THE SEIFERT-VAN KAMPEN THEOREM

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ABSTRACT. We propose a detailed self-contained survey of the Seifert-Van Kampen's theorem, along with a brief summary of topological and algebraic preliminaries, culminating in a proof accesible to the non-expert reader. The last section contains some notable computations.

1. Topological preliminaries.

We recall here the main topological concepts needed for the next sections.

A *topological space* is a pair (X, τ_X) where X is a set and τ_X a topology on X; when no danger of confusion may arise (namely, always) we will forget to specify the underlying topology of X. Elements of τ_X are called *open sets*; we say $C \subseteq X$ is a *closed set* if $X \setminus C \in \tau_X$. A set $U \subseteq X$ is a *neighborhood* of a point $x \in X$ if there is a $A \in \tau_X$ such that $x \in A \subseteq U$; the family $\mathscr{U}_x$ of all the neighborhoods of x in X is the *filter of neighborhoods*.

Definition 1.1. A *basis* for the topology τ_X on X is a subfamily $\mathfrak{B}_X$ of τ_X such that for each $A \in \tau_X$ there exist $B_i \in \mathfrak{B}_X$ such that A is the union of the B_i.

Definition 1.2. Given $x \in X$, a *basis of neighborhoods* for x in X is a subfamily $\mathfrak{V}_x$ of $\mathscr{U}_x$ such that, for each $U \in \mathscr{U}_x$ there exists a $V \in \mathfrak{V}_x$ such that $V \subseteq U$.

The *euclidean topology* over $\mathbf{R}^n$ is the topology having basis $\mathfrak{E}^n = \{B(x, \rho) \mid \rho > 0, x \in \mathbf{R}^n\}$. Unless differently stated, when dealing with subsets of euclidean spaces, we henceforth assume that they are endowed with the euclidean topology.

Given a topological space X, a family $\{U_i\}_{i \in I} \subseteq \tau_X$ is an *open cover* of X if the union of all the U_i is the whole X.

Definition 1.3. A topological space X is *connected* if it can not be written as union of two disjoint open subsets.

The maximal connected sets in X are called *connected components* and form a partition of X.

Definition 1.4. A topological space X is *quasi-compact* if every open cover of X has a finite open sub-covering.

Definition 1.5. Let X be a topological space.Then X is said to be

- T_0 (or *Kolmogoroff*) if for each pair of distinct points $x, y \in X$ there are $U \in \mathscr{U}_x$ and $V \in \mathscr{U}_y$ such that $y \notin U$ or $x \notin V$;
- T_1 (or *Fréchet*) if for each pair of distinct points $x, y \in X$ there are $U \in \mathscr{U}_x$ and $V \in \mathscr{U}_y$ such that $x \notin V$ and $y \notin U$;
- T_2 (or *Hausdorff*) if for each pair of distinct points $x, y \in X$ there are $U \in \mathscr{U}_x$ and $V \in \mathscr{U}_y$ such that $U \cap V = \varnothing$.

Euclidean topology is Hausdorff (so *a fortiori* even Fréchet and Kolmogoroff).

Recall that, given a function between topological spaces

$$f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$$

we say that f is *continuous* if $f^{-1}(A) \in \tau_X$ for each $A \in \tau_Y$. Compactness and connectedness is preserved under continuous maps. A continuous map which is invertible and whose inverse is continuous is a homeomorphism. We will often identify topological spaces up to homeomorphism.

Let I be the real closed interval

$$I = \{t \in \mathbf{R} \mid 0 \leq t \leq 1\}$$

Definition 1.6. A topological space X is *path connected* if, given any $x_0, x_1 \in X$, there is a continuous map $f : I \longrightarrow X$ such that $f(0) = x_0$, $f(1) = x_1$. This map is called *path* (or *arc*) *joining* x_0 *to* x_1.

Paths can be linked together in the following way. If $f : I \longrightarrow X$ is a path joining x_0 to x_1 and $g : I \longrightarrow X$ is a path joining x_0 to x_2, then we can define $h : I \longrightarrow X$ as follows:

$$h(t) = \begin{cases} f(1-2t) & 0 \leq t \leq 1/2 \\ g(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

Clearly h is a well defined path in X joining x_1 to x_2.

