1$; for every $p_1, \ldots, p_{2g-1-d} \in X$ general points, one can write

$$\mathscr{L} = \mathscr{M}(-p_1 - \ldots - p_{2g-1-d})$$

Therefore $h^0(\mathscr{M}) = h^1(\mathscr{M}) + 2g - 1 + 1 - g = 2g - 1 + 1 - g = g$. In particular, $h^0(\mathscr{M}) \geq 2g - 1 - d > 0$ holds, and using the previous Lemma

$$h^0(\mathscr{L}) = h^0(\mathscr{M}(-p_1 - \ldots - p_{2g-1-d})) = h^0(\mathscr{M}) - (2g - 1 - d) = g - (d-1)$$

that is the thesis. □

A different, yet equivalent, definition of degree can be given. Since X is a projective variety, there exists a closed immersion $i : X \hookrightarrow \mathbf{P}^r_k$, depending uniquely by the sheaf $i^*\mathscr{O}_{\mathbf{P}^r}(1)$ and by the section which generate it, precisely

$$s_i := i^* X_i$$

where X_i are homogeneous coordinates for $\mathbf{P}^r_k$. One then sets

$$\deg(X) := \deg(i^*\mathscr{O}_{\mathbf{P}^r}(1))$$

Let us remark that the definition is well posed, for $i^*\mathscr{O}_{\mathbf{P}^r}(1)$ is an invertible sheaf. More formally, since fixing a projective variety means giving a pair

$(X, \mathscr{L})$, where $\mathscr{L}$ is a very ample invertible sheaf, one could define $\deg(X)$ as the (divisor) degree of $\mathscr{L}$. More details can be found in [**1**].

CHAPTER 5

Free resolutions.

In this chapter, we will assume that k is a fixed field and we will study the projective space $\mathbf{P}^r_k$ as a classical algebraic variety. We recall that the homogeneous coordinate ring of $\mathbf{P}^r_k$ is the ring $S = k[x_0, \ldots, x_r]$. It has a natural graduation, which turns it into a graded ring giving degree 1 to each variable.

Let us recall that a graded modules morphism is a modules morphism preserving degrees. In general, if a morphism shifts degree by a fixed integer$p$, we call it a *degree p morphism.*

5.1. Syzygies.

Let M be a finitely generated S-module, endowed with a grading $M = \bigoplus_{d\in\mathbb{Z}} M_d$. Since M is finitely generated, every M_d is a finitely generated vector k-space, and

$$\mathrm{Hilb}_M(d) := \dim_k M_d$$

defines the *Hilbert's function* of M. Hilbert's idea was to compute $\mathrm{Hilb}_M(d)$ comparing M and its graded parts with certain free modules, by means of a *free resolution.* For every graded module M, let be $M(a)$ the twisted module, having homogeneous parts

$$M(a)_d := M_{d+a}$$

Given homogeneous elements $m_i \in M$, each having degree a_i and generating M as S-module, one can define a morphism from the free module $F_0 = \bigoplus S(-a_i)$ to M, sending the i-th generator in m_i. Twisting F_0 is necessary to guarantee the morphism preserves degrees. Let $M_1 \subseteq F_0$ be the kernel of such morphism; by Hilbert's Basis Theorem, M_1 is also finitely generated, and its elements are called *linear syzygies* over the generators m_i, or simply *syzygies over M.*

Choosing a finite number of generators for M_1, one can also define a map from a free module F_1 in F_0, having image M_1. Going on this way, one builds an exact sequence of graded free modules, that is called a *free graded resolution* of M:

$$\cdots \longrightarrow F_i \xrightarrow{\varphi_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

This sequence contains degree 0 morphisms φ_i, with the additional property $M = \mathrm{coker}(\varphi_1)$. Since the φ_i are degree-preserving, taking any homogeneous part of degree d we obtain also many exact sequences of finitely generated vector k-spaces. Therefore, we determine

$$\mathrm{Hilb}_M(d) = \sum_{i=0}^{\infty}(-1)^i \mathrm{Hilb}_{F_i}(d)$$

THEOREM 5.1. (HILBERT'S SYZYGY THEOREM) *Every graded finitely generated S-module M has a free graded resolution of finite length $m \leq r+1$.*

PROOF. See [**9, 10, 16**]. □

Actually, one can concretely compute Hilbert's function by means of *degrees* of a free resolution.

COROLLARY 5.1. *Let $S = k[x_0, \dots, x_r]$ be a polynomial ring. If the graded S-module M has the following finite free resolution*

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

in a way that F_i is a finitely generated free S-module of the form $F_i = \bigoplus_j S(-a_{i,j})$, then

$$\mathrm{Hilb}_M(d) = \sum_{i=0}^{m}(-1)^i \sum_j \binom{r+d-a_{i,j}}{r}$$

PROOF. Using the preceding remarks, it is enough to prove that

$$\mathrm{Hilb}_{F_i}(d) = \sum_j \binom{r+d-a_{i,j}}{r}$$

In particular, decomposing F_i as a direct sum, it suffices to prove that $\mathrm{Hilb}_{S(-a)}(d) = \binom{r+d-a}{r}$, and, removing twists, it suffices to prove that $\mathrm{Hilb}_S(d) = \binom{r+d}{r}$. This is straightforward: a degree d monomial is uniquely determined by the sequence of exponents of each degree 1 one element; one can choose a particular monomial order such that the sequence of exponents is an increasing sequence of d integers, each one between 0 and r. Adding i to the i-th element of this sequence, causes the monomial to be identified with a sequence of d elements in $\{1, 3, \dots, r+d\}$, so we can enumerate

$$\binom{r+d}{d} = \frac{(r+d)!}{d!r!} = \binom{r+d}{r}$$

□

COROLLARY 5.2. *In the previous hypotheses, there exists a polynomial $P_M(d)$, the* Hilbert's polynomial *of M, such that $P_M(d) = \mathrm{Hilb}_M(d)$ for $d \geq \max\{a_{i,j} - r\}$.*

PROOF. Note that, if $d + r - a \geq 0$, we have

$$\binom{d+r-a}{r} = \frac{(d+r-a)(d+r-a-1)\cdots(d+1-a)}{r!}$$

and it is a degree d polynomial in r; so, if the condition holds, Hilbert's function is a polynomial by means of its binomial expression. □

5.2. Minimal resolutions.

Every graded finitely generated S-module owns a free graded *minimal* resolution, and it is unique up to isomorphism. The degree of generators of its free modules not only determines Hilbert's function, like every other free resolution, but gives rise to an other much finer invariant that we will discuss further.

Intuitively, minimal resolutions can be defined in this way. Let M be a graded finitely generated S-module M and choose a minimal set of generators m_i; define then a map from a free module F_0 to M, sending a basis for F_0 in the set of m_i. Let then M_1 be the kernel of this map; it will be finitely generated too. Choosing again a minimal set of generators for M_1, we determine a map $F_1 \longrightarrow F_0$ whose image is M_1. Continuing this way, one builds the desired minimal resolution.

Nevertheless, many properties of these peculiar resolution can be derived with more ease by a characterizing property, that we will assume as the leading definition. To simplify notations, let us call $\mathfrak{m}$ the homogeneous maximal ideal $(x_0, \ldots, x_r) \subseteq S = k[x_0, \ldots, x_r]$.

DEFINITION 5.1. A graded S-module complex

$$\cdots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \cdots$$

is called *minimal* if $\operatorname{im}(\delta_i) \subseteq \mathfrak{m}F_{i-1}$ for every i.

Heuristically, a complex is minimal if every coboundary operator can be represented by a matrix, whose entries belong in the maximal ideal. The link between this definition and the concept of minimality is discovered via Nakayama's lemma.

LEMMA 5.1. (NAKAYAMA) *Let M be a graded finitely generated S-module and $m_1, \ldots, m_n \in M$ elements representing generators for $M/\mathfrak{m}M$. Therefore $m_1, \ldots, m_n$ generate M.*

It follows, then, the full characterization of minimal free graded resolutions.

PROPOSITION 5.1. *A free graded resolution*

$$\mathbf{F}: \quad \cdots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \cdots$$

is a minimal complex if and only if δ_i takes a basis for F_i in a minimal set of generators for $\operatorname{im}(\delta_i)$, for every choice of $i \in \mathbb{N}$.

PROOF. Let us consider the right-exact sequence

$$F_{i+1} \xrightarrow{\delta_{i+1}} F_i \longrightarrow \operatorname{im}(\delta_i) \longrightarrow 0 \tag{5.2.1}$$

The complex $\mathbf{F}$ is minimal if and only if the quotient map $\delta'_{i+1} : F_{i+1}/\mathfrak{m}F_{i+1} \longrightarrow F_i/\mathfrak{m}F_i$ is the zero map for every i. In fact, if $\mathbf{F}$ is minimal, then $\delta_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i$ and δ'_{i+1} vanishes. The converse is straightforward. However, this holds, by means of (5.2.1), if and only if the map $F_i/\mathfrak{m}F_i \longrightarrow \mathrm{im}(\delta_i)/\mathfrak{m}(\mathrm{im}(\delta_i))$ is an isomorphism. By Nakayama's Lemma, this happens if and only if a basis for F_i is brought into a minimal set of generators for $\mathrm{im}(\delta_i)$. □

The following results guarantees that the construction does not depend from the choices we made.

THEOREM 5.2. *Let M be a graded finitely generated S-module. If $\mathbf{F}$ and $\mathbf{G}$ are two minimal free graded resolution for M, then there exists a graded isomorphism of complexes $\mathbf{F} \longrightarrow \mathbf{G}$ inducing the identity over M. Moreover, every free resolution of M contains the minimal free graded resolution of M as direct summand.*

PROOF. See [**9**]. □

The most significant aspect of uniqueness of minimal resolutions is that, if $\mathbf{F} : \ \ldots . \longrightarrow F_1 \longrightarrow F_0$ is a such resolution for M, then the number of generators required for F_i in every degree depends only by M. The simplest way to express this property in a precise statement is to use the torsion functor. Let us recall that, for every S-module N, the functor $\mathrm{Tor}_n^S(N, M)$ is the n-th homology group of the complex $N \otimes_S \mathbf{Pr}_M$, being $\mathbf{Pr}_M$ any projective resolution for M.

PROPOSITION 5.2. *If $\mathbf{F} : \ \cdots \longrightarrow F_1 \longrightarrow F_0$ is the minimal free resolution of a graded finitely generated S-module M, then every minimal set of generators for F_i contains exactly $\dim_k \mathrm{Tor}_i^S(k, M)_j$ degree j generators.*

PROOF. The vector space $\mathrm{Tor}_i^S(k, M)_j$ is the degree j homogeneous part of the i-th homology module taken on the complex $k \otimes_S \mathbf{F}$, as $\mathbf{F}$ is also a projective resolution of M. Since $\mathbf{F}$ is minimal, every map in $k \otimes \mathbf{F}$ is zero (recall that $k \otimes F_i = (S/\mathfrak{m}) \otimes F_i = F_i/\mathfrak{m}F_i$), hence we have $\mathrm{Tor}_i^S(k, M) = k \otimes_S F_i$ and by Nakayama Lemma, F_i requires exactly $\dim_k \mathrm{Tor}_i^S(k, M)_j$ generators of degree j. □

COROLLARY 5.3. *If M is a graded finitely generated S-module, projective dimension of M equals the length of its minimal free graded resolution.*

PROOF. By definition, projective dimension is the least length of a projective resolution of M; since minimal free resolution is a projective resolution, one inequality is trivial; it remains to prove that the length of minimal free resolution is at most $\dim_S^{(\mathrm{proj})}(M)$. Let us note that $\mathrm{Tor}_i^S(k, M) = 0$ if $i > \dim_S^{(\mathrm{proj})}(M)$, then over that integer the terms in the minimal free resolution have zero generators. This means exactly that its length is $\leq \dim_S^{(\mathrm{proj})}(M)$, as we wished. □

Let

$$\mathbf{F}: \cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0$$

be a free S-modules complex such that $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$, that is F_i requires $\beta_{i,j}$ minimal generators of degree j. If $\mathbf{F}$ is the minimal free resolution of a graded finitely generated S-module M and has length m, then the integers $\beta_{i,j}$, sometimes written $\beta_{i,j}(M)$, are called *graded Betti numbers.* The previous arguments show clearly that $\beta_{i,j}(M) = \dim_k \operatorname{Tor}_i^S(k, M)_j$.

For example, the number $\beta_{0,j}$ is the number of degree j elements required to generate M; since we will often take M as the projective coordinate ring S_X of some non-empty projective algebraic variety X, it is convenient to show related examples. With its S-module structure, the ring S_X is generated by unity, so $\beta_{0,0} = 1$ and $\beta_{0,j} = 0$ for $j \neq 1$. Also, the $\beta_{1,j}$ is the number of independent elements required to generate the ideal I_X of X. If $S_X \neq 0$ (that is to say, $X \neq \varnothing$), the ideal does not contain degree 0 elements, so $\beta_{1,0} = 0$. In general, we can prove the following result.

PROPOSITION 5.3. *Let $\{\beta_{i,j}\}$ be the graded Betti number of a graded finitely generated S-module. If, for a given i, there exists d such that $\beta_{i,j} = 0$ for every $j < d$, then we have $\beta_{i+1,j+1} = 0$ for every $j < d$.*

PROOF. Let $\cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0$ be the minimal free resolution. By minimality condition, every generator for F_{i+1} must be mapped in a non-zero element of the same degree in $\mathfrak{m}F_i$. Claiming that $\beta_{i,j} = 0$ for every $j < d$ means that every generator for F_i (and then every non-zero element) has degree at least d. So, every non-zero element in $\mathfrak{m}F_i$ has degree at least $d+1$. It follows that, F_{i+1} has only generators of degree at least $d+1$. Hence it follows that $\beta_{i+1,j+1} = 0$ for every $j < d$. □

COROLLARY 5.4. *If $\{\beta_{i,j}\}$ are the graded Betti number of a graded finitely generated S-module M, the alternating sum*

$$B_j := \sum_{i=0}^{\infty} (-1)^i \beta_{i,j}$$

determines Hilbert's function of M by means of the following formula:

$$\operatorname{Hilb}_M(d) = \sum_j B_j \binom{r+d-j}{r}$$

Moreover, the values of B_j can be deduced inductively from $\operatorname{Hilb}_M(d)$*, that is to say*

$$B_j = \operatorname{Hilb}_M(j) - \sum_{n<j} B_n \binom{r+j-n}{r}$$

5.3. Determinantal ideals.

Assume that $S = k[x_0, x_1, x_2]$ is the graded polynomial ring; we will deal with graded and finitely generated S-modules only. Such modules have a minimal free resolution. Moreover, by Corollary 5.3, we know also that the minimal free resolution has length $\dim_S^{(\mathrm{proj})}(M)$.

PROPOSITION 5.4. *Let $I \subseteq S$ be the homogeneous ideal defining a finite set of points in $\mathbf{P}^2$. Therefore I has a minimal free resolution of length 1.*

PROOF. Because of the previous arguments, it is enough to show that S/I has projective dimension 1. By Auslander-Buchsbaum formula in the graded cases, one has

$$\operatorname{depth}(\mathfrak{m}, S/I) + \dim^{(\mathrm{proj})}(S/I) = \operatorname{depth}(\mathfrak{m}, S)$$

But $\operatorname{depth}(\mathfrak{m}, S/I) \leq \dim(S/I) = 1$ and the irrelevant ideal $\mathfrak{m}$ of S is not associated to I: in fact I can be viewed as intersection of prime ideals $\mathfrak{p}_x$, each one containing polynomials vanishing on the point x, where x varies among the finite set given; hence, I can not contain a copy of $k = S/\mathfrak{m}$ and this leads to say that $\operatorname{depth}(\mathfrak{m}, S/I) > 0$. Moreover, the indeterminates x_0, x_1, x_2 form a maximal regular sequence in S, so $\operatorname{depth}(\mathfrak{m}, S) = 3$. It follows that $\dim^{(\mathrm{proj})} S/I = 2$, but in a free resolution for S/I, the ideal I is the first module of syzygies for S/I, hence $\dim^{(\mathrm{proj})} I = 1$. □

In the following, we will settle in a more general context, assuming that R is a general noetherian ring; for every matrix Ψ with coefficients in R and arbitrary dimensions, we will write $I_t(\Psi)$ for the ideal in R generated by determinants of order t submatrices taken from Ψ. Such ideals are called *determinantal ideals* and own very remarkable properties: in the following classical theorem we show how they relate to free resolutions.

THEOREM 5.3. (HILBERT-BURCH) *Let us assume that an ideal I in a noetherian ring R has a free resolution of length 1, precisely*

$$0 \longrightarrow R^m \xrightarrow{\Psi} R^n \longrightarrow I \longrightarrow 0$$

Therefore:

(1) $n = m + 1$;
(2) $I = a \cdot I_t(\Psi)$, *for some non zero-divisor $a \in R$;*
(3) $\operatorname{depth}(I_t(\Psi), R) = 2$.

Conversely, given a non zero-divisor $a \in R$ and a $(t+1) \times t$-dimensional matrix Ψ with coefficients in R, such that $\operatorname{depth}(I_t(\Psi), R) \geq 2$, the ideal $I = a \cdot I_t(\Psi)$ admits a free resolution of length 1 like the above. Moreover, $\operatorname{depth}(I, R) = 2$ if and only if a is an unit in R.

Let us call *i-th order t minor t* the number $(-1)^i \det \Psi_i$, where Ψ_i is a the submatrix taken from Ψ removing the i-th row; therefore, we claim that the generator for I corresponding to the i-th of the chosen basis for G is a times the i-th order t minor of Ψ. We postpone the Hilbert-Burch

Theorem's proof to the end of the section, in order to derive some important tools and results regarding free resolutions.

If φ is a R-modules morphisms, we write $\mathrm{rk}(\varphi)$ for the *rank* of φ, that is to say the order of the maximal non-zero minor, and $I(\varphi)$ for the determinantal ideal $I_{\mathrm{rk}(\varphi)}(\Phi)$, where Φ is an arbitrary matrix representation for φ. Conventionally, we put $I_0(\varphi) := R$, in line with the characteristics of zero morphisms. Finally, we put also $\mathrm{depth}(R, R) = \infty$, in order to obtain $\mathrm{depth}(I_0(\varphi), R) = \infty$.

THEOREM 5.4. (BUCHSBAUM-EISENBUD) *A free modules complex*

$$\mathbf{F}: \quad 0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

over a noetherian ring R is a resolution if and only if for every $i \geq 0$ the following conditions hold:

(1) $\mathrm{rk}(\varphi_{i+1}) + \mathrm{rk}(\varphi_i) = \mathrm{rk}(F_i)$;
(2) $\mathrm{depth}(I(\varphi_i), R) \geq i$.

PROOF. See [**9**], or [**4**] for further details. □

In the remarkable case in which $R = k[x_0, \dots, x_r]$ for an algebraically closed field k, Buchsbaum-Eisenbud's Theorem has geometric meaning. We can think R as the function ring of $\mathbf{A}_k^{r+1} (= k^{r+1})$ (in the graded case, one could think to $\mathbf{P}^r$ in similar fashion) and, being $p \in \mathbf{A}_k^{r+1}$ a point, we let be $I(p)$ the ideal in R of functions vanishing in p. Let be $\mathbf{F}$ a complex satisfying the above hypotheses and let be

$$\mathbf{F}(p): \ 0 \longrightarrow F_m(p) \xrightarrow{\varphi_{m,p}} F_{m-1}(p) \longrightarrow \cdots \longrightarrow F_1(p) \xrightarrow{\varphi_{1,p}} F_0(p)$$

the complex obtained tensoring with the residue field $k(p) := R/I(p)$; $\mathbf{F}(p)$ could be understood as a finitely generated vector $k(p)$-spaces complex. A matrix representation for each $\varphi_{i,p}$ can be obtained evaluating in p the coefficient of the corresponding matrix representation for φ_i. Buchsbaum-Eisenbud's Theorem explains the relation between exactness of $\mathbf{F}$ and exactness of $\mathbf{F}(p)$.

COROLLARY 5.5. *Let*

$$\mathbf{F}: \quad 0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

be a free R-modules complex, with $R = k[x_0, \dots, x_r]$ and k algebraically closed. Let $X_i \subseteq \mathbf{A}_k^{r+1}$ be the set of points such that the complex $\mathbf{F}(p) := \mathbf{F} \otimes_R \kappa(p)$ fails to be exact in $F_i(p)$. Therefore $\mathbf{F}$ is exact if and only if X_i is empty for every i or $\mathrm{codim}(X_i) \geq i$ *for every i.*

PROOF. Let us assume $\mathbf{F}$ is exact and let us define

$$r_i := \sum_{j=i}^{m} (-1)^j \mathrm{rk}(F_j)$$

Note that, since every F_i is a free module,

$$\mathrm{rk}(\varphi_i) = \mathrm{rk}(F_i) - \mathrm{rk}(\ker \varphi_i) = \mathrm{rk}(F_i) - \mathrm{rk}(\varphi_{i+1})$$

where the last equality holds by exactness. Inductively, one shows that $\mathrm{rk}(\varphi_i) = r_i$. Hence, the second condition in Buchbaum-Eisenbud's Theorem equals asking that $\mathrm{depth}(I_{r_i}(\varphi_i), R) \geq i$. Conversely, let us suppose that $\mathrm{depth}(I_{r_i}(\varphi_i), R) \geq i$ for every $i \in I$. In general, we know that

$$\mathrm{rk}(\varphi_i) = \mathrm{rk}(F_i) - \mathrm{rk}(\ker(\varphi_i)) \geq \mathrm{rk}(F_i) - \mathrm{rk}(\varphi_{i+1})$$

hence $\mathrm{rk}(\varphi_i) \geq r_i$. Moreover, also $\mathrm{rk}(\varphi_i) \leq \mathrm{rk}(F_i)$ holds, so again $r_i = \mathrm{rk}(\varphi_i)$ and Buchsbaum-Eisenbud's Theorem can be used.

Let us define, now,

$$Y_i = \{p \in k^{r+1} \mid \mathrm{rk}(\varphi_i) < r_i\}$$

namely, the algebraic affine set defined by the ideal $I_{r_i}(\varphi_i)$. Since R is Cohen-Macaulay, depth and codimension coincide, hence $\mathrm{depth}(I_{r_i}(\varphi_i), R) = \dim(A_{I_{r_i}(\varphi_i)}) = \mathrm{codim}(Y_i)$. By Buchsbaum-Eisenbud's Theorem, it follows that $\mathbf{F}$ is exact if and only if $\mathrm{codim}(Y_i) \geq i$ for every $i \geq 1$.

On the other hand, the complex $\mathbf{F}(p)$ is a complex of finitely generated vector k-space, and it is exact if and only if $\mathrm{rk}(\varphi_{j,p}) + \mathrm{rk}(\varphi_{j+1,p}) = \dim_k(F_j(p))$ holds for every j; this the same to ask $\mathrm{rk}(\varphi_{j,p}) + \mathrm{rk}(\varphi_{j+1,p}) \geq \dim_k(F_j(p))$ and this holds for every $j \geq i$ if and only if $\mathrm{rk}(\varphi_j) \geq r_j$ for every $j \geq i$. Therefore $\mathbf{F}(p)$ is exact in $F_j(p)$ for every $j \geq i$ if and only if $p \notin \bigcup_{j \geq i} Y_j = Y_{(i)}$. Now, codimension of $Y_{(i)}$ is the minimum among codimensions of Y_j for $j \geq i$, hence $\mathrm{codim}(Y_{(i)}) \geq i$ for every i if and only if $\mathrm{codim}(Y_i) \geq i$ for every i. By the previous arguments, we can conclude the proof. □

A consequence of Hilbert-Burch's Theorem is that every ideal having a length 1 free resolution contains a non zero-divisor. Buchsbaum-Eisenbud's Theorem allows us a more general statement.

Theorem 5.5. (Auslander-Buchsbaum) If an ideal I has a finite length free resolution, therefore it contains a non zero-divisor.

Proof. Let us consider the free resolution

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0$$

Now, determinantal ideal $I(\varphi_1)$ equals exactly I. Hence, by Buchsbaum-Eisenbud's Theorem, we have $\mathrm{depth}(I, R) = \mathrm{depth}(I(\varphi_1), R) \geq 1$, that is to say I contains a non zero-divisor. □

Before we begin the proof of Hilbert-Burch's Theorem, a preliminary linear algebra result is needed.

Lemma 5.2. *Let Φ be a $(t+1) \times t$-dimensional matrix having coefficients in a commutative ring R and let $a \in R$. Therefore the composite map*

$$R^t \xrightarrow{\Phi} R^{t+1} \xrightarrow{\Delta} R$$

is the zero map, where Δ is defined by row matrix $(a\Delta_1 \cdots a\Delta_{t+1})$, calling Δ_i the i-th order t minor taken from the matrix Φ.

PROOF. Let $\Phi = (a_{ij})$. Therefore $\Delta\Phi$ has coefficients of the form

$$a \cdot \sum_j \Delta_j a_{ij}$$

and this equals the Laplace expansion for the determinant of a $(t+1) \times (t+1)$-dimensional matrix, obtained repeating twice the i-th column. By determinant theory, it follows readily that $\Delta\Phi = 0$. □

Now we can prove Hilbert-Burch's Theorem.

PROOF. *(Hilbert-Burch's Theorem)* We prove the last statement first; let us assume $I_t(\Psi)$ has depth at least 2 and let a be a non zero-divisor. We have to show that $I = a \cdot I_t(\Psi)$ has a length 1 free resolution. Since $I_t(\Psi)$ has depth at least 2, the matrix Ψ has rank t (because Ψ can not have greater rank) and the map Δ defined in the above Lemma 5.2 has necessarily rank 1. Therefore $I(\Delta) = I_1(\Delta) = a \cdot I(\Psi)$ and $I(\Delta)$ has depth at least 1. By Buchsbaum-Eisenbud's Theorem, we conclude that the exact sequence

$$0 \longrightarrow F \xrightarrow{\Psi} G \longrightarrow I \longrightarrow 0 \tag{5.3.1}$$

is the desired resolution of $I = a \cdot I(\Psi)$.

Now let us prove the first part. Using inclusion $I \subseteq R$ we can state that there is a free resolution of R/I such that

$$0 \longrightarrow F \xrightarrow{\Psi} G \xrightarrow{A} R$$

where A is a non-zero map of rank 1. By Buchsbaum-Eisenbud's Theorem, it must follow that $\mathrm{rk}(M) = t$ and $\mathrm{rk}(G) = t+1$. Similarly, it must hold that $\mathrm{depth}(I_t(\Psi), R) = \mathrm{depth}(I(\Psi), R) \geq 2$. Moreover, one could prove that for every prime ideal $\mathfrak{p} \in \mathrm{Spec}(R/I_t(\Psi))$

$$\dim(R_{\mathfrak{p}}) \leq 2$$

holds, hence $\mathrm{depth}(I_t(\Psi), R) \leq \dim(R_{\mathfrak{p}}) \leq 2$; finally, $\mathrm{depth}(I_t(\Psi), R) = 2$. Let $\Delta = (\Delta_1, \ldots, \Delta_{t+1})$ be the map defined in Lemma 5.2; dualizing the sequence (5.3.1) and restricting, we find that

$$0 \longrightarrow \hom(R, R) \xrightarrow{\Delta^\vee} \hom(G, R) \xrightarrow{\Psi^\vee} \hom(F, R)$$

is a complex since $\Psi^\vee \Delta^\vee = (\Delta\Psi)^\vee = 0$. Moreover, by Buchsbaum-Eisenbud's Theorem, the above complex is exact: in fact $\mathrm{rk}(\hom(G, R)) = \mathrm{rk}(G) = \mathrm{rk}(\Psi) + \mathrm{rk}(A) = \mathrm{rk}(\Psi^\vee) + \mathrm{rk}(A^\vee)$ and depths respect to ideals are compatible. On the other hand, the range of Ψ is contained in the kernel of A, hence $\mathrm{im}(A^\vee) \subseteq \ker(\Psi^\vee)$, inducing thus a map $a : R \longrightarrow R$ which makes the following diagram to commute:

$$\begin{array}{ccccc}
\hom(R, R) & \xrightarrow{A^\vee} & \hom(G, R) & \xrightarrow{\Psi^\vee} & \hom(F, R) \\
\downarrow a & & \downarrow = & & \downarrow = \\
\hom(R, R) & \xrightarrow{\Delta^\vee} & \hom(G, R) & \xrightarrow{\Psi^\vee} & \hom(F, R)
\end{array}$$

The arrow a is represented by a 1×1 matrix, and we will call a its only coefficient, with a slight abuse of notation. By Corollary 5.5, the ideal I contains a non zero-divisor element; but looking at the previous diagram, it is clear that $I = a \cdot I_t(M)$ is contained in (a), so a is not a zero-divisor.

It remains to show the last sentence. For $I_t(\Psi)$ has depth 2, the ideal $a \cdot I_t(\Psi)$ keeps having depth 2 if and only if a is an unit in R. □

5.4. Invariants in resolutions.

Hilbert Burch's Theorem is an useful tool to study certain invariants that arise when we deal with resolutions of finite sets of points in the plane $\mathbf{P}^2$. In the following discussion, we will write $I_X \subseteq S$ for the defining ideal of a set $X \subseteq \mathbf{P}^2$ containing a finite number of points, and we will write $S_X = S/I_X$ for the homogeneous coordinate ring of X. By Proposition 5.3, we know that I_X has projective dimension 1 and that S_X has projective dimension 2. Let us assume that the minimal free resolution of S_X has the form

$$\mathbf{F}:\ 0 \longrightarrow F \xrightarrow{\Psi} G \longrightarrow S$$

where G is a $t+1$ rank free S-module; by Hilbert-Burch's Theorem it follows that F has rank t. One could write explicitly

$$G = \bigoplus_{i=1}^{t+1} S(-a_i), \quad F = \bigoplus_{i=1}^{t} S(-b_i)$$

where $S(-a)$ means the rank 1 free S-module generated by degree a elements; in other words, the numbers a_i are the degree of minimal generators for $I = I_t(\Psi)$. Hence, the degree of the (i,j)-th elements of matrix Ψ is $b_j - a_i$. As we will explain later, we are interested in the elements belonging to the main diagonals of M: then write $e_i := b_i - a_i$ and $f_i = b_i - a_{i+1}$ to indicate their degrees.

To avoid any confusion, let us assume that the basis chosen for F and G are ordered, that is $a_1 \geq \ldots \geq a_{t+1}$ and $b_1 \geq \ldots \geq b_t$, hence $f_i \geq e_i$, $f_i \geq e_{i+1}$. Since minimal free resolutions are unique up to isomorphism, the number a_i, b_i, e_i and f_i are thus invariant up to isomorphism; however, they can not assume arbitrary integer values, being determined by e_i and f_i.

Proposition 5.5. *If I is the defining ideal for a finite set of points in $\mathbf{P}^2$ and*

$$\mathbf{F}: \quad \cdots \longrightarrow \bigoplus_{i=1}^{t} S(-b_i) \xrightarrow{\Psi} \bigoplus_{i=1}^{t+1} S(-a_i) \longrightarrow S$$

is the minimal free resolution for S/I, and e_i, f_i denote the degrees of elements of Ψ belonging to the two main diagonals, therefore, for every i the following properties hold:

- $e_i \geq 1,\ f_i \geq 1$;
- $a_i = \sum_{j<i} e_i + \sum_{j \geq i} f_i$;

- $b_i = a_i + e_i$ *and also* $\sum_{i=1}^{t} b_i = \sum_{i=1}^{t+1} a_i$.

If, moreover, the basis are ordered in a way such that $a_1 \geq \ldots \geq a_{t+1}$ *and* $b_1 \geq \ldots \geq b_t$*, therefore* $f_i \geq e_i$ *and* $f_i \geq e_{i+1}$.

Proof. The ideal I has codimension 2 and S is Cohen-Macaulay, hence I has depth 2. Therefore, using Hilbert-Burch's Theorem, the non zero-divisor $a \in S$ associated to the resolution $\mathbf{F}$ is an unit in S; for S is a polynomial ring, a is a constant and a_i are actually the degrees of minors of Ψ.

Let us assume, without loss of generality, that basis are ordered as in hypothesis. We will prove that $e_i \geq 1$ (by order this will also imply that $f_i \geq 1$). Let $\Psi = (m_{i,j})$; by minimality of $\mathbf{F}$, no $m_{i,j}$ can be a non-zero constant (recall that $\delta(F_i) \subseteq \mathfrak{m}F_{i-1}$ where $\mathfrak{m}$ is the irrelevant ideal in S), hence if $e_i \leq 0$ then $m_{i,i} = 0$. Moreover, if $p \leq i$ and $q \geq i$ we have

$$\deg(m_{p,q}) = b_q - a_p \leq b_i - a_i = e_i$$

by the order chosen. Thus, if $e_i \leq 0$, we have $m_{p,q} = 0$ for every pair (p, q) such that $p \leq i$ and $q \geq i$. In this way, one can prove that at least one among the order t minors in Ψ vanishes; since by Hilbert-Burch's Theorem it should be a minimal generator for I, we find a contradiction. Hence $e_i \geq 1$.

The identity

$$a_i = \sum_{j<i} e_i + \sum_{j \geq i} f_i$$

follows from Buchsbaum-Eisenbud's Theorem. In fact a_i is the degree of the determinant Δ_i obtained from Ψ removing the i-th row and a term in the expansion of such determinant is

$$\prod_{j<i} m_{j,j} \cdot \prod_{j \geq i} m_{j+1,j}$$

Finally, since $e_i = b_i - a_i$, we find

$$\sum_{i=1}^{t} b_i = \sum_{i=1}^{t} a_i + \sum_{i=1}^{t} e_i = \sum_{i=1}^{t+1} a_i$$

finishing the proof. □

Above Proposition 5.5 gives an upper bound to the minimal number of generators required for the ideal of points lying over a curve of given degree. Such a bound was known even before the introduction of free resolution and can actually be proved separately.

Corollary 5.6. *If* I *is the defining ideal of a finite set of point in* $\mathbf{P}^2$ *lying over a curve of degree* d*, therefore it can be generated by* $d+1$ *elements.*

Proof. Let $t + 1$ be the least number of generators for I; then, by Proposition 5.5, the degree a_i of the i-th minimal generator is the sum of t numbers, each one being ≥ 1, so $t \leq a_i$. Since the curve has degree d, the ideal I must contain a polynomial of degree d, hence we have $a_i \leq d$ for some i. It follows $t+1 \leq d+1$ as we wished. $\square$

Computing Hilbert's function of a finite set X of points in $\mathbf{P}^2$ (namely, of its S-module of coordinates) using information collected from a free resolution, we can use the properties of invariants e_i, f_i. Function $\mathrm{Hilb}_X(d)$ is constant when $d \gg 0$, and its definitive value is the cardinality of the set X, that is to say, its degree $\deg X$.

If X is obtained as a complete intersection (namely, I_X has $t+1=2$ minimal generators) of two distinct curves having respectively degree e and f, therefore with previous notations one has $t=1$, $e_1=e$, $f_1=f$. By Bézout's Theorem, the degree of X thus should be $ef = e_1 f_1$. Indeed, we have the following generalization.

Corollary 5.7. *Let X be a finite set of points in $\mathbf{P}^2$. Therefore, with the previous notations,*

$$\deg X = \sum_{i \leq j} e_i f_j$$

Proof. See [**6**]. $\square$

5.5. Examples.

In this section we'll discuss some examples regarding the theory shown in the above sections.

Firstly, we may determine the possible free resolutions for a finite set of points lying over an irreducible conic curve. Let us assume that $X \subseteq \mathbf{P}^2$ is a finite set of points lying over an irreducible conic, defined by a quadratic form q. With the previous notations, we have $a_{t+1} = 2$ and, since $a_{t+1} = \sum_{i=1}^{t} e_i$, only two possibilities are allowed: if $t=1$, then $e_1 = 2$ and if $t=2$, then $e_1 = e_2 = 1$.

- In the case $t=1$, therefore X is the complete intersection between an irreducible conic and a curve of degree $a_1 = d$, defined by a polynomial g. One knows that $\deg X = 2d$ by Bézout's Theorem (or even by the above formula). Moreover, $b_1 = d+2$ and the desired resolution has the form:

$$0 \longrightarrow S(-d-2) \xrightarrow{\varphi_2} S(-2) \oplus S(-d) \xrightarrow{\varphi_1} S \longrightarrow S_X$$

  where $\varphi_2 = (g \ \ -q)$ and $\varphi_1 = (q, g)$.
- If, instead, one has $t=2$, we have $e_1 = e_2 = 1$; let us assume that the conic $q=0$ is irreducible. By Proposition 5.5, the resolution shall have the following form:

$$0 \longrightarrow S(-1-f_1-f_2) \oplus S(-2-f_2) \xrightarrow{\Psi} S(-f_1-f_2) \oplus S(-1-f_2) \longrightarrow S$$

  where we clearly assumed that $f_1 \geq e_1 = 1$ and $f_1 \geq e_2 = 1$ and $f_2 \geq e_2 = 1$. By Hilbert-Burch's Theorem, q is multiple of the order 2 minor obtained from Ψ deleting the third row; since q is irreducible, each of the four entries in the such submatrix obtained from Ψ must be non zero. Moreover, the element of M placed in the above right corner has degree $e_1 + e_2 - f_1 \leq 1$ and if it had zero degree then it should be equal to zero, contradicting the minimality of resolution. Therefor $e_1 + e_2 - f_1 = 1$, that is $f_1 = 1$. Moreover, $a_3 = \sum_{j<3} e_j = 2$ and we have

$$a_1 = a_2 = 1 + f_2, \quad b_1 = b_2 = 2 + f_2$$

Hebe, one can argue that the resolution has the following form:

$$0 \longrightarrow S^2(-2 - f_2) \longrightarrow S^2(-1 - f_2) \oplus S(-2) \longrightarrow S$$

Using the above formula for degree, we find that $\deg X = 2f_2 + 1$. The two cases in this situation are hence distinguished by the parity of degree.