Remark 1.7. If a topological space X has a point x_0 which can be joint to every other point $x \in X$ via a continuous path, then X is clearly parh connected. If X has this special property, sometimes it is called *path star-shaped* with respect to x_0. Not every path connected space is path star-shaped.

Remark 1.8. Recall that a continuous map sends connected sets to connected sets; hence, if $f : I \longrightarrow X$ is a path in X, then $f(I)$ is a connected set which must belong to the connected component of X containing $f(0)$ e $f(1)$. It follows readily that X must have only one connected component: namely, a path connected topological space is connected.

In general, the converse is not true. Let

$$P = \{(t, \sin(1/t)) \mid 0 < t \leq 1\} \subseteq \mathbf{R}^2$$

and $Y_0 = \{(0, y) \mid 0 \leq y \leq 1\}$, then $P \cup Y_0$ is the closure of the connected set P, so it is connected. Yet is can not be path connected as points in Y_0 can not be reached by continuous paths.

Definition 1.9. A topological space X is *locally path connected* if, for each $x_0 \in X$, there exists a basis of neighborhoods $\mathfrak{V}(x_0)$ for x_0 such that each $U \in \mathfrak{V}(x_0)$ is an open path connected set.

Remark 1.10. Locally path connectedness is much weaker than path connectedness: let us consider the sets

$$X_0 = \{(x, y) \in \mathbf{R}^2 \mid 0 \leq x \leq 1, y = 0\}$$
$$Y_0 = \{(x, y) \in \mathbf{R}^2 \mid x = 0, 0 \leq y \leq 1\}$$
$$Y_n = \{(x, y) \in \mathbf{R}^2 \mid x = 1/n, 0 \leq y \leq 1\}$$

and let

$$X = (X_0 \cup Y_0) \cup \left(\bigcup_{n=2}^{\infty} Y_n \right)$$

It is not difficult to picture that X is path connected. However X it is not locally path connected, as points $y_0 \in Y_0 \backslash \{(0, 0)\}$ do not admit a basis of neighborhoods made of path connected open sets.

Theorem 1.11. A topological space X is locally path connected if and only if its topology has a basis of open path connected sets.

Definition 1.12. Two continuous maps $f_0, f_1 : X \longrightarrow Y$ are called *homotopic* if there exists a continuous map $F : X \times I \longrightarrow Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for each $x \in X$. The map F is called *homotopy* between f_0 and f_1.If, moreover, given $X_0 \subseteq X, Y_0 \subseteq Y$ we have $f_0(X_0) \subseteq Y_0, f_1(X_0) \subseteq Y_0$ and $F(X_0 \times I) \subseteq Y_0$, therefore F is called *homotopy* between f_0 and f_1 *relative to* X_0, Y_0.

Proposition 1.13. Homotopy of continuous maps is an equivalence relation and it is well behaved with respect to composition of continuous functions.

Remark 1.14. Sometimes we will write $f_0 \sim f_1$ to mean that f_0 is homotopic to f_1. Relative homotopy is either an equivalence relation.

Definition 1.15. We say that two topological spaces X, Y have *the same homotopy type* (or that they are *homotopic*) if there exist two maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$.

We will write $X \sim Y$ to indicate that X and Y have the same homotopy type.

Proposition 1.16. Having the same homotopy type is an equivalence relation.

Remark 1.17. The "set of all topological spaces" does not exists. It is a (proper) class, but equivalences can still be defined in classes.

Definition 1.18. A topological space X is *contractible* if the identity map on X is homotopic to a constant map.

Theorem 1.19. X is contractible if and only if X has the same homotopy type of a point.

Proposition 1.20. Contractible spaces are path connected.

We now concentrate on homotopy relations between *paths* in a topological space X.

Suppose X is path connected; in this setting, the notion of paths inside X is meaningful. Recall that two paths $f, g : I \longrightarrow X$ linking respectively x_0 to x_1 and x_1 to x_2 can be linked together via

$$fg(t) := \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

which is a path joining x_0 a x_2. Since we are interested in base points, we will formally say that two paths $f, g : I \longrightarrow X$ such that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$ are *homotopic* if there exists an homotopy between f and g relative to $\{0, 1\}$ and $\{x_0, x_1\}$ in the sense specified above.