Let us now concentrate on points over conics with more detail. We know that vector space of quadratic forms in 3 indeterminates has dimension 5; it follows that 5 distinct points lie over an unique conic, since imposing the passage from a point lead to a single linear condition. We can thus use the ideas developed in the above in order to study resolution of sets containing up to 5 points. The most interesting case is to consider 4 non collinear points, $X = \{p_1, \ldots, p_4\}$.

Imposing the 4 passage condition over a conic leaves two free parameters; hence two distinct conics containing X must exist.

Let us assume firstly that no triple of points in X lies over a line. In this case, the only possibility is that X is contained in the intersection of the following conics, each one composed by the union of two lines:

$$C_1 := \overline{p_1 - p_2} \cup \overline{p_3 - p_4}, \qquad C_2 = \overline{p_1 - p_3} \cup \overline{p_2 - p_4}$$

In this case X is complete intersection of C_1 and C_2 and resolution has the form

$$0 \longrightarrow S(-4) \longrightarrow S^2(-2) \longrightarrow S$$

with Betti numbers $\beta_{0,1} = 1, \beta_{1,2} = 2, \beta_{2,4} = 1$.

Let us suppose, instead, that p_1, p_2, p_3 lie over a line L, without loss of generality. So let be L_1 and L_2 two lines by p_4 containing none of p_1, p_2, p_3. It follows that X is contained in the intersection of

$$C_1 = L \cup L_1, \quad C_2 = L \cup L_2$$

Since L belongs to both C_1, C_2, the set X is not obtained as complete intersection of conics C_1, C_2 and, by Corollary 5.6, the ideal of X requires exactly 3 generators. Hence, by Proposition 5.5,

$$a_1 = f_1 + f_2, \; a_2 = e_1 + f_2, \quad a_3 = e_1 + e_2$$

Since $a_3 = 2$, we have $e_1 = e_2 = 1$. By degree formula, we find that

$$4 = e_1 f_1 + e_1 f_2 + e_2 f_2 = f_1 + 2f_2$$

that is to say, $f_1 = 2$ and $f_2 = 1$. Degrees of generators are, thus,

$$a_1 = 3,\ a_2 = a_3 = 2, \qquad b_1 = 4,\ b_2 = 3$$

Therefore, the ideal of X is generated by quadric equations for C_1 and C_2, plus a cubic equation. In fact, the previous numbers mean that the resolution has the form

$$S(-3) \oplus S(-4) \longrightarrow S^2(-2) \oplus S(-3) \longrightarrow S$$

From a geometric point of view, we discover the following statement: any set of 4 non collinear points, 3 of which lying over a line, belongs to the intersection of two conics and a cubic curve.

Part 2

CASTELNUOVO-MUMFORD REGULARITY

CHAPTER 6

Local cohomology.

The following dissertation about local cohomology and about the many ways to link it with other cohomology theories can be found, in a condensed synthesis, in [**10**] or [**9**]. The most complete reference about local cohomology theory, however, rests [**14**].

6.1. Main definitions.

We can state a quite general definition. Let $(X, \mathscr{O}_X)$ be a ringed space, $Z \subseteq X$ a closed subscheme and let $\mathscr{F}$ be a $\mathscr{O}_X$-modules sheaf. We define the functor $\Gamma_Z : \mathbf{Sh}(X) \longrightarrow \mathbf{CRings}$ setting

$$\Gamma_Z(\mathscr{F}) = \ker(\rho^X_{X\backslash Z})$$

where $\rho^X_{X\backslash Z} : \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X\backslash Z, \mathscr{F})$ is the standard restriction map. Clearly, we have too

$$\begin{aligned} \Gamma_Z(\mathscr{F}) &= \{s \in \Gamma(X, \mathscr{F}) \mid \mathrm{supp}(s) \subseteq Z\} = \\ &= \{s \in \Gamma(X, \mathscr{F}) \mid s_x = 0 \text{ for every } x \in X\backslash Z\} \end{aligned}$$

Lemma 6.1. *The functor Γ_Z is left exact.*

Proof. Let $0 \longrightarrow \mathscr{F}_1 \xrightarrow{f} \mathscr{F}_2 \xrightarrow{f} \mathscr{F}_3 \longrightarrow 0$ be a $\mathscr{O}_X$-modules sheaves exact sequence. Therefore, we have a diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \Gamma_Z(\mathscr{F}_1) & \xrightarrow{p} & \Gamma_Z(\mathscr{F}_2) & \xrightarrow{q} & \Gamma_Z(\mathscr{F}_3) \\ & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_3} \\ 0 & \longrightarrow & \Gamma(X, \mathscr{F}_1) & \xrightarrow{f_X} & \Gamma(X, \mathscr{F}_2) & \xrightarrow{g_X} & \Gamma(X, \mathscr{F}_3) \\ & & \downarrow{\scriptstyle \rho^X_{X\backslash Z}} & & \downarrow{\scriptstyle \rho^X_{X\backslash Z}} & & \downarrow{\scriptstyle \rho^X_{X\backslash Z}} \\ & & \Gamma(X\backslash Z, \mathscr{F}_1) & \xrightarrow{f_{X\backslash Z}} & \Gamma(X\backslash Z, \mathscr{F}_2) & \xrightarrow{g_{X\backslash Z}} & \Gamma(X\backslash Z, \mathscr{F}_3) \end{array}$$

commuting because f, g are sheaves morphisms. Now, we have straightforwardly

$$\ker(p) = \ker(i_2 \circ p) = \ker(f_X \circ i_1) = 0$$

hence Γ_Z is left exact. □

Thanks to Lemma, the functor Γ_Z admits right derived functors, that we will call $H^i_Z(-)$; in particular, for every sheaf of $\mathscr{O}_X$-modules, let us define $H^i_Z(\mathscr{F})$ the i-th *local cohomology group of* $\mathscr{F}$ *with support in* Z. The same definition, in particular, can be arranged in the category of quasi-coherent sheaves over a general scheme X.

Let us suppose that $X = \mathrm{Spec}(A)$ is an affine scheme, with A any ring, and let $Z = V(I)$ for some ideal $I \subseteq A$. We know, hence, that $\mathscr{F} = M^\sim$ for some A-module M. We can give the following explicit definition of local cohomology. Define

$$H^0_I(M) := \{m \in M \mid mI^r = 0 \text{ for some } r \in \mathbb{N}\} = \bigcup_{n \geq 0} (0 :_M I^n)$$

An equivalent definition of the 0-th module can be given in the following way: note that every $m \in (0 : I^n)$ gives rise to a linear morphism $A/I^n \longrightarrow M$, setting $1 + I^n \mapsto m$; it is well defined, because if $a + I^n = b + I^n$ then $a - b \in I^n$ and

$$a + I^n = am, \quad b + I^n \mapsto bm$$

but $(b - a)m = 0$, since m is annihilated by I^n. Conversely, it is clear that every linear morphism $f : A/I^n \longrightarrow M$ sets an element $f(1 + I^n) \in (0 :_M I^n)$. More remarkably, if $n \leq m$, it is clear that $A/I^n \subseteq A/I^m$. Since $(0 :_M I^n) \simeq \hom_A(A/I^n, M)$, we can thus write

$$H^0_I(M) \simeq \lim_{n \to \infty} \hom_A(A/I^n, M)$$

where the inductive limit is trivial, every arrow being an injection. We obtain, this way, a left exact functor $\mathbf{Mod}_A \longrightarrow \mathbf{Mod}_A$ setting $M \mapsto H^0_I(M)$, and it gives rise to right derived functors $H^i_I(-)$; the i-th right derived functor $H^i_I(M)$ is the i-th module of local cohomology of M with support in $V(I)$. If $(A, \mathfrak{m})$ is a local ring, then one simply calls $H^i_{\mathfrak{m}}(M)$ the i-th *local cohomology module of* M. It is rather clear that

$$H^0_I(M) = \ker(M \longrightarrow \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M}))$$

so actually we have $H^0_I(M) = \Gamma_{V(I)}(\widetilde{M})$, accordingly with the previous definition.

Now one could note that the i-th right derived functor of $\hom_A(A/I^n, M)$ is exactly $\mathrm{Ext}^i_A(A/I^n, M)$ namely

$$H^i_I(M) \simeq \lim_{n \to \infty} \mathrm{Ext}^i_A(A/I^n, M)$$

Because of this description, local cohomology modules preserve many of the properties typical of Ext^i_A functors; amongst them, the long sequence property seems one of the most significant.

Lemma 6.2. *Every element in $H^i_I(M)$ is annihilated by a power of I.*

Proof. By definition of $H^i_I(M)$, each one of its elements belongs to the homomorphic image of some $\mathrm{Ext}^i_A(R/I^n, M)$, and this whole module is annihilated by I^n. □

LEMMA 6.3. *Assume that* $\{J_n\}_{n\geq 0}$ *is a decreasing sequence of ideals, and that it is cofinal with* $\{I_n\}$ *(namely, for every* I_n *there exists a* $J_{\alpha(n)}$ *such that* $I_n \subseteq J_{\alpha(n)}$*). Therefore*

$$H^i_I(M) = \lim_{n\to\infty} \operatorname{Ext}^i_A(A/J^n, M)$$

PROOF. Cofinal sets have the same inductive limit. □

PROPOSITION 6.1. *If two ideals* $I, J \subseteq A$ *have the same radical, therefore* $H^i_I(M) \simeq H^i_J(M)$ *for every* $i \geq 0$*. Moreover, if* $I = (x_1, \dots, x_n)$ *and* $I_s = (x_1^s, \dots, x_n^s)$*, then* $H^i_I(M) = H^i_{I_s}(M)$ *for every* $i \geq 0$ *and for every* $s > 0$*.*

PROOF. Since $\sqrt{I} = \sqrt{J}$, any power of I is contained in some power of J, hence the sequences $\{J_n\}$ and $\{I_n\}$ are each other cofinal. By Lemma 6.3, local cohomology does not change:

$$H^i_I(M) = \lim_{n\to\infty} \operatorname{Ext}^i_A(A/J^n) = H^i_J(M)$$

Moreover, one see that $\sqrt{I_s} = \sqrt{I}$, so the last statements follows readily. □

The following theorem contains the relevant property.

THEOREM 6.1. *Let* M', M, M'' *be three finitely generated* A*-modules and let* $I \subseteq A$ *be an ideal. Let us assume that*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence. Therefore, there exists a long exact sequence of local cohomology modules:

$$\cdots \longrightarrow H^n_I(M') \longrightarrow H^n_I(M) \longrightarrow H^n_I(M'') \longrightarrow H^{n+1}_I(M') \longrightarrow \cdots$$

PROOF. Given the above short exact sequence of modules, it is known that there exists a long exact sequence involving extension functors, for every $n \in \mathbb{N}$:

$$\cdots \longrightarrow \operatorname{Ext}^p_A(A/I^n, M') \longrightarrow \operatorname{Ext}^p_A(A/I^n, M) \longrightarrow \operatorname{Ext}^p_A(A/I^n, M'') \longrightarrow$$
$$\longrightarrow \operatorname{Ext}^{p+1}_A(A/I^n, M) \longrightarrow \cdots$$

Because inductive limits are exact functors, one can readily take to limit for $n \to \infty$, like above; then the desired exact sequence is found. □

One could prove that local cohomology relates to depth, in the following way.

PROPOSITION 6.2. *Let* A *be a noetherian ring and* M *a finitely generated* A*-module. If* $I \subseteq A$ *is an ideal such that* $IM \neq M$*, therefore*

$$\operatorname{depth}(I, M) = \min\{i \in \mathbb{N} \mid H^i_I(M) \neq 0\}$$

PROOF. Proceed by induction over $s = \min\{i \in \mathbb{N} \mid H^i_I(M) \neq 0\}$. If $s = 0$, then for some $n > 0$ the module $\hom_A(A/I^n, M)$ is not trivial, hence I^n contains at least a non M-regular element, that is to say that there exists $x \in I^n$ such that $xM = 0$. Hence $I^n \subseteq \mathfrak{p}$ for some associated prime ideal $\mathfrak{p} \in \mathrm{Ass}(M)$. Then, by primality, we have also $I \subseteq \mathfrak{p}$ and

$$0 \neq \hom_A(A/\mathfrak{p}, M) \subseteq \hom_A(A/I, M)$$

This proves $\mathrm{depth}(I, M) = 0$. Conversely, if $\mathrm{depth}(I, M) = 0$, then

$$0 \neq \hom_A(A/I, M) \subseteq \lim_{n\to\infty} \hom_A(A/I^n, M) = H^0_I(M)$$

and this proves the claim.

Now suppose $s > 0$; so necessarily $\mathrm{depth}(I, M) > 0$, by the above considerations. Let $x \in I$ be a M-regular element (namely, a non zero-divisor); we have thus the following exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

and this leads the long sequence

$$\cdots \longrightarrow H^{i-1}_I(M) \longrightarrow H^{i-1}_I(M/xM) \longrightarrow H^i_I(M) \longrightarrow \cdots$$

By inductive hypothesis, $H^{i-1}_I(M/xM) = 0$, so we have the exact sequence

$$0 \longrightarrow H^{s-1}_I(M/xM) \longrightarrow H^s_I(M) \xrightarrow{x} H^s_I(M) \longrightarrow \cdots$$

Since $H^s_I(M) \neq 0$, the map acting as multiplication by x can not be injective; in fact, x^n annihilates $\mathrm{Ext}^s_A(A/I^n, M)$ so also $H^i_I(M)$. It follows that $H^{s-1}_I(M/xM) \neq 0$. Using again inductive hypothesis, we find $\mathrm{depth}(I, M/xM) = s - 1$ and, since x is a M-regular element, $\mathrm{depth}(I, M) = s$. □

REMARK 6.1. Recall that, with above hypotheses,

$$\mathrm{depth}(I, M) = \min\{i \in \mathbb{N} \mid \mathrm{Ext}^i_A(A/I, M) \neq 0\}$$

holds. This allows us to transform the most part of theorems involving local cohomology in equivalent results involving extension functors.

6.2. Local cohomology, Čech complex and sheaf cohomology.

This section is devoted to explain how it is possible to express local cohomology modules using a Čech-like complex, in order to show the relation with coherent sheaf cohomology. Let A be a noetherian ring and let M be a finitely generated A-module. If $I = (x_1, \ldots, x_n)$ is an ideal and $s > 0$ is an integer, let us write $x^s_\bullet$ for the sequence $x^s_1, \ldots, x^s_n$ and let us consider the Koszul complex $\mathbf{Kosz}(x^s_\bullet, M) := \mathbf{Kosz}(x^s_\bullet) \otimes_A M$ as a cohomological complex. For every $s > 0$, we have

$$H^0(\mathbf{Kosz}(x^s_\bullet, M)) = (0 :_M (x^s_\bullet))$$

Moreover, the Koszul complexes built in this way can be organized in an inductive system, using the natural maps

$$\mathbf{Kosz}(x^s_\bullet) \longrightarrow \mathbf{Kosz}(x^{s+1}_\bullet)$$

induced in degree 0 by the linear map $A^n \longrightarrow A^n$ which acts multiplicating the i-th entry by x_i. In the 0-th cohomology module, these maps induce an inclusion

$$(0 :_M (x_\bullet^s)) \subseteq (0 :_M (x_\bullet^{s+1}))$$

Let now

$$\mathbf{Kosz}(x_\bullet^\infty) := \lim_{s\to\infty} \mathbf{Kosz}(x_\bullet^s)$$

and let $\mathbf{Kosz}(x_\bullet^\infty, M) := \mathbf{Kosz}(x_\bullet^\infty) \otimes_A M$. Therefore we obtain

$$H_I^0(M) = \lim_{s\to\infty} H^0(\mathbf{Kosz}(x_\bullet^s, M)) = H^0(\mathbf{Kosz}(x_\bullet^\infty, M)) \tag{6.2.1}$$

using the definition of 0-th local cohomology module. Hence we have isomorphisms for every $i \geq 0$:

$$H_I^i(M) \simeq H^i(\mathbf{Kosz}(x_\bullet^\infty, M))$$

since both members are the right derived functor of the same functor, as shown in (6.2.1). It is not difficult to prove that $\mathbf{Kosz}(x_\bullet^\infty)$ coincide actually with the Čech complex that follows ([**9**]):

$$\check{\mathbf{C}}(x) : \quad 0 \longrightarrow A \longrightarrow \bigoplus_{1\leq i\leq n} A_{x_i} \longrightarrow \bigoplus_{1\leq i_1<i_2\leq n} A_{x_{i_1}x_{i_2}} \longrightarrow \cdots$$

One has, therefore

$$H_I^i(M) = H^i(\check{\mathbf{C}}(x) \otimes_A M) \tag{6.2.2}$$

Now, let us explain how these ideas are used to characterize local cohomology using sheaf cohomology.

If we take the graded polynomial ring $A = k[x_0, \dots, x_r]$ and $I = \mathfrak{m}$ the irrelevant ideal, let M be an A-module and $\widetilde{M}$ the usual coherent sheaf over $X = \mathbf{P}_k^r$. It is therefore possible to establish a relation between cohomology of $\widetilde{M}$ as a sheaf over X and local cohomology of M, as A-module. Let $\mathfrak{U} = \{U_i\}$ be the open cover of X made up by the open subsets $U_i := X\backslash V(x_i)$; hence, we can build the Čech chain complex for the sheaf $\widetilde{M}$ and relative to $\mathfrak{U}$, namely

$$\check{\mathbf{C}}(\mathfrak{U}, \widetilde{M}) : 0 \longrightarrow \bigoplus_{0\leq i_1\leq r} M_{(x_{i_1})} \longrightarrow \bigoplus_{0\leq i_1<i_2\leq r} M_{(x_{i_1}x_{i_2})} \longrightarrow \cdots$$

Clearly, this complex is the degree 0 part taken out of the complex $\check{\mathbf{C}}(x)_{\mathrm{tr}} \otimes_A M$, where

$$(\check{\mathbf{C}}(x)_{\mathrm{tr}})^i := (\check{\mathbf{C}}(x))^{i+1}$$

is the truncated Čech complex. Hence, using additivity of cohomology,

$$H^i(\check{\mathbf{C}}(\mathfrak{U}, \widetilde{M})) \simeq H^i(\check{\mathbf{C}}(x)_{\mathrm{tr}} \otimes_A M)_0$$

Since also

$$H^i(\mathbf{P}_k^r, \widetilde{M}) \simeq H^i(\check{\mathbf{C}}(\mathfrak{U}, \widetilde{M})) \simeq H^i(\check{\mathbf{C}}(x)_{\mathrm{tr}} \otimes_A M)_0 \simeq H_{\mathfrak{m}}^{i+1}(M)_0$$

twisting the modules we find

$$H^i(\mathbf{P}_k^r, \widetilde{M}(n)) \simeq H_{\mathfrak{m}}^{i+1}(M)_n$$

It follows, therefore, that

$$H^{i+1}_{\mathfrak{m}}(M) \simeq \bigoplus_{n\in\mathbb{Z}} H^i(\mathbf{P}^r_k, \widetilde{M}(n)) \tag{6.2.3}$$

holds for every $i \geq 1$.

Note that the above isomorphism fails for $i = 0$; in fact, note that the functor

$$\mathbf{QCohSh}(\mathbf{P}^r_k) \longrightarrow \mathbf{Mod}_A$$
$$\mathscr{F} \mapsto \bigoplus_{n\in\mathbb{Z}} H^0(\mathbf{P}^r_k, \mathscr{F}(n))$$

and the functor

$$\mathbf{Mod}_A \longrightarrow \mathbf{QCohSh}(\mathbf{P}^r_k)$$
$$M \mapsto \widetilde{M}$$

are not each the inverse of the other, since in general

$$\bigoplus_{n\in\mathbb{Z}} H^0(\mathbf{P}^r_k, \widetilde{M}(n)) \neq M$$

Anyway, one could prove that the behaviour in degree 0 is ruled by the following exact sequence:

$$0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow \bigoplus_{n\in\mathbb{Z}} H^0(\mathbf{P}^r_k, \widetilde{M}(n)) \longrightarrow H^1_{\mathfrak{m}}(M) \longrightarrow 0 \tag{6.2.4}$$

6.3. Further results.

The most part of what follows here is essentially obtained as application of the previous results.

Corollary 6.1. *If $I = (x_1, \dots, x_t)$ therefore $H^i_I(M) = 0$ for $i > t$.*

Proof. We know that $H^i_I(M) = H^i(\check{\mathbf{C}}(x) \otimes_A M)$ if $i > 0$ and the Čech complex has length $t > 0$. □

Corollary 6.2. *Let M be a graded S-module of finite length, with $S = k[x_0, \dots, x_r]$. Then $H^0_I(M) = M$ and $H^i_I(M) = 0$ for every $i > 0$.*

Remark 6.2. (Change of ring) Let $\varphi : A \longrightarrow B$ be a rings morphism and let $I \subseteq A$ be an ideal. Recall that the *extension* of I under φ is the ideal $e(I)$ generated by $\varphi(A)$ in B. Now, if M is a B-module, it can be understood as an A-module by means of φ. However, the relation of *change of ring* (or basis) behave in a more cumbersome way than expected: in fact, is not clear how to establish a link between $\mathrm{Ext}^i_A(A/I^n, M)$ and $\mathrm{Ext}^i_B(B/e(I)^n, M)$ without the use of spectral sequences. Surprisingly everything get fixed when taking the inductive limit for $n \to \infty$.

Proposition 6.3. *Let $\varphi : A \longrightarrow B$ be a noetherian rings morphism. With the above notation, the isomorphism $H^i_I(M) \simeq H^i_{e(I)}(M)$ holds for every $i \geq 0$.*

PROOF. If $x \in A$ is an arbitrary element, therefore the localization M_x does not change under change of rings: in fact, M_x is the set of ordered pairs (m, x^n) modulo the equivalence relation that identifies (m, x^n) with $(m', x^{n'})$ if and only if $x^r(mx^{n'} - m'x^n) = 0$ for some $r \geq 0$. Hence, without touching the module structure, the Čech complex rests unchanged and the local cohomology does. □

Quite remarkable are certain results of *local duality* for local cohomology (see [**10**]).

PROPOSITION 6.4. *Let $S = k[x_0, \dots, x_r]$ and let $\mathfrak{m} = (x_0, \dots, x_s)$ be the irrelevant ideal. Therefore we have $H^i_{\mathfrak{m}}(S) = 0$ for $i < r+1$ and $H^{r+1}_{\mathfrak{m}}(S) \simeq S(-r-1)^\vee$, where $^\vee$ means the graded dual module.*

THEOREM 6.2. *Let $S = k[x_0, \dots, x_r]$ and let $\mathfrak{m} = (x_0, \dots, x_r)$ be the irrelevant ideal. If M is a graded finitely generated S-module, therefore $H^i_{\mathfrak{m}}(M)$ is (as a S-module) a vector k-space dual to $\mathrm{Ext}^{r+1-i}_S(M, S(-r-1))$.*

Finally, we present a vanishing result that involves depth and Krull dimension of modules.

THEOREM 6.3. *Let M be a graded finitely generated S-module. Then*

(1) (GROTHENDIECK) *if $i < \mathrm{depth}(\mathfrak{m}, M)$ or $i > \dim_S(M)$, therefore $H^i_{\mathfrak{m}}(M) = 0$;*
(2) *if $i = \mathrm{depth}(\mathfrak{m}, M)$ or $i = \dim(M)$, therefore $H^i_{\mathfrak{m}}(M) \neq 0$.*

PROOF. Since

$$H^i_{\mathfrak{m}}(M) = \lim_{n \to \infty} \mathrm{Ext}^i_S(S/\mathfrak{m}^n, M)$$

and since $\mathrm{Ext}^i_S(S/\mathfrak{m}^n, M) = 0$ for every $i > \mathrm{depth}(\mathfrak{m}, M)$ with $\mathrm{Ext}^{\mathrm{depth}(\mathfrak{m},M)}_S(S/\mathfrak{m}^n, M) \neq 0$, the depth part is straightforward. The dimension part, instead, can be proved noting that S is Cohen-Macaulay ring, hence $\mathrm{codim}_S(M) := \dim(S_{\mathrm{Ann}_S(M)}) = \mathrm{depth}(\mathrm{Ann}_S(M), S)$ holds. □

CHAPTER 7

Eagon-Northcott complex.

7.1. Symmetric algebra.

This section briefly recalls the introductory aspects of symmetric algebra, the symmetric counterpart of exterior algebra used before. Details can be found in [**3**].

Let us recall that, given a ring A and an A-module M, one defines the *tensor algebra* of M setting the A-module

$$T(M) := \bigoplus_{n\geq 0} T^n(M)$$

which indeed acts as a graded A-algebra with $T^n(M) := \bigotimes_A^n M$.

Definition 7.1. The *symmetric algebra* of M is the quotient algebra $\bigodot M$ obtained by $T(M)$ modulo the ideal I generated by elements of the form $x \otimes y - y \otimes x \in T(M)$, for every $x, y \in M$.

We will set $x \odot y$ for the equivalence class of symbols $x \otimes y \in T(M)$. Since the ideal I is generated by homogeneous elements, it is a graded ideal; in fact, setting $I_p := I \cap T^p(M)$ for every $p \geq 0$, one determines a grading for $\bigodot M$, called *canonical grading*; explicitly, the degree p term is

$$\bigodot^p M := T^p(M)/I_p$$

Since $I_0 = I_1 = \{0\}$, we can set, formally, $\bigodot^0 M \simeq A$ and $\bigodot^1 M \simeq T^1(M) = M$. There exists also a canonical mapping $\varphi_M = \varphi : M \longrightarrow \bigodot M$ defined as a consequence of the construction.

Since we have $\varphi(x) \odot \varphi(y) = \varphi(y) \odot \varphi(x)$ for every $x, y \in M$ and since elements $\varphi(x)$ generate $\bigodot M$, we are allowed to claim that the symmetric algebra is a commutative algebra. Moreover, the construction is *universal* in the following sense.

Proposition 7.1. (Universal property of symmetric algebra) *Let G be an A-algebra and let $f : M \longrightarrow G$ be an A-linear morphism such that $f(x)f(y) = f(y)f(x)$ for every $x, y \in M$. Then there exists an unique A-algebras morphism $g : \bigodot M \longrightarrow G$ such that $f = g \circ \varphi_M$.*

Remark 7.1. Let us suppose that G is a *graded* A-algebra and let us suppose that $f : M \longrightarrow G$ is a morphism such that $f(M) \subseteq G_1$. Therefore, the identity

$$g(x_1 \odot \ldots \odot x_p) = f(x_1) \cdots f(x_p)$$

for every $x_i \in M$ shows that $g\left(\bigodot^p M\right) \subseteq G_p$ for every $p \geq 0$; that is, in other words, g is a graded algebras morphism.

As a matter of fact, one could also show that

$$\bigodot : \mathbf{Mod}(A) \longrightarrow \mathbf{Alg}(A)$$

set by $M \mapsto \bigodot M$ extends to a (covariant) functor, in the sense that for every A-modules morphism f there exists an unique A-algebras morphism $\bigodot f$ satisfying the standard functorial properties.

In the significant case M is a finitely generated module, it can easily be shown that $\bigodot M$ is finitely generated too; in particular, given a set of generator $\{m_1, \dots, m_r\}$ for M, every homogeneous part $\bigodot^p M$ can be generated by products $x_{i_1} \odot \dots \odot x_{i_p}$ for every $1 \leq i_1 \leq \dots \leq i_p \leq r$. It follows that the rank of $\bigodot^p M$ can be computed as

$$\binom{r+p-1}{p}$$

where r is the rank of M.

Finally, we show the behaviour of symmetric algebra when paired to a direct sum. Let

$$M := \bigoplus_{n \geq 0} M_n$$

be a A-module defined as direct sum of A-modules M_n and let $j_n : M_n \longrightarrow M$ be the canonical injections. There are thus functorially induced A-algebras morphisms $J_n : \bigodot M_n \longrightarrow \bigodot M$; since $\bigodot M$ is commutative, we can use universal property of symmetric algebra to claim the existence of an unique map

$$g : \bigotimes_{n \geq 0} \left(\bigodot M_n\right) \longrightarrow \bigodot M$$

such that $J_n = g \circ f_n$ where

$$f_n : \bigodot M_n \longrightarrow \bigotimes_{n \geq 0} \left(\bigodot M_n\right)$$

is the canonical morphism. Furthermore, one can also show that g is a *graded* isomorphism, that is to say

$$\bigodot \left(\bigoplus_{n \geq 0} M_n\right) \simeq \bigotimes_{n \geq 0} \left(\bigodot M_n\right)$$

7.2. Construction of the complex.

Let R be a ring and let $F = R^n, G = R^m$ be two free R-modules, where $n \geq m$. Let $f : F \longrightarrow G$ be a R-modules morphism. The *Eagon-Northcott* *complex* for f (indeed, for each one of its matrix representations) is the R-modules sequence

$$\begin{aligned}\mathbf{EN}(f):\ 0 \longrightarrow \left(\bigodot^{n-m} G\right)^{\vee} \otimes_R \left(\bigwedge^{n} F\right) \xrightarrow{\partial} \\ \xrightarrow{\partial} \left(\bigodot^{n-m-1} G\right)^{\vee} \otimes_R \left(\bigwedge^{n-1} F\right) \longrightarrow \\ \longrightarrow \cdots \longrightarrow \left(\bigodot^{2} G\right)^{\vee} \otimes_R \left(\bigwedge^{m+2} F\right) \xrightarrow{\partial} \\ \longrightarrow G^{\vee} \otimes_R \left(\bigwedge^{m+1} F\right) \xrightarrow{\partial} \bigwedge^{m} F \xrightarrow{\bigwedge^m f} \bigwedge^{m} G \simeq R\end{aligned}$$

where $M^{\vee} := \mathrm{Hom}_R(M, R)$; the coboundary maps are defined as following: set firstly a diagonal map

$$\Delta : \left(\bigodot^{p} G\right) \longrightarrow G^{\vee} \otimes_R \left(\bigodot^{p-1} G\right)^{\vee}$$

as the dual of the natural map

$$\begin{gathered} G \otimes_R \left(\bigodot^{p-1} G\right) \longrightarrow \bigodot^{p} G \\ u \otimes (u_1 \odot \ldots \odot u_{p-1}) \mapsto u \odot u_1 \odot \ldots \odot u_{p-1} \end{gathered}$$

Then we define a similar map

$$\nabla : \bigwedge^{p} F \longrightarrow F \otimes \left(\bigwedge^{p-1} F\right)$$

dualizing the multiplication map

$$\begin{gathered} F^{\vee} \otimes_R \left(\bigwedge^{p-1} F\right)^{\vee} \longrightarrow \left(\bigwedge^{p} F\right)^{\vee} \\ u^* \otimes (u_1^* \wedge \ldots \wedge u_{p-1}^*) \mapsto u^* \wedge u_1^* \wedge \ldots \wedge u_{p-1}^* \end{gathered}$$

The action of two maps can be expressed cleanly in components:

$$\Delta(u) := \sum_i u_i' \otimes u_i'', \qquad \nabla(v) := \sum_i v_i' \otimes v_i''$$

where $u_i' \in G^{\vee}, u_i'' \in \left(\bigodot^{p-1} G\right)^{\vee}$ and $v_j' \in F, v_j'' \in \bigwedge^{p-1} F$. These notations allow us to define the p-th differential as the morphism

$$\begin{aligned} \partial_p : \left(\bigodot^{p-1} G\right)^{\vee} \otimes_R \bigwedge^{n+p-1} F \longrightarrow \left(\bigodot^{p-2} G\right)^{\vee} \otimes_R \bigwedge^{n+p-2} F \\ \xi \otimes \omega \mapsto \sum_i \left(f^{\vee}_{u_i'}(v_i')u_i''\right) \otimes v_i'' \end{aligned}$$

where $f^{\vee} : G^{\vee} \longrightarrow F^{\vee}$ is induced by f and $f^{\vee}_{u_i''} : F \longrightarrow R$ is the image of u_i' in $F^{\vee}$ under $f^{\vee}$. Carrying out explicit calculations, we can prove that $\mathbf{EN}(f)$ is a complex of R-modules.

Let us note that the Eagon-Northcott complex deals with the same kind of information that Koszul complex does, namely the cokernel of the map

$\bigwedge^m f : \bigwedge^m F \longrightarrow \bigwedge^m G$. The subtlety is to recognize that Koszul complex deals with sequences of elements in R because it can be obtained as a particular case of the Eagon-Northcott complex, setting $m = 1$.

The above arguments can be replicated in the class of vector bundles over a scheme X. In particular, given a morphism $\varphi : \mathscr{F} \longrightarrow \mathscr{G}$ of vector bundles, one can define the Eagon-Northcott complex $\mathbf{EN}(\varphi)$ setting

$$\mathbf{EN}(\varphi)_p := \left(\bigodot^{p-1} \mathscr{G}\right)^\vee \otimes_{\mathscr{O}_X} \bigwedge^{\mathrm{rk}(\mathscr{G})+p-1} \mathscr{F}$$

for every $p > 0$ and $\mathbf{EN}(\varphi)_0 := \mathscr{O}_X$, where $\mathscr{M}^\vee := \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X)$ is intended.

Further details can be found in [**8**].

CHAPTER 8

Regularity for modules and sheaves.

8.1. Regularity and local cohomology for modules.

Let us give the following algebraic definition.

Definition 8.1. Let M be a graded finitely generated S-module and let us consider the minimal free resolution for M:

$$\mathbf{F}: \quad \cdots \longrightarrow F_i \longrightarrow F_{i+1} \longrightarrow \cdots \longrightarrow F_0$$

where we have set

$$F_i = \bigoplus_j S(-a_{i,j})^{\beta_{i,j}}$$

Therefore, we define the *Castelnuovo-Mumford regularity* of M as

$$\operatorname{reg}(M) := \max\{a_{i,j} - i \mid i \geq 0, j \geq 0\}$$

By Betti numbers properties, we can express $\operatorname{reg}(M)$ as the greatest integer q such that $\beta_{i,i+q}(M) \neq 0$, with $i \geq 0$. In the following we will carry over a characterization of this algebraic version of regularity, by means of local cohomology. Some first application are also presented.

Theorem 8.1. *Let M be a graded finitely generated S-module and let d be an integer. Let $\mathfrak{m} = (x_0, \ldots, x_r)$ be the irrelevant ideal. Therefore, the following statements are equivalent:*

(1) $d \geq \operatorname{reg}(M)$*;*
(2) $d \geq \max\{e \mid H^i_{\mathfrak{m}}(M)_e \neq 0\} + i$ *for every* $i \geq 0$*;*
(3) $d \geq \max\{e \mid H^0_{\mathfrak{m}}(M)_e \neq 0\}$ *and* $H^i_{\mathfrak{m}}(M)_{d-i+1} = 0$ *for every* $i > 0$.

Let us introduce the following terminology: a S-module M is *weakly d-regular* if $H^i_{\mathfrak{m}}(M)_{d-i+1} = 0$ for every $i > 0$ and it is *d-regular* if it is weakly d-regular and $d \geq \operatorname{reg}(H^0_{\mathfrak{m}}(M))$.

Remark 8.1. With the new terminology, the Theorem states that M is d-regular if and only if $d \geq \operatorname{reg}(M)$. In fact, we know that M is d-regular if and only if

$$\begin{cases} H^i_{\mathfrak{m}}(M)_{d-i+1} = 0 & i > 0 \\ d \geq \operatorname{reg}(H^0_{\mathfrak{m}}(M)) \end{cases}$$

On the other hand, by definition, one has $d \geq \operatorname{reg}(H^0_{\mathfrak{m}}(M))$ if and only if $H^0_{\mathfrak{m}}(M)_e = 0$ for every $e > d$. So, $d \geq \max\{e \mid H^i_{\mathfrak{m}}(M)_e \neq 0\} + i$ for every $i \geq 0$ if and only if $H^i_{\mathfrak{m}}(M)_{d-i+1} = 0$ for every $i > 0$ and $H^0_{\mathfrak{m}}(M)_{d+1} = 0$, and this holds if and only if M is d-regular. This remark will help us understanding the importance of such a characterization. In fact, the theorem allows us to define regularity as

$$\operatorname{reg}(M) = \min\{d \mid M \text{ is } d\text{-regular}\}$$

and this is a much simpler definition to work with, being completely determined by local cohomology. However, before we use this definition it is required to check that it bears no ambiguity. In other terms, we need to show that d-regularity *does not really involve* the definition of Castelnuovo-Mumford regularity itself; this is achieved proving that regularity for local cohomology modules can be obtained in a "sufficiently trivial way". This will follow by the Artin property of these modules, which we will see involved in the next section.