Lemma 1.21. Let $f, f_1 : I \longrightarrow X$ be two paths joining x_0 and x_1 and let $g, g_1 : I \longrightarrow X$ be two paths joining x_1 to x_2. If $f \sim f_1$ and $g \sim g_1$ therefore $fg \sim f_1 g_1$.

Given a path $f : I \longrightarrow X$ joining x_0 to x_1, let us define the *reverse path* of f as the map

$$\begin{aligned} f^{-1} : I &\longrightarrow X \\ t &\mapsto f(1-t) \end{aligned}$$

which is a path in X joining x_1 and x_0.

Lemma 1.22. Let $f_0, f_1 : I \longrightarrow X$ two paths joining the same points. If $f_0 \sim f_1$, then $f_0^{-1} \sim f_1^{-1}$.

Lemma 1.23. Let $f, g, h : I \longrightarrow X$ paths joining x_0 to x_1, x_1 to x_2 and x_2 to x_3 respectively. Therefore

$$(fg)h \sim f(gh)$$

Let $x_0 \in X$ be a point. Define the *constant path* as

$$\begin{aligned} e_{x_0} : I &\longrightarrow X \\ t &\mapsto x_0 \end{aligned}$$

Lemma 1.24. Let $f : I \longrightarrow X$ a path joining x_0 to x_1. Therefore $e_{x_0} f \sim f \sim f e_{x_1}$.

Lemma 1.25. Let $f : I \longrightarrow X$ a path joining x_0 to x_1. Therefore $ff^{-1} \sim e_{x_0}$ e $f^{-1} f \sim e_{x_1}$.

Let us now consider a particular class of paths, called *loops*, which are characterised by the property of having the same start and end points. We will indicate with $\Omega(X, x_0)$ the set of loops X having *origin* at x_0.

The above lemmas allow us to define a composition law inside $\Omega(X, x_0)$, given by $(f, g) \mapsto fg$, the link of two loops. This operation is well behaved with respect to homotopy of loops (namely, homotopy relative to the origin) and it gives rise to a composition rule amongst the homotopy equivalence classes:

$$[f] * [g] := [fg]$$

This operation is well defined, is associative and has an identity element $[e_{x_0}]$, with inverse given by

$$[f]^{-1} = [f^{-1}]$$

Hence we can give the following definition.

Definition 1.26. Let X be a path connected topological space. The set $\Omega(X, x_0)$ of loops in X with origin x_0 modulo the equivalence relation of path homotopy is a group with the above composition law $*$ and it is called *first homotopy group* or *fundamental group* of X. It is denoted with $\pi_1(X, x_0)$.

If X is path connected, the choice of the base point is not relevant.

Theorem 1.27. *Let X be a path connected topological space and let $x_0, x_1 \in X$. Then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.*

Corollary 1.28. If X is any topological space, its fundamental space is unique, up to isomorphism, in each path connected component of X.

The construction made is functorial in the category of topological spaces.

Proposition 1.29. *Let $\varphi : X \longrightarrow Y$ be a continuous map such that $\varphi(x_0) = y_0$. Therefore the map*

$$\varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$
$$[f] \mapsto [\varphi \circ f]$$

is a group morphism. Moreover, if $\psi : Y \longrightarrow Z$ is another continuous map such that $\psi(y_0) = z_0$, then

$$(\psi \circ \varphi)_* = (\psi_*) \circ (\varphi_*)$$

Remark 1.30. The above proposition shows that, if X and Y are homeomorphic, then $\pi_1(X, x_0) \simeq \pi_1(Y, y_0)$ for a compatible choice of base-points.

Proposition 1.31. Let $\varphi_0, \varphi_1 : X \longrightarrow Y$ two homotopic continuous maps. Therefore there is an isomorphism $\lambda_\# : \pi_1(Y, \varphi_0(x_0)) \longrightarrow \pi_1(Y, \varphi_1(x_0))$ such that $(\varphi_1)_* = (\lambda_\#) \circ (\varphi_0)_*$.

Corollary 1.32. If X are Y path connected topological spaces with the same homotopy type, the corresponding fundamental groups are isomorphic.