Recall that, for every $x \in S$, one defines

$$(0 :_M x) := \{m \in M \mid xm = 0\}$$

as a S-submodule in M; it becomes trivial when x is a regular element for M. More generally, if $(0 :_M x)$ has finite length, the element x is called *quasi-regular.*

LEMMA 8.1. *Let M be a graded finitely generated S-module, and let us suppose that the ground field k in S is infinite. Then, there exists an homogeneous polynomial f having degree d that is a quasi-regular element for M.*

PROOF. The module $(0 :_M f)$ has finite length if and only if the annihilator $\operatorname{Ann}_S((0 :_M f))$ is not contained in any relevant prime ideal $\mathfrak{p} \subseteq S$ (see Theorem 1.3). This is the same to ask $(0 :_M f)_\mathfrak{p} = 0$ for every prime ideal $\mathfrak{p} \neq \mathfrak{m}$, namely x is regular for $M_\mathfrak{p}$. To show this, it suffice to prove that f is not contained in any associated prime, except $\mathfrak{m}$.

Every relevant prime $\mathfrak{p}$ in S meets S_d in a proper subspace, otherwise $\mathfrak{p} \supseteq \mathfrak{m}^d$ and we should have $\mathfrak{m} = \mathfrak{p}$ by maximality. Since the number of associated primes for M is finite (for M is finitely generated), the element f has the required property if it avoids a certain finite number of proper subspaces. □

PROPOSITION 8.1. *Let M be a graded finitely generated S-module and let $x \in S$ be a linear homogeneous polynomial that is quasi-regular for M. Therefore*

1. *if M is weakly d-regular, M/xM is weakly d-regular too;*
2. *if M is (weakly) d-regular, M is (weakly) $(d+1)$-regular too;*
3. *M is d-regular if and only if M/xM and $H^0_\mathfrak{m}(M)$ are d-regular.*

PROOF. (1) Lemma 8.1 shows that a linear homogeneous polynomial x sufficiently general forces $(0 :_M x)$ to have finite length. Then we set $M' = M/(0 :_M x)$ and we consider the following exact sequence

$$0 \longrightarrow (0 :_M x) \longrightarrow M \longrightarrow M' \longrightarrow 0$$

Switching to the long sequence in local cohomology, one finds

$$\cdots \longrightarrow H^i_{\mathfrak{m}}((0:_M x)) \longrightarrow H^i_{\mathfrak{m}}(M) \longrightarrow H^i_{\mathfrak{m}}(M') \longrightarrow H^{i+1}_{\mathfrak{m}}((0:_M x)) \longrightarrow \cdots$$

But $(0:_M x)$ has finite length, so by Corollary 6.2 its local cohomology vanishes in degree $i > 0$; this shows that $H^i_{\mathfrak{m}}(M) \simeq H^i_{\mathfrak{m}}(M')$ for every $i > 0$. Let us now consider the following exact sequence

$$0 \longrightarrow M'(-1) \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0 \tag{8.1.1}$$

where the first non-trivial arrow represents multiplication by x. Local cohomology long exact sequence is thus made up by sequences of the form (8.1.2)

$$\cdots \longrightarrow H^i_{\mathfrak{m}}(M)_{d-i+1} \longrightarrow H^i_{\mathfrak{m}}(M/xM)_{d-i+1} \longrightarrow H^{i+1}_{\mathfrak{m}}(M'(-1)_{d+i-1} \longrightarrow \cdots$$

By definition, one has also

$$H^{i+1}_{\mathfrak{m}}(M'(-1))_{d+i-1} \simeq H^{i+1}(M)_{d+i}$$

Hence, if M is weakly d-regular, in exact sequence (8.1.2) the first module vanishes for $i > 0$, and the third vanishes for $i > 0$ by above isomorphism; it follows then $H^i_{\mathfrak{m}}(M/xM)_{d-i+1} = 0$ for every $i > 0$, and this means M/xM is weakly d-regular.

(2) Let us assume M is weakly d-regular and let us prove that M is weakly $(d+1)$-regular by induction over $\dim_S M$. If $\dim_S M = 0$, then M has finite length by Theorem 1.3, hence $H^i_{\mathfrak{m}}(M) = 0$ for every $i > 0$; is this case, M is weakly p-regular for every p and there is nothing to be proved. Now let us assume that $\dim_S M > 0$. Since $(0:_M x)$ has finite length, it follows that Hilbert's polynomial of M/xM can be obtained subtracting 1 by the Hilbert's polynomial of M; on the other hand, by Krull's principal ideal theorem, one finds that $\dim_S(M/xM) = \dim_S M - 1$. The previous point shows that M/xM is already weakly d-regular; now, by inductive hypothesis, we know also that it is weakly $(d+1)$-regular. Finally, the exact sequence (8.1.1) induces the long sequence in local cohomology

$$\cdots \longrightarrow H^i_{\mathfrak{m}}(M'(-1))_{(d+1)-i+1} \longrightarrow H^i_{\mathfrak{m}}(M)_{(d+1)-i+1} \longrightarrow H^i_{\mathfrak{m}}(M/xM)_{(d+1)-i+1} \longrightarrow \cdots$$

For every $i \geq 1$ we know that $H^i_{\mathfrak{m}}(M'(-1)) = H^i_{\mathfrak{m}}(M)$; the first term thus vanishes, since M is weakly d-regular, while the third one vanishes because M/xM is weakly $(d+1)$-regular. It follows that $H^i_{\mathfrak{m}}(M)_{(d+1)-i+1} = 0$ and M is weakly $(d+1)$-regular.

Moreover, if M is d-regular too, therefore by above arguments M is weakly $(d+1)$-regular. But one has also

$$d + 1 > d \geq \operatorname{reg} H^0_{\mathfrak{m}}(M)$$

hence M is $(d+1)$-regular too.

(3) Firstly, let us assume that M is d-regular; therefore $d \geq \operatorname{reg} H^0_{\mathfrak{m}}(M)$. From this inequality, it follows that $H^0_{\mathfrak{m}}(M)_p = 0$ for every $p > d$, that is $H^0_{\mathfrak{m}}(M)$ is d-regular. It remains to show that M/xM is d-regular. By point (1) we already know that M/xM is weakly d-regular, so it suffices to prove

that $H^0_{\mathfrak{m}}(M/xM)_p = 0$ for every $p > d$. Using again the exact sequence (8.1.1), one studies the long cohomology sequence

$$\cdots \longrightarrow H^0_{\mathfrak{m}}(M)_e \longrightarrow H^0_{\mathfrak{m}}(M/xM)_e \longrightarrow H^1_{\mathfrak{m}}(M'(-1))_e \longrightarrow \cdots$$

But, assuming $e > d$, the leftmost term vanishes by hypothesis, while $H^1_{\mathfrak{m}}(M'(-1))_e = H^1_{\mathfrak{m}}(M)_{e-1}$. Since point (2) claims M is weakly e-regular for every $e \geq d$, the rightmost term vanishes too. Hence, M/xM is d-regular.

Conversely, let us assume that M/xM is d-regular and $H^0_{\mathfrak{m}}(M)_p = 0$ for every $p > d$. In order to prove that M is d-regular, it suffices to prove that $H^i_{\mathfrak{m}}(M)_{d-i+1} = 0$ for every $i > 0$. Again, using (8.1.1) we obtain the sequence

$$\cdots \longrightarrow H^{i-1}_{\mathfrak{m}}(M/xM)_{p+1} \longrightarrow H^i_{\mathfrak{m}}(M')_p \xrightarrow{f_p} H^i_{\mathfrak{m}}(M)_{p+1} \longrightarrow \cdots$$

where we used $H^i_{\mathfrak{m}}(M'(-1))_{p+1} = H^i_{\mathfrak{m}}(M')_p$. Since M/xM is d-regular by hypothesis, point (2) shows that it is p-regular for every $p \geq d$ too, hence the first term vanishes for every $p \geq d-i+1$, forcing the arrow f_p to be injective. Recalling that $H^i_{\mathfrak{m}}(M') \simeq H^i_{\mathfrak{m}}(M)$, we obtain a sequence of monomorphisms

$$H^i_{\mathfrak{m}}(M)_{d-i+1} \longrightarrow H^i_{\mathfrak{m}}(M)_{d-i+2} \longrightarrow \cdots$$

induced by multiplication by x on $H^i_{\mathfrak{m}}(M)$. But by Lemma 6.2, every elements in $H^i_{\mathfrak{m}}(M)$ is annihilated by some power of x, hence the composition of these maps vanishes definitively; by injectivity, it follows $H^i_{\mathfrak{m}}(M)_{d-i+1} = 0$ for every $i > 0$. This finishes the proof. □

Now we are able to produce a proof of the characterization theorem.

Proof. *(of Theorem 8.1)* We need to prove that $d \geq \operatorname{reg}(M)$ if and only if M is d-regular, as we noted in Remark 8.1, linking the various statements.

We start proving that (1) ⇒ (2), by induction on projective dimension $\dim^{(\mathrm{proj})}(M)$ of M. Let $\dim^{(\mathrm{proj})}(M) = 0$: indeed in this case one has

$$M = \bigoplus_j S(-a_j)$$

namely, M is a graded free S-module, and the thesis follows readily: by definition, $\operatorname{reg}(M) = \max\{a_j \mid j \geq 0\}$ (since M has trivial free resolution). Moreover, M is d-regular if and only if $d \geq a_j$ for every j, by Proposition 6.4. This suffices to prove that $d \geq \operatorname{reg}(M)$ in this case.

Let us suppose, now, that $\dim^{(\mathrm{proj})}(M) > 0$ and let us assume M has the following minimal free resolution

$$\cdots \longrightarrow L_1 \xrightarrow{\varphi_1} L_0 \longrightarrow M \longrightarrow 0$$

Call $M' = \operatorname{im}(\varphi_1)$ the first syzygy module for M; by definition of regularity, it is clear that $\operatorname{reg}(M') \leq \operatorname{reg}(M) + 1$: in fact, if

$$L_i = \bigoplus_j S(-a_{i,j})^{\beta_{i,j}}$$

therefore $\operatorname{reg}(M) = \max\{a_{i,j} - i \mid i, j \geq 0\}$ while

$$\operatorname{reg}(M') = \max\{a_{i,j} - i \mid i \geq 1, j \geq 0\} \leq 1 + \operatorname{reg}(M)$$

Inducting on projective dimension, we claim thus that M' is $(d+1)$-regular; in fact, being $\dim^{(\text{proj})}(M') < \dim^{(\text{proj})}(M)$, one notes that if $e \geq \operatorname{reg}(M')$, then M' is e-regular. But by above inequality

$$d \geq \operatorname{reg}(M) \geq \operatorname{reg}(M') - 1$$

so M' is $(e+1)$-regular for every $e \geq d$. Now let us consider the following exact sequence

$$0 \longrightarrow M' \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

and let us switch to degree $e \geq d$ part of the long local cohomology sequence: we have

$$\cdots \longrightarrow H^i_{\mathfrak{m}}(L_0)_{e-i+1} \longrightarrow H^i_{\mathfrak{m}}(M)_{e-i+1} \longrightarrow H^{i+1}_{\mathfrak{m}}(M')_{e-i+1} \longrightarrow 0$$

It is already known that L_0 is e-regular by first step, so the first term vanishes for every $i \geq 0$; moreover, also $H^i_{\mathfrak{m}}(M')_{(e+1)-i+1} = 0$ holds for every $i \geq 0$, hence $0 = H^{i+1}_{\mathfrak{m}}(M')_{(e+1)-(i+1)+1} = H^{i+1}_{\mathfrak{m}}(M')_{e-i+1}$. This proves that M is e-regular for every $e \geq d$, then $d \geq \max\{e \mid H^i_{\mathfrak{m}}(M)_e = 0\} + i$ for every $i \geq 0$, and this is what we need.

The (2) $\Rightarrow$ (3) is straightforward: if (2) holds, one has in particular $d \geq \max\{e \mid H^0_{\mathfrak{m}}(M)_e \neq 0\}$ and $H^i_{\mathfrak{m}}(M)_{d-i+1} = 0$ for every $i > 0$.

It remains to show that (3) $\Rightarrow$ (1). Let us assume (3) holds, namely M is d-regular. Then it suffices to prove that $d \geq \operatorname{reg}(M)$. Since field extension commute with local cohomology, we can assume without loss of generality that k is an infinite field. Let us assume also that M has the following minimal free resolution:

$$\cdots \longrightarrow L_1 \xrightarrow{\varphi_1} L_0 \longrightarrow M \longrightarrow 0$$

We prove first that every generator for L_0 has degree at most d (that is to say, $a_{0,j} \leq d$ for every j). This is the same to prove that M is generated by elements of degree at most d; hence, we proceed by induction on $\dim_S(M)$. If $\dim_S(M) = 0$, then the thesis is trivial: M has finite length, so by d-regularity if follows that $M_e = H^0_{\mathfrak{m}}(M)_e = 0$ for every $e > d$.

Now let us assume that $\dim_S(M) > 0$ and set $M' = M/H^0_{\mathfrak{m}}(M)$; by short exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow M' \longrightarrow 0$$

we note that it is enough to show that generators for $H^0_{\mathfrak{m}}(M)$ and M' have degree at most d. By d-regularity thus we can claim that $H^0_{\mathfrak{m}}(M)_e = 0$ for every $e > d$. By Lemma 8.1 we can choose a linear homogeneous polynomial x such that it is not a zero-divisor in M' and by Proposition 8.1 it follows that M'/xM' is d-regular. Since $\dim_S(M'/xM') < \dim_S(M')$ holds, inductive hypothesis shows that M'/xM' is generated by elements having degree at most d; the same follows readily also for $M'/\mathfrak{m}M'$. By Nakayama's Lemma in the graded case, the generators for M' have degree at most d too.

Now return to the main proof. If M is free, the above arguments conclude the proof. Otherwise, we proceed by induction on projective dimension $\dim^{(\text{proj})}(M)$. Let be $M' = \text{im}(\varphi_1)$ the first syzygy module for M and let us consider the same exact sequence as before:

$$0 \longrightarrow M' \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

Switching to long local cohomology sequence, one shows that M' is $(d+1)$-regular. Since $\dim_S(M') < \dim_S(M)$, we can apply inductive hypothesis and it follows that $\text{reg}(M') \leq d+1$. This means, indeed, that the part of the minimal free resolution for M beginning with L_1 satisfies the conditions under which, if completed, we have $\text{reg}(M) \leq d$. □

We are now ready to carry out some results that will help us studying regularity; they are all achieved as consequences of the characterization theorem. The first one provides a different formula for regularity.

Corollary 8.1. *Let M be a graded finitely generated S-module and let $x \in M$ be a quasi-regular element. Therefore*

$$\text{reg}(M) = \max\{\text{reg}(H^0_{\mathfrak{m}}(M)),\ \text{reg}(M/xM)\}$$

Proof. We know that $\text{reg}(M) \leq d$ if and only if M is d-regular, and this happens if and only if $H^0_{\mathfrak{m}}(M)$ and M/xM are d-regular; indeed, this holds if and only if $d \geq \text{reg}(H^0_{\mathfrak{m}}(M))$ and $\text{reg}(M/xM)$. □

8.2. Artinian modules case.

This case is particularly simple to treat, so it deserves a deeper explanation.

Corollary 8.2. *If M is a graded finitely generated S-module having finite length, therefore we have* $\text{reg}(M) = \max\{d \mid M_d \neq 0\}$.

Proof. It is known that, in these hypotheses, $H^0_{\mathfrak{m}}(M) = M$ and $H^i_{\mathfrak{m}}(M) = 0$ holds for every $i > 0$; by Theorem 8.1, $\text{reg}(M) \leq d$ if and only if M is d-regular, that is $H^0_{\mathfrak{m}}(M)_e = 0$ for every $e > d$. It follows that $\text{reg}(M) = \max\{d \mid M_d \neq 0\}$. □

Corollary above suggests a new definition for regularity, at least in some nice cases. In fact, let M be a graded artinian S-module; we set

$$\text{reg}(M) := \max\{d \mid M_d \neq 0\}$$

Note that this definition does not contrast with the former one; in fact a finitely generated artinian module over a noetherian ring has finite length (by Theorem 1.3) and thus satisfies formula by Corollary 8.2. Moreover, by local duality Theorem 6.2, local cohomology module for a graded finitely generated S-module M act all as graded artinian S-modules of finite length; this makes the following theorems sensible.

Corollary 8.3. *Let M be a graded finitely generated S-module. Therefore*

$$\begin{aligned}\operatorname{reg}(M) &= \max\{\operatorname{reg}\operatorname{Tor}_i^S(M,k) - i \mid i \geq 0\} = \\ &= \max\{\operatorname{reg} H^i_{\mathfrak{m}}(M) + i \mid i \geq 0\}\end{aligned}$$

Proof. The formula $\operatorname{reg}(M) = \max\{\operatorname{reg} H^i_{\mathfrak{m}}(M) + i \mid i \geq 0\}$ follows immediately by point (2) of Theorem 8.1. In order to prove the other formula, let $\mathbf{F} : \cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots$ be the minimal free resolution for M. Then modules $\operatorname{Tor}^i_S(M,k) = F_i \otimes_S k = F_i/\mathfrak{m}F_i$ are finitely generated vector k-spaces, in particular finite length modules. By Nakayama's Lemma, Betti numbers $\beta_{i,j}$, namely the degrees of generators for F_i, are also the degrees of non-zero generators for $\operatorname{Tor}^i_S(M,k)$. Therefore, $\operatorname{reg}\operatorname{Tor}^i_S(M,k) - i = \max\{\beta_{i,j} \mid i,j \geq 0\} - i \leq \operatorname{reg}(M)$. Taking maximum on both parts, thesis follows. □

Looking more deeply at Corollary 8.2 we deduce a remarkable property: regularity for finite length modules does not depend on their S-modules structure, that is to say it does not depend on the ring S but only on its grading. We can state this property in a precise way, in the following result. We will write $\operatorname{reg}_S(M)$ to mean regularity of M as a S-module.

Corollary 8.4. *Let M be a graded finitely generated S-module and let $\varphi : S' \longrightarrow S$ be a graded rings morphism, where S', S are generated in degree 1. If M acts as a finitely generated S'-module too (by means of φ), therefore $\operatorname{reg}_S(M) = \operatorname{reg}_{S'}(M)$.*

Proof. A very well known results in commutative algebra claims that M is a finitely generated S'-module if and only if S if a finitely generated S'-module (by means of scalar restriction), and this happens if and only if the irrelevant ideal in S is nilpotent module the ideal generated by the irrelevant ideal in S' and the annihilator of M. But by change of ring property, local cohomology of M does not change upon this operation; it follows then by characterization theorem that neither regularity changes. □

8.3. Regularity for arithmetically Cohen-Macaulay varieties.

Let M be a graded finitely generated S-module. Assume that $x \in S$ is a regular element for M, that is to say x is not a zero-divisor in M. Therefore $\operatorname{depth}(M) \geq 1$, and by means of Proposition 6.2 one has $H^0_{\mathfrak{m}}(M) = 0$. Using part (3) of Proposition 8.1 and the characterization Theorem 8.1, one proves then that $\operatorname{reg}(M) = \operatorname{reg}(M/xM)$.

Furthermore, dealing with Cohen-Macaulay modules, a similar property holds for any regular sequence.

Proposition 8.2. *Let M be a graded finitely generated S-module satisfying the Cohen-Macaulay property. Let moreover $y_1, \dots, y_t$ be a maximal regular M-sequence of linear polynomials. Therefore*

$$\operatorname{reg}(M) = \max\{d \mid (M/(y_1, \dots, y_t)M)_d \neq 0\}$$

PROOF. If $\dim_S(M) = 0$ the result is obvious: M has finite length and there are no regular M-sequences (since M has depth 0) hence by Corollary 8.2 the formula is obtained. Now suppose that $\dim_S(M) > 0$ and proceed by induction. Let $y_1, \dots, y_t$ be a regular M-sequence and, up to rearranging the terms, let us assume that y_1 is M-regular, namely it is not a zero-divisor. Hence by previous arguments, $\operatorname{reg}(M) = \operatorname{reg}(M/y_1M)$ holds and, being $\dim_S(M/y_1M) < \dim_S(M)$, we can use induction. Calling $M_1 = M/y_1M$, inductive hypothesis forces

$$\begin{aligned}\operatorname{reg}(M) = \operatorname{reg}(M_1) &= \max\{d \mid (M_1/(y_2, \dots, y_t)M_1)_d \neq 0\} \\ &= \max\{d \mid (M/(y_1, \dots, y_t)M)_d \neq 0\}\end{aligned}$$

□

If $X \subseteq \mathbf{P}^r_k$ is a projective variety, we define its *regularity* as the regularity of its defining ideal I_X; sometimes is useful to remember that regularity of I_X can be calculated by means of regularity of $S_X := S/I_X$. Now we will show that, if X is arithmetically Cohen-Macaulay (namely, S_X is a Cohen-Macaulay module) there exists an upper bound for its regularity that can be derived with geometric techniques.

COROLLARY 8.5. *Let $X \subseteq \mathbf{P}^r_k$ be an arithmetically Cohen-Macauly variety not contained in a hyperplane. Therefore*

$$\operatorname{reg}(S_X) \leq \deg(X) - \operatorname{codim}(X)$$

PROOF. Let $t = \dim(X)$, such that $t+1$ is the dimension of S_X as S-module. Up to field extensions, it is possible (without changing local cohomology modules) to suppose k is an infinite algebraically closed field. Hence one can assume there exists a regular S_X-sequence of linear homogeneous polynomials $y_0, \dots, y_t$. Let $S'_X := S_X/(y_0, \dots, y_t)$ and note that $\dim_k(S_X)_1 = r+1$ since X is not contained in a hyperplane; hence $\dim_k(S'_X)_1 = r - t = \operatorname{codim}(X)$.

If we call $d = \operatorname{reg}(S_X)$, by Proposition 8.2 we have that

$$\operatorname{Hilb}_{S'_X}(d) \neq 0$$

and this implies $\operatorname{Hilb}_{S'_X}(e) \neq 0$ for every $e \in \{0, \dots, d\}$. On the other hand, $\deg(X)$ can be thought as the number of points X meets a general t-codimensional linear subspace. Let us consider the following exact sequence

$$0 \longrightarrow S_X/(y_1, \dots, y_t)(-1) \xrightarrow{y_0} S_X/(y_1, \dots, y_t) \longrightarrow S'_X \longrightarrow 0$$

and compute by induction

$$\operatorname{Hilb}_{S_X/(y_1,\dots,y_t)}(e) = \sum_{p=0}^{e} \operatorname{Hilb}_{S'_X}(p)$$

Therefore, for very large e,

$$\deg(X) = \sum_{p=0}^{e} \mathrm{Hilb}_{S'_X}(p) \geq 1 + \mathrm{codim}(X) + (\mathrm{reg}(X) - 1)$$

since there are at least $\mathrm{reg}(X) - 1$ non-zero terms of $\mathrm{Hilb}_{S'_X}(e) \neq 0$ for $e \in \{2, \dots, e\}$. □

Unfortunately, the above proof won't work loosening the arithmetic hypothesis over X, neither this results has deep consequences: in fact, arithmetic Cohen-Macaulay property is very much stronger that the equivalent "geometric" property (that is to say, every local ring over a closed point is Cohen-Macaulay), and this is satisfied by a large class of algebraic curves (for example, every smooth curve is a Cohen-Macaulay variety, but not necessarily *arithmetically*). As a matter of fact, one can determine ideals $I \subseteq S$ such that regularity of S/I is arbitrarily bigger than the degrees of generators for I.

As we will explain in the next chapter, in the case X is a smooth irreducible curve over an algebraically closed field, the upper bound still holds.

8.4. Regularity for coherent sheaves.

In this section we summarize the main topics about original regularity theory, that Mumford developed exclusively for coherent algebraic sheaves; further, we discuss how it's possible to link this "geometric" notion of regularity with the preceding arguments.

Let $\mathbf{P}^r = \mathbf{P}^r_k$ be a (schematic) projective space over a field k and let $\mathscr{F}$ be an algebraic coherent sheaf over $\mathbf{P}^r$. For every integer $m \in \mathbb{Z}$, we say that $\mathscr{F}$ is *m-regular* if

$$H^i(\mathbf{P}^r, \mathscr{F}(m-i)) = 0$$

for every $i > 0$. Recall that $\mathscr{F}(p) := \mathscr{F} \otimes_{\mathscr{O}_{\mathbf{P}^r}} \mathscr{O}_{\mathbf{P}^r}(p)$ for every integer p. In general, the definition of m-regularity can be extended to any variety endowed with a very ample line bundle; in particular, it can be applied in the case of a projective k-variety X.

By Serre's vanishing theorem, every coherent sheaf is m-regular for some m: we know, in fact, there exists an integer n_0 such that $H^i(X, \mathscr{F}(n)) = 0$ for every $i > 0, n \geq n_0$ and such that $H^i(X, \mathscr{F}(n)) = 0$ for every $i > \dim X$ and $n \in \mathbb{Z}$. Therefore it is enough to put $m = n_0 + \dim X$ to obtain $H^i(X, \mathscr{F}(m-i)) = 0$ for every $i > 0$.

In order to simplify notations, let us reduce to the case $X = \mathbf{P}^r$; if an arbitrary projective variety is involved, similar results can be obtained by pulling back the corresponding over $\mathbf{P}^r$. We are now able to give the following definition.

Definition 8.2. Let $\mathscr{F}$ be an algebraic coherent sheaf over $\mathbf{P}^r$. The minimum, if it exists, amongst integers m such that $\mathscr{F}$ is m-regular is called *Castelnuovo-Mumford regularity* of $\mathscr{F}$ and it's written $\mathrm{reg}(\mathscr{F})$.

The following result shows the connection between the above definition of regularity and the regularity for graded modules introduced in the previous sections.

PROPOSITION 8.3. *Let M be a graded finitely generated S-module and let $\widetilde{M}$ be the associated coherent sheaf over $\mathbf{P}^r_k$. Therefore, the module M is d-regular if and only if:*

(1) *$\widetilde{M}$ is a d-regular sheaf;*
(2) *$H^0_{\mathfrak{m}}(M)_e = 0$ for every $e > d$;*
(3) *the canonical mapping $M_d \longrightarrow H^0(\widetilde{M}(d))$ is a surjection.*

PROOF. We know that $H^i_{\mathfrak{m}}(M)_e = H^{i-1}(\mathbf{P}^r_k, \widetilde{M}(e))$ for every $i \geq 2$. Then M is d-regular if and only if it satisfies conditions (1), (2) and $H^1_{\mathfrak{m}}(M)_e = 0$ holds for every $e \geq d$. But using the exact sequence (6.2.4) in degree e part we have

$$0 \longrightarrow H^0_{\mathfrak{m}}(M)_e \longrightarrow M_e \longrightarrow H^0(\mathbf{P}^r_k, \widetilde{M}(e)) \longrightarrow H^1_{\mathfrak{m}}(M)_e \longrightarrow 0$$

and this implies that $H^0(M)_e = 0$ for every $e > d$ if and only if $H^1(M)_e = 0$ for every $e \geq d$. ☐

COROLLARY 8.6. *If M is a graded finitely generated S-module, therefore* $\operatorname{reg}(M) \geq \operatorname{reg}(\widetilde{M})$*; in particular, equality holds if and only if*

$$M = \bigoplus_{n \in \mathbb{Z}} H^0(X, \widetilde{M}(n))$$

PROOF. The second statement follows straightforwardly by the exact sequence 6.2.4 and by the first one. To prove $\operatorname{reg}(M) \geq \operatorname{reg}(\widetilde{M})$, it is enough to prove that $\widetilde{M}$ is $\operatorname{reg}(M)$-regular, that is

$$H^p(\mathbf{P}^r_k, \widetilde{M}(\operatorname{reg}(M) - p)) = 0$$

for every $p > 0$. Now, for every $p \geq 2$ we have

$$H^{p-1}(\mathbf{P}^r_k, \widetilde{M}(\operatorname{reg}(M) - p + 1)) \simeq H^p_{\mathfrak{m}}(M)_{\operatorname{reg}(M)-p+1}$$

by isomorphism 6.2.3. But using Theorem 8.1 we know also that

$$H^p(M)_{\operatorname{reg}(M)-p+1} = 0$$

for every $p \geq 1$. Hence it follows that $H^p(\mathbf{P}^r_k, \widetilde{M}(\operatorname{reg}(M) - p)) = 0$ for every $p \geq 1$, that is the thesis. ☐

The main results proven by Mumford in its early papers was attributed to Castelnuovo. We state it here in the Mumford's modern terminology.

THEOREM 8.2. *Let $\mathscr{F}$ be a coherent sheaf over $\mathbf{P}^r_k$ and let us assume $\mathscr{F}$ is m-regular. Therefore*

(1) *$\mathscr{F}$ is n-regular for every $n \geq m$;*

(2) *there exists a surjection*

$$H^0(\mathbf{P}^r_k, \mathscr{F}(p-1)) \otimes H^0(\mathbf{P}^r_k, \mathscr{O}_{\mathbf{P}^r_k}(1)) \longrightarrow H^0(\mathbf{P}^r_k, \mathscr{F}(p))$$

for every $p > m$.

PROOF. The results follows from characterization 8.1, together with the above considerations about sheaf regularity. A more direct proof can be found in [**22**]. □

CHAPTER 9

Regularity of projective curves.

The whole chapter is devoted to the detailed proof of a Theorem by Gruson, Lazarsfeld and Peskine, treated in the paper [**17**]; its purpose is to establish an upper bound for the regularity of a projective curve in a similar way we did in Corollary 8.5, assuming the curve is irreducible over an algebraically closed field.

We will discuss the proof restricting to smooth curves only. The generalization can be carried over with some technical tools and remains available in the original paper.

9.1. Preamble.

The result we are going to prove is the following.

THEOREM 9.1. (GRUSON, LAZARSFELD, PESKINE) *Let k be an algebraically closed field and let $X \subseteq \mathbf{P}^r_k$ be a projective smooth k-curve that is irreducible and non degenerate. Therefore* $\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim}(X) + 1$.

Let us recall that, with our terminology, by $\operatorname{reg}(X)$ we mean the regularity of the homogeneous saturated ideal of X, namely

$$I_X := \bigoplus_{n \geq 0} H^0(\mathbf{P}^r_k, \mathscr{I}_X(n))$$

being $\mathscr{I}_X$ the ideal sheaf of X. In such hypotheses, $\operatorname{reg}(X) = \operatorname{reg}(\mathscr{I}_X)$ holds by Corollary 8.6.

9.2. Fitting ideals.

The first reduction we will make to prove the result is about the ideal of X, or equivalently its ideal sheaf.

Let $X \subseteq \mathbf{P}^r := \mathbf{P}^r_k$ be an irreducible, non degenerate, smooth curve and let $\mathscr{L}$ be a line bundle over X. Let us consider the following finitely generated S-module

$$F := \bigoplus_{n \geq 0} H^0(X, \mathscr{L}(n))$$

It has a minimal free presentation, that is the beginning of the minimal free resolution for F:

$$L_1 \xrightarrow{\psi} L_0 \longrightarrow F \longrightarrow 0$$

Call t the rank of L_0 and call $I(\psi)$ the ideal generated by order t minors taken out of some matrix representation of ψ. Therefore $I(\psi)$ is by definition the 0-th Fitting ideal for ψ and does not depend on the presentation of F. Since Fitting ideal commute with localization, we can sheafify and, setting $\mathscr{I}(\psi) := \widetilde{I(\psi)}$, we obtain the sheaf of Fitting ideals for $\mathscr{L}$. More precisely, if we sheafify the minimal free presentation of F we obtain

$$\bigoplus_{i=1}^{s} \mathscr{O}_{\mathbf{P}^r}(-h_i) \xrightarrow{\widetilde{\psi}} \bigoplus_{j=1}^{t} \mathscr{O}_{\mathbf{P}^r}(-k_j) \longrightarrow \mathscr{L} \longrightarrow 0$$

and the sheaf of Fitting ideals for $\widetilde{\psi}$ is exactly $\mathscr{I}(\psi)$.

Lemma 9.1. *With the above notations,* $\operatorname{reg}(\mathscr{I}(\psi)) \geq \operatorname{reg}(\mathscr{I}_X)$.

Proof. Let us note that support of $\mathscr{L}$ is contained in X, hence the ideal sheaf $\mathscr{I}(\psi)$ has support in X and is forced to be a sub-sheaf of $\mathscr{I}_X$. Now, one has the following exact sequence

$$0 \longrightarrow \mathscr{I}(\psi) \longrightarrow \mathscr{I}_X \longrightarrow \mathscr{I}_X/\mathscr{I}(\psi) \longrightarrow 0$$

that, up to twist, induces the following long exact sequence

$$\cdots \longrightarrow H^i(\mathbf{P}^r, \mathscr{I}_X/\mathscr{I}(\psi)(m-i)) \longrightarrow H^{i+1}(\mathbf{P}^r, \mathscr{I}(\psi)(m-i)) \longrightarrow \\ \longrightarrow H^{i+1}(\mathbf{P}^r, \mathscr{I}_X(m-i)) \longrightarrow \cdots$$

Indeed, the support of $\mathscr{I}_X/\mathscr{I}(\psi)$ is 0-dimensional, so $H^i(\mathbf{P}^r, \mathscr{I}_X/\mathscr{I}(\psi)(m-i)) = 0$ for every $i > 0$ and for every $m \in \mathbb{Z}$, proving thus $H^i(\mathbf{P}^r, \mathscr{I}(\psi)(m-i)) \simeq H^i(\mathbf{P}^r, \mathscr{I}_X(m-i))$ for every m and for every $i > 1$. For $i = 1$ one has

$$\cdots \longrightarrow H^0(\mathbf{P}^r, \mathscr{I}_X/\mathscr{I}(\psi)(m)) \longrightarrow H^1(\mathbf{P}^r, \mathscr{I}(\psi)(m-1)) \longrightarrow \\ \longrightarrow H^1(\mathbf{P}^r, \mathscr{I}_X(m-1)) \longrightarrow 0$$

hence the obstruction $H^0(\mathbf{P}^r, \mathscr{I}_X/\mathscr{I}(\psi)(m)) \neq 0$ suffices to to claim that $\operatorname{reg}(\mathscr{I}_X) \leq \operatorname{reg}(\mathscr{I}(\psi))$. □

Lemma 9.1, allows us to study regularity of the sheaf $\mathscr{I}(\psi)$, generated by maximal minors obtained by a minimal free presentation associated to an invertible sheaf $\mathscr{L}$ in the above way.

9.3. Linear presentations.

Let A be a graded ring and let M be a graded A-module. Recall that M is said to be *generated in degree* j if, for every i, one has $M_{i+j} = A_i M_j$.

Definition 9.1. Let M be a graded finitely generated S-module. We say that M has a *free linear presentation* if the minimal free resolution

$$\cdots \longrightarrow L_1 \xrightarrow{\varphi_1} L_0 \longrightarrow M \longrightarrow 0$$

has the property that L_i is generated in degree i, for $i = 0, 1$.

Equivalently, one says M has a free linear presentation if $L_0 = S^{b_0}$ and $L_1 = S(-1)^{b_1}$, namely if and only if M is generated in degree zero and the arrow φ_1 can be represented with a matrix of linear homogeneous polynomials.

REMARK 9.1. If M has a free linear presentation, of course $M_d = 0$ holds for $d < 0$. Conversely, if M is a graded finitely generated S-module such that $M_d = 0$ for every $d < 0$ and the minimal free presentation for M is $L_1 \longrightarrow L_0 \longrightarrow M \longrightarrow 0$, therefore L_0 is generated in degree at least zero. By Nakayama's Lemma, the kernel of the arrow $L_0 \longrightarrow F$ is contained in $(x_0, \dots, x_r)L_0$, so it is generated in degree at least 1; by minimality of presentation, even L_1 must be generated in degree at least 1. Hence, a S-module M generated in degree at least 0 admits a free linear presentation if and only if L_i does not need generators having degree greater than i, for $i = 0, 1$.

In the following, we will make use of the notion of *tautological rank r sub-bundle* over $\mathbf{P}^r := \mathbf{P}^r_k$; it is defined as the sub-bundle $\mathscr{M}$ of $\mathscr{O}_{\mathbf{P}^r}^{r+1}$ that makes Euler's sequence to be exact:

$$0 \longrightarrow \mathscr{M} \longrightarrow \mathscr{O}_{\mathbf{P}^r}^{r+1} \longrightarrow \mathscr{O}_{\mathbf{P}^r}(1) \longrightarrow 0$$

The second map is induced by the homogeneous coordinated that generate globally $\mathscr{O}_{\mathbf{P}^r}(1)$, the sheaf of linear form. We already established that $\mathscr{M}$ can be identified with the cotangent sheaf $\Omega^1_{\mathbf{P}^r}(1)$.