Definition 1.33. A topological space with trivial fundamental group is called *simply connected.*

2. Free products and amalgamation.

Let $\{G_i\}_{i\in I}$ a collection (possibly infinite) of arbitrary groups and let

$$E = \bigcup_{i\in I} G_i$$

be the set of *letters* over G_i. The set E is often described as a choice of an *alphabet*. We will denote as$W(E)$ the set of *words* with respect to the alphabet E, namely the set of arbitrary finite strings made of elements in E; formally

$$W(E) = \{a_1 a_2 \dots a_n = w \mid a_1, \dots, a_n \in E, n \in \mathbf{N}\}$$

The *length* of a word $w \in W(E)$ of the above form is defined as $\lg(w) = n$, namely the number of letters of which w is made of. The empty word w_0, that is the word which does not contain any letter and acts like a blank space, has formally zero length. It is natural to introduce a composition law in $W(E)$ by setting:

$$(w, w') \mapsto ww'$$

namely, operating a plain juxtaposition of the two original words, without further changes. This law is clearly associative and has an identity element (given by w_0), which is however the only invertible word. Indeed, the elements of $W(E)$ need to be selected with more care in order to extract some significant information from $W(E)$.

Let us say that w and w' are *equivalent words* ($w \approx w'$) if and only if w can be turned into w' by means of a *finite* number of elementary operations of the following two kinds:

(1) removing an identity element $e_j \in G_j$ namely $a_1 e_j a_2 \approx a_1 a_2$;
(2) replacing two consecutive elements belonging to the same group with their internal product: namely if $a_1, a_2 \in G_j$ and $a = a_1 \cdot a_2 \in G_j$ then $a_1 a_2 \approx a$;

The relation $\approx$ defined above introduces an equivalence in $W(E)$. It could be verified, even if very tediously, that the juxtaposition operation is stable under $\approx$ and indeed turns $W(E)/\approx$ in a group, called *free product* of the $\{G_j\}_{j\in J}$. The free product is usually denoted as

$$\mathop{\ast}_{j\in J} G_j$$

or, if $J = \{1, \dots, n\}$ is finite, as $G_1 * G_2 * \dots * G_n$. Let us now explore some properties of this newly introduced object.

Definition 2.1. A word $w \in W(E)$ of the form $w = a_1 a_2 \dots a_n$ is said to be *reduced* if $a_j \neq e_j$ for each $j = 1, \dots, n$ and if no couple of adjacent letters belong to the same group.

Note that length is a well defined function from the set of reduced words $R(E)$ to $\mathbf{Z}$. Moreover, reduced words are a good choice for a system of representatives for the free product.

Lemma 2.2. *In every equivalence class of $W(E)/\approx$ lies exactly one reduced word.*

Proof. Assume $w = a_1 \dots a_n$ is a reduced word and let $a \in E$ be a letter. Therefore, the new word $w' = aw$ can fall in the following cases:

(1) if a is any identity element, then $w' \approx w$;
(2) if a is not an identity element and does not belong to G_{j_1}, the group to which a_1 belongs. In this case we get $w' = aa_1 \dots a_n$;
(3) if a is not an identity element, belongs to G_{j_1} as before, but $a \neq a_1^{-1}$. In this case $w' = ba_2 \dots a_n$ where $b = a \cdot a_1$ internally in G_{j_1};
(4) if a is not an identity element, but $a = a_1^{-1}$. Then $w' = a_2 \dots a_n$.

Given $a \in E$, let us define the operator T_a defined of the set of reduced words $R(E)$ in the following way: $T_a(w) = aw$. Then we extend T to arbitrary reduced words $w_1 = a_1 \dots a_n$ by setting $T_{w_1} := T_{a_1} \circ \dots \circ T_{a_n}$. Note that if w_0 is the empty word, then $T_w(w_0)$ is the same reduced word w. Moreover, if $z = xy$ is a reduced word that can be split as juxtaposition of two reduced word, then

$$T_z = T_x \circ T_y$$

Also, if e is an identity element, then T_e is the identity over $R(E)$. It follows that if $w_1 \approx w_2$, then $T_{w_1} = T_{w_2}$as operators. But then let $w \approx w'$ be two reduced equivalent words. Hence

$$w = T_w(w_0) = T_{w'}(w_0) = w'$$

This proves that each equivalence class contains only one reduced word. □

Thus, the free product can be fully described by working on reduced words. However, the group structure is heavily anabelian.