Let us introduce some machinery involving exterior powers. Let

$$\mathbf{K} :\ 0 \longrightarrow K^{r+1} \longrightarrow \cdots \longrightarrow K^0$$

be the minimal free resolution of the residue field $k = S/(x_0, \dots, x_r)$ thought as S-module. By self-duality properties of Koszul complexes, $\mathbf{K}$ can be identified, as a *non-graded* complex, with the Koszul dual complex for $(x_0, \dots, x_r) \in (S^{r+1})^\vee$. To take into account grading we shall set

$$K_i := \bigwedge^i (S^{r+1}(-1)) = \left(\bigwedge^i S^{r+1}\right)(-i)$$

in order to obtain the following complex

$$\mathbf{K} :\ \cdots \xrightarrow{\varphi_3} \left(\bigwedge^2 S^{r+1}\right)(-2) \xrightarrow{\varphi_2} S^{r+1}(-1) \xrightarrow{\varphi_1} S$$

where φ_1 is again represented by the row matrix $(x_0 \ \dots \ x_r)$. Let be $M_i = (\ker \varphi_i)(i)$ the kernel, twisted in a way such that it can be seen as a sub-module in $\bigwedge^i S^{r+1}$. Note that the tautological sub-bundle $\mathscr{M}$ over $\mathbf{P}^r$ is the sheafification of M_1: this can be proved sheafifying the sequence

$$0 \longrightarrow M_1(-1) \longrightarrow S^{r+1}(-1) \longrightarrow S \longrightarrow 0$$

and comparing with Euler's sequence. In fact, sheafification is given by

$$0 \longrightarrow \widetilde{M_1}(-1) \longrightarrow \mathscr{O}_{\mathbf{P}^r}^{r+1}(-1) \longrightarrow \mathscr{O}_{\mathbf{P}^r} \longrightarrow 0$$

so we must have $\widetilde{M_1} \simeq \mathscr{M}$. This fact has a relevant generalization.

LEMMA 9.2. *With above notations, the sheaf $\bigwedge^i \mathscr{M}$ is the sheafification of M_i.*

PROOF. Since sheafification of Koszul complex is exact, every M_i sheafifies in a vector bundle, hence it suffices to show that $(\widetilde{M_i})^\vee \simeq (\bigwedge^i \mathscr{M})^\vee$. Now, duality functor is left exact, so M_i is dual to $N_i := (\operatorname{coker} \varphi_i^\vee)(-i)$, which sheafifies in a vector bundle; in particular, in a reflexive sheaf[1] so

$$(\widetilde{M_i})^\vee \simeq (\widetilde{N_i})^{\vee\vee} \simeq \widetilde{N_i}$$

Therefore it is enough to show that

$$N_i \simeq \bigwedge^i N_1$$

Now, Koszul complex $\mathbf{K}$ is dual to Koszul complex for the element $x = (x_0, \ldots, x_r) \in (S^{r+1})^\vee(1)$ and the arrow $\varphi_i^\vee$ is induced by exterior product by x, so that grading is preserved. Since we have

$$N_1 = \frac{(S^{r+1})^\vee(1)}{Sx}$$

one deduces that

$$\bigwedge^i N_1 = \frac{(\bigwedge^i S^{r+1})^\vee(1)}{x \wedge (\bigwedge^{i-1} S^{r+1})^\vee(1)} = \operatorname{coker} \varphi_i^\vee$$

and this suffices to prove the thesis. □

We need one more technical result.

LEMMA 9.3. *Let $\mathscr{F}$ be a coherent sheaf over $\mathbf{P}^r$ and*

$$F = \bigoplus_{n \geq 0} H^0(\mathbf{P}^r, \mathscr{F}(n))$$

Let $\mathscr{M}$ be the rank r tautological sub-bundle over $\mathbf{P}^r$ and fix an integer i. If $d \geq i+1$ there exists an exact sequence of the form

$$0 \longrightarrow \operatorname{Tor}_i^S(F,k)_d \longrightarrow H^1\Big(\mathbf{P}^r, \bigwedge^{i+1} \mathscr{M} \otimes \mathscr{F}(d-i-1)\Big) \xrightarrow{\alpha}$$
$$\xrightarrow{\alpha} H^1\Big(\mathbf{P}^r, \bigwedge^{i+1} \mathscr{O}_{\mathbf{P}^r}^{r+1} \otimes \mathscr{F}(d-i-1)\Big) \longrightarrow \cdots$$

where α is induced by immersion $\mathscr{M} \hookrightarrow \mathscr{O}_{\mathbf{P}^r}^{r+1}$.

PROOF. $\operatorname{Tor}_i^S(F,k)$ is the i-th homology group of the extended Koszul complex $\mathbf{K} \otimes_S F$; in fact $\mathbf{K}$ is a projective resolution for k. Concretely, $\operatorname{Tor}_i^S(F,k)_d$ is computed as the i-th homology group of the sequence

$$\cdots \longrightarrow \Big(\bigwedge^{i+1} S^{r+1}(-i-1) \otimes F\Big)_d \longrightarrow \Big(\bigwedge^{i} S^{r+1}(-i) \otimes F\Big)_d \longrightarrow$$
$$\longrightarrow \Big(\bigwedge^{i-1} S^{r+1}(-i+1) \otimes F\Big)_d \longrightarrow \cdots$$

[1]A coherent sheaf $\mathscr{F}$ is *reflexive* if $\mathscr{H}\!om(\mathscr{H}\!om(\mathscr{F}, \mathscr{O}_X)) \simeq \mathscr{F}$. Locally free sheaves are all reflexive sheaves.

Now, for every t the module $\bigwedge^t S^{r+1}(-t) \otimes F$ is direct sum of copies of $F(-t) = F \otimes S(-t)$, hence recalling that $F_{d-t} = H^0(\mathbf{P}^r, \mathscr{F}(d-t))$ for every $d \geq t$, we find

$$\Big(\bigwedge^t S^{r+1}(-t) \otimes F\Big)_d = \Big(\bigwedge^t S^{r+1} \otimes F\Big)_{d-t} = H^0\Big(\mathbf{P}^r, \bigwedge^t \mathscr{O}_{\mathbf{P}^r}^{r+1} \otimes \mathscr{F}(d-t)\Big)$$

By this identity, we can calculate torsion functors using sheaf cohomology. Since the sheafification of $\mathbf{K}$ is locally split, it remains exact if tensored with any sheaf, for example $\mathscr{F}(d)$. Recalling Lemma 9.2, the following short exact sequence is obtained

$$0 \longrightarrow \bigwedge^t \mathscr{M} \otimes \mathscr{F}(d-t) \longrightarrow \bigwedge^t \mathscr{O}_{\mathbf{P}^r}^{r+1} \otimes \mathscr{F}(d-t) \longrightarrow \bigwedge^{t-1} \mathscr{M} \otimes \mathscr{F}(d-t+1) \longrightarrow 0$$

and it has the property to be compatible with Koszul complex. Switching to cohomology and integrating the sequence obtained with Koszul complex cohomology, we find that $\mathrm{Tor}_i^S(F,k)_d$ is the cokernel of the diagonal map, so thesis follows. □

We are now ready to prove the main result of the section.

THEOREM 9.2. *Let $\mathscr{F}$ be a coherent sheaf over $\mathbf{P}^r$, for $r \geq 2$, and $\mathscr{M}$ the rank r tautological sub-bundle over $\mathbf{P}^r$. If $\mathscr{F}$ has support in dimension at most* 1 *and if*

$$H^1\Big(\mathbf{P}^r, \bigwedge^2 \mathscr{M} \otimes \mathscr{F}\Big) = 0$$

holds, therefore the graded S-module

$$F := \bigoplus_{n \geq 0} H^0(\mathbf{P}^r, \mathscr{F}(n))$$

has a free linear presentation.

PROOF. Let $\mathbf{L}: \ \cdots \longrightarrow L_1 \xrightarrow{\varphi_1} L_0 \longrightarrow F \longrightarrow 0$ be the minimal free resolution of F. By definition of F, it is clear that L_0 has no generators with negative degree. With a similar argument to Remark 9.1, one can easily show thus that L_1 does not have generators of degree less than 1. Since $\mathscr{F}$ has support of dimension at most 1, the support of $\bigwedge^2 \mathscr{M} \otimes \mathscr{F}$ has dimension at most 1 too, so $H^p(\mathbf{P}^r, \bigwedge^2 \mathscr{M} \otimes \mathscr{F}(1-p)) = 0$ for every $p \geq 2$. Since hypothesis guarantee that identity holds for $p = 1$ too, it follows that $\bigwedge^2 \mathscr{M} \otimes \mathscr{F}$ is a 1-regular sheaf. Therefore it also is s-regular for every $s \geq 2$, and in particular this means that

$$H^1\Big(\mathbf{P}^r, \bigwedge^2 \mathscr{M} \otimes \mathscr{F}(t)\Big) = 0$$

for every $t \geq 0$. Therefore, by Lemma 9.3 we can state that $\mathrm{Tor}_1^S(F,k)_d = 0$ for every $d \geq 2$ and $\mathrm{Tor}_i^S(F,k)$ can be computed looking at the homology of complex $\mathbf{L} \otimes k$; by minimality of $\mathbf{L}$, the boundary operators in $\mathbf{L} \otimes k$ are zero maps, hence $\mathrm{Tor}_i^S(F,k) = L_i \otimes k$ for every i. Because of that, L_1 can not have generators of degree greater than 2. Hence, using the previous arguments, generators for L_1 have necessarily degree 1.

Now, F is a torsion S-module, so it has no free summands; this implies that, for every summand L'_0 of L_0, the composite arrow $L_1 \longrightarrow L_0 \longrightarrow L'_0$ is not zero, otherwise L'_0 should be the cokernel for $L_1 \longrightarrow L_0$, namely F up to isomorphism. Hence, being L_1 generated in degree 1, it follows that L_0 can not have generators of degree greater than 1. Using the above arguments, L_0's generators are necessarily of zero degree; since F is generated in degree zero, by Remark 9.1, we are able to prove that F has a free linear presentation. □

9.4. Evaluation of regularity.

Let us firstly recall that, given a complex $\mathbf{E}$ of algebraic sheaves over a ringed space X, we can define its *homology* $H_\bullet(\mathbf{E})$ in the category $\mathbf{CC}(\mathbf{Sh}(X))$ of (co)chain complexes with values in the category of sheaves over X. Such homology, thus, is by all means a functor $\mathbf{CC}(\mathbf{Sh}(X)) \longrightarrow \mathbf{Sh}(X)$ and every $H^i(\mathbf{E})$ is a sheaf, so it makes sense to compute its cohomology $H^\bullet(X, H_i(\mathbf{E}))$.

Lemma 9.4. *Let* $\mathbf{E}:\ 0 \longrightarrow E_t \xrightarrow{\varphi_t} E_{t-1} \longrightarrow \dots \longrightarrow E_0$ *be a complex of algebraic sheaves over* $\mathbf{P}^r_k$ *and fix an integer* d*. Let us assume that, for every* $i > 0$*, homology* $H_i(\mathbf{E})$ *is supported in dimension at most* 1*. Therefore, if* $\operatorname{reg}(E_s) - s \leq d$ *for every* $s \geq 0$ *, one has*

$$\operatorname{reg}(\operatorname{coker}\varphi_1) \leq d, \quad \operatorname{reg}(\operatorname{im}\varphi_1) \leq d+1$$

Proof. Let us proceed by induction on t. The case $t = 0$ is trivial so let us assume $t > 0$ and let us consider the following exact sequence, for every integers d, i

$$0 \longrightarrow \operatorname{im}\varphi_1(d-i) \longrightarrow E_0(d-i) \longrightarrow \operatorname{coker}\varphi_1(d-i) \longrightarrow 0$$

from which we can determine the long cohomology sequence:

$$\cdots \longrightarrow H^i(\mathbf{P}^r, E_0(d-i)) \longrightarrow H^i(\mathbf{P}^r, \operatorname{coker}\varphi_1(d-i)) \longrightarrow$$
$$\longrightarrow H^{i+1}(\mathbf{P}^r, \operatorname{im}\varphi_1(d-i)) \longrightarrow \cdots$$

Thus, if we have $\operatorname{reg}(\operatorname{im}\varphi_1) \leq d+1$ it follows that

$$H^{i+1}(\mathbf{P}^r, \operatorname{im}\varphi_1(d-i)) = H^{i+1}(\mathbf{P}^r, \operatorname{im}\varphi_1(d+1-i-1)) = 0$$

for every $i > 0$; moreover, since by hypothesis $\operatorname{reg}(E_0) \leq d$, one has also

$$H^i(\mathbf{P}^r, E_0(d-i)) = 0$$

for every $i > 0$. Combining the two identities, we obtain

$$H^i(\mathbf{P}^r, \operatorname{coker}\varphi_1(d-i)) = 0$$

for every $i < 0$, so that $\operatorname{reg}(\operatorname{coker}\varphi_1) \leq d$. This implies that the inequality for $\operatorname{reg}(\operatorname{im}\varphi_1)$ implies the one for $\operatorname{reg}(\operatorname{coker}\varphi_1)$.

Since $H_1(\mathbf{E})$ has support in dimension at most 1, we can claim that $H^i(\mathbf{P}^r, H_1(\mathbf{E})(p)) = 0$ for every $i > 1$ and for every $p \in \mathbb{Z}$. So, considering the short exact sequence

$$0 \longrightarrow H_1(\mathbf{E}) \longrightarrow \operatorname{coker}\varphi_2 \longrightarrow \operatorname{im}\varphi_1 \longrightarrow 0$$

and its associated long cohomology sequence, we can determine $\operatorname{reg}(\operatorname{im}\varphi_1) \leq \operatorname{reg}(\operatorname{coker}\varphi_2)$. But by inductive hypothesis we find that $\operatorname{reg}(\operatorname{coker}\varphi_2) \leq d+1$, and the thesis is proved. □

The above result is useful to derive an appropriate estimate for the regularity of an ideal sheaf.

Proposition 9.1. *Let us assume that $\varphi : \mathscr{F}_1 \longrightarrow \mathscr{F}_0$ is a vector bundles morphism over $\mathbf{P}^r := \mathbf{P}^r_k$, and that*

$$\mathscr{F}_0 = \bigoplus_{i=1}^{m} \mathscr{O}_{\mathbf{P}^r_k}, \quad \mathscr{F}_1 = \bigoplus_{i=1}^{n} \mathscr{O}_{\mathbf{P}^r_k}(-1)$$

If the ideal sheaf $\mathscr{I}_m(\varphi)$ generated by order m minors taken out of φ defines a closed subscheme in $\mathbf{P}^r_k$ with dimension at most 1, therefore

$$\operatorname{reg}(\mathscr{I}_m(\varphi)) \leq m$$

Proof. Let us consider the Eagon-Northcott complex $\mathbf{EN}(\varphi) := (E_p)_{p\geq 0}$ relatively to the morphism φ: the zeroth term is isomorphic to $\mathscr{O}_{\mathbf{P}^r}$ while for $p>0$ the general term has the form

$$E_p = \left(\bigodot^{p-1}\mathscr{F}_0\right)^{\vee} \otimes \bigwedge^{m+p-1}\mathscr{F}_1$$

Because both symmetric algebra and exterior algebra are free over free objects, we have that

$$\bigodot^{p-1}\mathscr{O}^m_{\mathbf{P}^r} \simeq \mathscr{O}_{\mathbf{P}^r}^{\binom{m+p-2}{p-1}}$$

and

$$\bigwedge^{m+p-1}\mathscr{O}_{\mathbf{P}^r}(-1)^n \simeq \left(\bigwedge^{m+p-1}\mathscr{O}^n_{\mathbf{P}^r}\right)(-m-p+1) \simeq$$
$$\simeq \mathscr{O}_{\mathbf{P}^r}(-m-p+1)^{\binom{n}{m+p-1}}$$

That is to say, the p-th term of the complex $\mathbf{EN}(\varphi)$ is direct sum of copies of $\mathscr{O}_{\mathbf{P}^r}(-m-p+1)$, which has regularity $m+p-1$. Hence, by Lemma 9.4 with $d=m-1$, one finds that

$$\operatorname{reg}(\operatorname{coker}\partial_1) = \operatorname{reg}(\mathscr{I}_m(\varphi)) \leq m-1 < m$$

□

The next result unites the progress made until now. Let us call $h^j(\mathscr{F}) := \dim_k H^j(X,\mathscr{F})$.

Theorem 9.3. *Let $X \subseteq \mathbf{P}^r := \mathbf{P}^r_k$ be an irreducible smooth curve, with $r \geq 3$, and let $\mathscr{L}$ be a line bundle over X; let also $\mathscr{M}$ be the rank r tautological sub-bundle over $\mathbf{P}^r$. If*

$$H^1\left(X, \bigwedge^2 \mathscr{M} \otimes \mathscr{L}\right) = 0$$

therefore $\operatorname{reg}(\mathscr{I}_X) \leq h^0(\mathscr{L})$.

PROOF. Using the stated hypotheses, by Theorem 9.2 the S-module

$$F = \bigoplus_{n \geq 0} H^0(X, \mathscr{L}(n))$$

has a free presentation, say $S(-1)^n \longrightarrow S^m \longrightarrow F \longrightarrow 0$, proving that $\mathscr{L}$ is the cokernel of a locally linear morphism $\varphi : \mathscr{O}^n_{\mathbf{P}^r}(-1) \longrightarrow \mathscr{O}^m_{\mathbf{P}^r}$. Hence, by Proposition 9.1 it follows that $\operatorname{reg}(\mathscr{I}_m(\varphi)) \leq h^0(\mathscr{L})$. Finally, Lemma 9.1 shows readily that $\operatorname{reg}(\mathscr{I}_X) \leq \operatorname{reg}(\mathscr{I}_m(\varphi))$. □

9.5. Filtering the tautological sub-bundle.

Thanks to the reduction made in Theorem 9.3, the main result is proved exhibiting an invertible sheaf $\mathscr{L}$ that satisfies the conditions required by the Theorem and such that $h^0(\mathscr{L})$ is accurately controlled. To obtain the vanishing of cohomology of $\bigwedge^2 \mathscr{M} \otimes \mathscr{L}$ we have to use some specific properties of vector bundles over smooth curves.

LEMMA 9.5. *Let $\mathscr{N}$ be a vector bundle over a smooth curve C, over an algebraically closed field k. If $\mathscr{N}$ is contained in the direct sum of copies of $\mathscr{O}_C$ and if $H^0(C, \mathscr{N}) = 0$, there exists a filtration*

$$\mathscr{N} = \mathscr{N}_0 \supset \mathscr{N}_1 \supset \ldots \supset \mathscr{N}_{r+1} = 0$$

whose factorials $\mathscr{N}_i / \mathscr{N}_{i+1} = \mathscr{L}_i$ are line bundles of strictly negative degree.

PROOF. It is enough, and necessary, to determine a morphism $\mathscr{N} \longrightarrow \mathscr{L}_1$, where $\mathscr{L}_1$ is a line bundle of strictly negative degree. In this condition, the kernel $\mathscr{N}'$ satisfies immediately the hypothesis and a filtration can be hence produced by induction.