Theorem 2.3. *If G is a free product of the $\{G_j\}_{j\in J}$ and J contains at least two elements, then the centre $Z(G)$ is trivial.*

Proof. Let, by contradiction, $w = a_1 \dots a_n$ be the reduced representative of a non trivial central class, with $a_i \in G_{j_i}$ for each $i = 1, \dots, n$. Since G has at least two factors, there exists $a \notin G_{j_1}$ which is not an identity element. Let $g = aa_1^{-1}$; as w is central, we have that $wg = gw$, but this would imply, after reducing the words, $\lg(wg) = n + 1 > n = \lg(gw)$ and this is a contradiction. □

Note, finally, that each G_j is canonically identified to a subgroup of $\mathop{\ast}_{j\in J} G_j$: it is mapped in the subgroup generated by the empty word and all the elements of G_j. Hence, the free product has attached a canonical family of monomorphisms $i_j : G_j \hookrightarrow \mathop{\ast}_{j\in J} G_j$.

Now we come to the most important property.

Theorem 2.4. (Universal property of free product) *Let $\{G_j\}_{j\in J}$ be a collection of groups, G a group and $\{h_j\}_{j\in J}$ a family of group morphisms*

$h_j : G_j \longrightarrow G$. Then, there exists an unique group morphism

$$h : \mathop{\ast}_{j\in J} G_j \longrightarrow G$$

such that the following diagram commutes for each $j \in G$:

$$\begin{array}{ccc} G_j & \xrightarrow{h_j} & G \\ \downarrow{\scriptstyle i_j} & \nearrow{\scriptstyle h} & \\ \mathop{\ast}_{j\in J} G_j & & \end{array}$$

where i_j is the canonical inclusion of G_j in the free product.

Proof. Note, on a first instance, that every function $g : E \longrightarrow G$ can be easily extended to a function $\tilde{g} : W(E) \longrightarrow G$ by splitting each word into its letters:

$$\tilde{g}(a_1 \dots a_n) := g(a_1) \cdot g(a_2) \cdot \dots \cdot g(a_n)$$

In our situation, we can define the morphism $\tilde{h} : W(E) \longrightarrow G$ in the following way:

$$\tilde{h}(a_1 \dots a_n) = h_{j_1}(a_1) \cdot h_{j_2}(a_2) \cdot \dots \cdot h_{j_n}(a_n)$$

for each word $w = a_1 \dots a_n \in W(E)$ with $a_k \in G_{j_k}$. Note that $\tilde{h}$ quotients modulo $\approx$: indeed if $e_j \in G_j$ is an identity element, then $\tilde{h}(e_j) = h_j(e_j) = e_G := [w_0]$ is the identity element of G; moreover, if $a_k, a_{k+1} \in G$ then clearly

$$\tilde{h}(a_k a_{k+1}) = h_{j_k}(a_k) \cdot h_{j_k}(a_{k+1}) = h(a_k \cdot a_{k+1})$$

as each h_j is a group morphism. Hence, the morphism

$$h : \mathop{\ast}_{j\in J} G_j \longrightarrow G$$

is clearly well defined. Furthermore, it is easy to see that h satisfies the requested commutativity properties. Uniqueness is obvious by the fact that $h \circ i = h_j$, since in this way h depends only on the h_j. □

Definition 2.5. The group $\mathop{\ast}_{j\in G} G_j$ is called *free group* if each factor G_j is an infinite cyclic group (in particular, $G_j \simeq \mathbf{Z}$).

Example 2.6. $\mathbf{Z} * \mathbf{Z}$ is a free group, while $(\mathbf{Z}/2) * (\mathbf{Z}/2)$ o $\mathbf{Z} * (\mathbf{Z}/4)$ are not free groups. A free group has no relations on the generators, neither *between* them.

Sometimes it can happen that, while the generators itself are kept free, we want to impose conditions that allow the words to *mix* together in some controlled way. This is the aim of this new construction.