Now, we know there exists an immersion $\mathscr{N} \longrightarrow \mathscr{O}^n_C$ for some n. Let us prove that one can choose n as the rank r of $\mathscr{N}$. Since $\mathscr{N}$ is a locally free sheaf, it suffices to show this locally; that is to say, given an integral domain A, if M is an A-module contained in some direct sum A^n, then n can be chosen exactly the rank of M. In fact, calling K the quotient field of A, we know $M \otimes_A K$ is a vector K-space having exactly dimension r, the rank of M. Now, let $\{b_1, \ldots, b_r\}$ be a basis for $M \otimes_A K$ over K: since $M \otimes_A K \simeq K^r$, we can write also $b_i = (x_{i,1}/y_{i,1}, \ldots, x_{i,r}/y_{i,r})$. Hence the morphism $K^r \longrightarrow A^r$ set by $b_i \mapsto (x_{i,1}, \ldots, x_{i,r})$ for every i is an isomorphism and gives an immersion $M \subseteq A^r$, as we wished.

Now, let us consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathscr{N} & \xrightarrow{f} & \mathscr{O}^r_C \\ & & & & \big\downarrow \pi \\ & & & & \mathscr{O}_C \\ & & & & \big\downarrow \\ & & & & 0 \end{array}$$

where π is the canonical projection. The arrow f fails to be an isomorphism, since $H^0(C, \mathscr{O}_C^r) \neq 0$; moreover $\mathscr{N}$ is not contained in $\ker(\pi) = \mathscr{O}_C^{r-1}$, or it would contradict the fact that $\mathrm{rk}(\mathscr{N}) = r$. So the sheaf $\mathscr{J} := \mathrm{im}(\pi \circ f)$ is a non-zero algebraic coherent subsheaf in $\mathscr{O}_C$, and it makes the following diagram to commute:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathscr{N} & \xrightarrow{f} & \mathscr{O}_C^r \\ & & \downarrow & & \downarrow \pi \\ 0 & \longrightarrow & \mathscr{J} & \longrightarrow & \mathscr{O}_C \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array}$$

Necessarily, $\mathscr{J}$ is an ideal sheaf of rank at most 1. It remains to show that $\mathscr{J}$ is locally free. Since C is a smooth curve, for every $p \in C$ the local ring $\mathscr{O}_{C,p}$ is a regular local ring of dimension 1. By Auslander-Buchsbaum formula, it follows that

$$\dim^{(\mathrm{proj})}(\mathscr{J}_p) + \mathrm{depth}(\mathscr{J}_p) = \dim(\mathscr{O}_{C,p}) = 1$$

Since $\mathscr{J}_p \subseteq \mathscr{O}_{C,p}$, every $\mathscr{O}_{C,p}$-regular element must be $\mathscr{J}_p$-regular too, hence $\mathscr{J}_p$ admits a regular element in $\mathfrak{m}_p$. This proves $\mathrm{depth}(\mathscr{J}_p) = 1$ so $\mathscr{J}_p$ is projective. But a projective module over a local ring is compelled to be a free module, hence $\mathscr{J}_p$ is free for every $p \in C$. By Proposition 2.1, this is enough to show that $\mathscr{J}$ is locally free of rank 1. ☐

Lemma 9.6. *If $\mathscr{N}$ is a vector bundle over a variety X, such that there exists a filtration*

$$\mathscr{N} = \mathscr{N}_0 \supset \mathscr{N}_1 \supset \ldots \supset \mathscr{N}_{r+1} = 0$$

whose factorials are line bundles $\mathscr{L}_i$, therefore the bundle $\bigwedge^2 \mathscr{N}$ has a similar filtration, and its factorials are isomorphic to the line bundles $\mathscr{L}_i \otimes \mathscr{L}_j$ for $1 \leq i < j \leq r$.

Proof. Let us proceed by induction on the rank r of $\mathscr{N}$. If $r = 1$, therefore $\bigwedge^2 \mathscr{N} = 0$ so there is nothing to prove. Let $r > 1$ and let us assume $\mathscr{N}$ has a filtration like in hypothesis. Therefore, by short exact sequence

$$0 \longrightarrow \mathscr{N}_r \longrightarrow \mathscr{N} \longrightarrow \mathscr{N}/\mathscr{N}_r \longrightarrow 0$$

one can find the following exact sequence with exterior powers, obtained using 3.10:

$$0 \longrightarrow (\mathscr{N}/\mathscr{N}_r) \otimes \mathscr{N}_r \longrightarrow \bigwedge^2 \mathscr{N} \longrightarrow \bigwedge^2 \mathscr{N}/\mathscr{N}_r \longrightarrow 0$$

Bundles $(\mathscr{N}_i/\mathscr{N}_r) \otimes \mathscr{N}_r$ make a filtration of $(\mathscr{N}/\mathscr{N}_r) \otimes \mathscr{N}_r$ with factorials

$$\frac{\mathscr{N}_i/\mathscr{N}_r \otimes \mathscr{N}_r}{\mathscr{N}_{i+1}/\mathscr{N}_r \otimes \mathscr{N}_r} \simeq \mathscr{L}_i \otimes \mathscr{N}_r$$

Similarly, since bundles $\mathscr{N}_i/\mathscr{N}_r$ make a filtration for $\mathscr{N}/\mathscr{N}_r$ with factorials

$$\frac{\mathscr{N}_i/\mathscr{N}_r}{\mathscr{N}_{i+1}/\mathscr{N}_r} \simeq \mathscr{N}_i/\mathscr{N}_{i+1} \simeq \mathscr{L}_i$$

the bundle $\bigwedge^2 \mathscr{N}/\mathscr{N}_r$ has a filtration with factorials $\mathscr{L}_i \otimes \mathscr{L}_j$, for $1 \leq i < j \leq r-1$. □

9.6. Summary of the proof.

We are finally able to glue together all the various parts in order to arrange a complete proof for Theorem 9.1.

Proof. Call $d = \deg X$ and let $\mathscr{M}$ be the rank r tautological sub-bundle over $\mathbf{P}^r$; we know that $\mathscr{M}$ is contained in the direct sum of $r+1$ copies of $\mathscr{O}_X$ so fulfills the hypotheses required to apply Lemma 9.6; therefore, we can claim that $\bigwedge^2 \mathscr{M}$ has a finite filtration, whose factorial are isomorphic to vector bundles $\mathscr{L}_i \otimes \mathscr{L}_j$, for $i, j \in \{1, \ldots, r\}$. Moreover, every $\mathscr{L}_j$ has strictly negative degree.

To obtain the vanishing of $H^1(X, \bigwedge^2 \mathscr{M} \otimes \mathscr{L})$, one can note that it suffices to obtain the vanishing of every $H^1(X, \mathscr{L}_i \otimes \mathscr{L}_j \otimes \mathscr{L})$. But by Theorem 4.4, this is achieved choosing a general line bundle $\mathscr{L}$ of degree e and such that

$$g - 1 \leq \deg(\mathscr{L}_i \otimes \mathscr{L}_j \otimes \mathscr{L}) = \deg \mathscr{L}_i + \deg \mathscr{L}_j + e$$

Now, it is known that

$$\bigwedge^r \mathscr{M} \simeq \mathscr{O}_{\mathbf{P}^r}(-1)$$

so restricting to X one has that

$$\sum_i \deg(\mathscr{L}_i) = \deg(\mathscr{M}) = \deg(\mathscr{O}_X(-1)) = -d$$

Hence, for every distinct i, j we have

$$\deg \mathscr{L}_i + \deg \mathscr{L}_j = -d - \sum_{p \neq i,j} \deg \mathscr{L}_p \geq -d + r - 2$$

since $\deg \mathscr{L}_j \leq -1$ for every $j = 1, \ldots, r$. Therefore, it is enough to choose

$$e = g - 1 + d - r + 2 = g + d - (r-1)$$

We have proved that, if $\mathscr{L}$ is a general line bundle of degree $g + d - (r-1)$, therefore Theorem 9.3 holds, that is to say $\operatorname{reg}(\mathscr{I}_X) \leq h^0(\mathscr{L})$. By Riemann-Roch's Theorem, one has

$$\begin{aligned} h^0(\mathscr{L}) &= h^0(K_X - \mathscr{L}) + \deg(\mathscr{L}) + 1 - g = \\ &= h^1(\mathscr{L}) + (g - 1 + d - r + 2) + 1 - g = \\ &= h^1(\mathscr{L}) + d - r + 2 \end{aligned}$$

Recalling Corollary 4.1 we find that $\deg(X) \geq r = 1 + \operatorname{codim}(X)$, hence we may choose $\mathscr{L}$ such that $\deg(\mathscr{L}) \geq g + 1$, so that Theorem 4.4 holds; hence $\mathscr{L}$ is a non special bundle and we can claim $h^0(\mathscr{L}) = d - (r-1) + 1 =$

$\deg(X) - \text{codim}(X) + 1$. This, together with the above considerations, brings directly to the thesis. □

9.7. Examples.

We may try to show the results of Theorem 9.1 in a more geometric context.

Let $X \subseteq \mathbf{P}^r := \mathbf{P}^r_k$ be a smooth, irreducible and non degenerate curve and let $s = \text{reg}(X)$, $d = \deg(X)$. By definition, this means that the ideal sheaf $\mathscr{I}_X$ of X is s-regular, namely

$$H^i(\mathbf{P}^r, \mathscr{I}_X(s-i)) = 0$$

for every $i \geq 1$, and that $H^i(\mathbf{P}^r, \mathscr{I}_X(t - i_0)) \neq 0$ for some $t < s$, $i_0 \geq 1$. Therefore, let us consider the following exact sequence:

$$0 \longrightarrow \mathscr{I}_X(s-i) \longrightarrow \mathscr{O}_{\mathbf{P}^r}(s-i) \longrightarrow \mathscr{O}_X(s-i) \longrightarrow 0$$

Taking cohomology, the resulting long exact sequence shows that $\mathscr{I}_X$ is s-regular if and only if

$$\begin{aligned} &H^1(\mathbf{P}^r, \mathscr{I}_X(p)) = 0, \qquad p \geq s-1 \\ &H^1(\mathbf{P}^r, \mathscr{O}_X(s-2)) = 0 \end{aligned} \tag{9.7.1}$$

Hence, by these considerations, Theorem 9.1 states that

$$H^1(\mathbf{P}^r, \mathscr{I}_X(s+1)) = 0, \quad H^1(\mathbf{P}^r, \mathscr{O}_X(s)) = 0$$

for every $s \geq d - r$. By Theorem 8.2, it is not necessary to verify the statement for big s, since the regularity property holds definitively; essentially, Theorem shows that

$$H^1(\mathbf{P}^r, \mathscr{I}_X(d-r+1)) = 0, \quad H^1(\mathbf{P}^r, \mathscr{O}_X(d-r)) = 0 \tag{9.7.2}$$

Let us investigate some remarkable concrete cases in which this can be achieved; the depth of the results varies together with dimension r of the underlying projective space.

For the ground case, let us consider a smooth curve X over the plane $\mathbf{P}^2$; calling $d = \deg(X)$, the theorem claims simply that $\mathscr{I}_X$ is d-regular (the other condition is trivial). Let us prove the theorem actually holds: first condition in (9.7.1) is trivial, since the ideal sheaf of X is $\mathscr{O}_{\mathbf{P}^2}(-d)$. The second relation is the most interesting: by adjunction formula 3.5 we know that $\omega_X \simeq \omega_{\mathbf{P}^2}(d) \simeq \mathscr{O}_{\mathbf{P}^2}(d-3)$, hence

$$H^1(\mathbf{P}^r, \mathscr{O}_X(d-2)) = 0$$

agreeing perfectly with Theorem 9.1. Let us remark that, even if the result is not very significant, the Theorem still makes a very neat prediction.

The first non trivial case shows for $r = 3$. Theorem claims that

$$H^1(\mathbf{P}^3, \mathscr{I}_X(d-2)) = 0, \quad H^1(\mathbf{P}^3, \mathscr{O}_X(d-3)) = 0$$

These conditions are not easily double-checked in the general case; let us verify them in some significant additional hypotheses on X. Recall that a projective curve $X \subseteq \mathbf{P}^r$ is said *complete intersection* is its ideal I_X is generated by exactly $r-1$ elements. In particular, there exists hypersurfaces $F_{d_1}, \ldots, F_{d_{r-1}} \subseteq \mathbf{P}^r$ such that $X = F_{d_1} \cap \ldots \cap F_{d_{r-1}}$. If, moreover, one calls $d = \deg(X)$, it is clear that $d = \deg(F_{d_1}) \cdots \deg(F_{d_{r-1}}) = d_1 \cdots d_{r-1}$.

Thus, if $X \subseteq \mathbf{P}^3$ is complete intersection of hypersurfaces F_{d_1}, F_{d_2} having degrees d_1, d_2 respectively, therefore by Hilbert Burch's Theorem, the ideal $\mathscr{I}_X$ has a free resolution of the following form:

$$0 \longrightarrow \mathscr{O}_{\mathbf{P}^3}(-d_1 - d_2) \longrightarrow \mathscr{O}_{\mathbf{P}^3}(-d_1) \oplus \mathscr{O}_{\mathbf{P}^3}(-d_2) \xrightarrow{\psi} \mathscr{I}_X \longrightarrow 0$$

where the arrow ψ is induced by multiplication by matrix (F_{d_1}, F_{d_2}), where $F_{d_i} = V(F_i)$. Taking cohomology over $\mathbf{P}^3$, this implies that

$$H^1(\mathbf{P}^3, \mathscr{I}_X(s)) = 0$$

for every s, and in particular for $s = d-2$. To show the other condition, let us note that $\omega_X \simeq \omega_{\mathbf{P}^3}(d_1+d_2) \simeq \mathscr{O}_{\mathbf{P}^3}(d_1+d_2-4)$; therefore $H^1(\mathbf{P}^3, \mathscr{O}_X(p)) = 0$ if $p \geq d_1 + d_2 - 3$. But if $d_1, d_2 \geq 1$ it is easy to obtain the inequality over integers

$$d = d_1 d_2 \geq d_1 + d_2 - 1$$

even more so $H^1(\mathbf{P}^3, \mathscr{O}_X(p)) = 0$ holds if $p \geq d - 3$.

An other case in which it is not difficult to check the theorem is the case of a rational normal curve $X \subseteq \mathbf{P}^3$ of degree $d = 3$. We know that X is birationally equivalent to $\mathbf{P}^1$ and that it could be identified with the closed immersion $\mathbf{P}^1 \hookrightarrow \mathbf{P}^3$ defined by complete linear system $|\mathscr{O}_{\mathbf{P}^1}(3)|$. Moreover, one has the identifications

$$H^p(\mathbf{P}^3, \mathscr{O}_X(s)) \simeq H^p(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(3s)) \tag{9.7.3}$$

for every $p \geq 0$ and for every s. In particular

$$H^1(\mathbf{P}^3, \mathscr{O}_X(d-3)) = H^1(\mathbf{P}^3, \mathscr{O}_X) \simeq H^1(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}) = 0$$

holds. To show the other condition, let us consider the exact sequence

$$0 \longrightarrow \mathscr{I}_X(1) \longrightarrow \mathscr{O}_{\mathbf{P}^3}(1) \longrightarrow \mathscr{O}_X(1) \longrightarrow 0$$

Taking cohomology over $\mathbf{P}^3$, the long exact sequence shows that

$$H^0(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(1)) \longrightarrow H^0(\mathbf{P}^3, \mathscr{O}_X(1)) \longrightarrow H^1(\mathbf{P}^3, \mathscr{I}_X(1)) \longrightarrow 0$$

Recalling isomorphisms (9.7.3) and that $H^0(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(1)) \simeq H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(3))$ holds by definition of curve, one proves that $H^1(\mathbf{P}^3, \mathscr{I}_X(d-2)) = H^1(\mathbf{P}^3, \mathscr{I}_X(1)) = 0$.

The last interesting case we treat is a rational quartic over $\mathbf{P}^3$, namely a degree 4 curve X that is birationally equivalent to $\mathbf{P}^1$. In this case, the theorem claims that

$$H^1(\mathbf{P}^3, \mathscr{I}_X(2)) = 0, \quad H^1(\mathbf{P}^3, \mathscr{O}_X(1)) = 0$$

One can identify X with the linear system $|V|$ associated to a 4-dimensional vector subspace $V \subseteq \Gamma(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(4))$. In particular, this leads immediately that

$$H^1(\mathbf{P}^3, \mathscr{O}_X(1)) \simeq H^1(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(4)) = 0$$

To obtain the other condition, let us note that the following short exact sequence

$$0 \longrightarrow \mathscr{I}_X(2) \longrightarrow \mathscr{O}_{\mathbf{P}^3}(2) \longrightarrow \mathscr{O}_X(2) \longrightarrow 0$$

induces the following long exact sequence in cohomology:

$$0 \longrightarrow H^0(\mathbf{P}^3, \mathscr{I}_X(2)) \longrightarrow H^0(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(2)) \xrightarrow{f}$$

$$\xrightarrow{f} H^0(\mathbf{P}^3, \mathscr{O}_X(2)) \longrightarrow H^1(\mathbf{P}^3, \mathscr{I}_X(2)) \longrightarrow 0$$

So it is enough to show that the arrow f is a surjection; since

$$h^0(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(2)) = \binom{5}{2} = 10, \quad h^0(\mathbf{P}^3, \mathscr{O}_X(2)) = h^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(8)) = 9$$

it suffices to show that $h^0(\mathbf{P}^3, \mathscr{I}_X(2)) = 1$. Let us assume by contradiction that $h^0(\mathbf{P}^3, \mathscr{I}_X(2)) \geq 2$; in this case, X is contained in the intersection $Q_1 \cap Q_2$ of two smooth quadrics. Therefore, using adjunction formula

$$\omega_{Q_j} = \omega_{\mathbf{P}^3} \otimes \mathscr{O}_{\mathbf{P}^3}(2) \otimes \mathscr{O}_{Q_j} = \mathscr{O}_{Q_j}(-2),$$
$$\omega_{Q_1 \cap Q_2} = \omega_{Q_j} \otimes \mathscr{O}_{Q_j}(2) \otimes \mathscr{O}_X = \mathscr{O}_X$$

that is $g(Q_1 \cap Q_2) = h^0(X, \mathscr{O}_X) = 1$. Comparing Euler-Poincaré's characteristics, it should hold that $\chi(\mathscr{O}_X(p)) \leq \chi(\mathscr{O}_{Q_1 \cap Q_2}(p))$, but the above arguments show instead that

$$\chi(\mathscr{O}_X(p)) = 4p + 1 > 4p = \chi(\mathscr{O}_{Q_1 \cap Q_2}(p))$$

and this is a contradiction.

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