Let $\{G_j\}_{j\in J}$ and $\{F_{jk}\}_{(j,k)\in J^2}$ be families of groups (assume $F_{jk} = F_{kj}$) and suppose $\alpha_{jk} : F_{jk} \longrightarrow G_j$ are group morphisms for each $j, k \in J$. We define the *amalgamated product* of the G_j with respect to the relations α_{jk} as the quotient group

$$\mathrm{Am}(G_j; \alpha_{jk}) = \mathop{\ast}_{j\in J} G_j \Big/ N$$

where

$$N = \langle \alpha_{jk}(x)\alpha_{kj}(x)^{-1} \mid j,k \in J, x \in F_{jk} \rangle$$

is called *amalgamation* subgroup. There is a slight abuse of notation: $\alpha_{jk}(x)$ is not an element of the free product, but it is intended so via the canonical inclusion i_j. It is not difficult to prove that N is a normal subgroup of the free product, so that the amalgamated product is well defined.

It is useful to see how the amalgamation works in the case of the free products of two groups $G_1 * G_2$. Suppose $\alpha : F_1 \longrightarrow G_1$ and $\beta : F_2 \longrightarrow G_2$ are group morphisms. Then

$$\mathrm{Am}(G_i; \{\alpha, \beta\}) =: G_1 *_{F_i} G_2 = G_1 * G_2/N$$

where $N = \langle \alpha(x)\beta(y)^{-1} \mid x \in F_1, y \in F_2 \rangle$ is the amalgamation subgroup. In practice, $G_1 *_{F_i} G_2$ is obtained by $G_1 * G_2$ imposing the relation

$$\alpha(x) = \beta(y)$$

for each $x \in F_1$ and $y \in F_2$, inside the free products. Heuristically, this forces an identification between $\alpha(F_1)$ and $\beta(F_2)$, element by element. This is exactly what is needed to reconstruct the fundamental group of a space from two separate parts: there must be an identification in the intersection, which translates theoretically in the concept of amalgamation.

More formally, in the notable case in which G_1, G_2 and F_1, F_2 are finitely generated and finitely presented, namely

$$\begin{aligned} G_1 &= \langle g_1, \ldots, g_n \mid R_1, \ldots, R_s \rangle \\ G_2 &= \langle h_1, \ldots, h_m \mid L_1, \ldots, L_t \rangle \\ F_1 &= \langle f_1, \ldots, f_p \mid U_1, \ldots, U_w \rangle \\ F_2 &= \langle e_1, \ldots, e_q \mid V_1, \ldots, V_z \rangle \end{aligned}$$

we have that

$$G_1 *_{F_i} G_2 = \langle (g_i), (h_i) \mid (R_i), (L_i), \alpha(f_k)\beta(e_l)^{-1}, k = 1, \ldots, p, l = 1, \ldots, q \rangle$$

Remark 2.7. The amalgamated product is either an universal construction. In the category theory setting, it is group A such that the diagram

$$\bigsqcup_{(j,k)\in J^2} F_{jk} \rightrightarrows \bigsqcup_{j\in J} G_j \to A$$

is a *coequaliser* in the category of groupoids.

3. Seifert-Van Kampen's theorem.

We present two versions of Seifert-Van Kampen's theorem, the general statement and then a weaker version which is more useful in the applications. Firstly, note that we will make use of the following Lemma.

Lemma 3.1. (Lebesgue number Lemma) *Let (X, d) be a sequentially compact metric space and let $\mathfrak{V} = \{U_\alpha\}_{\alpha \in A}$ an open cover of X. Then, there exists a $\delta > 0$, called* Lebesgue number, *such that for each $Y \subseteq X$ having smaller diameter than δ, there exists $\alpha_0 \in A$ such that $Y \subseteq U_{\alpha_0}$.*

Theorem 3.2. *Let X be a path connected topological spaces, with an open cover $\{A_j\}_{j \in J}$ such that*

(1) *the A_j are not disjoint;*
(2) *each $A_j \cap A_k$ is path connected;*
(3) *each $A_i \cap A_j \cap A_k$ is path connected.*

Let x_0 be a point in the intersection, namely $x_0 \in A_j$ for each $j \in J$ and let $p_j : A_j \longrightarrow X$, $p_{jk} : A_j \cap A_k \longrightarrow A_j$ and $p_{kj} : A_j \cap A_k \longrightarrow A_k$ be, for each $j, k \in J$ the canonical open inclusions which induce the morphisms $p_k^* : \pi_1(A_k, x_0) \longrightarrow \pi_1(X, x_0)$, $p_{jk}^* : \pi_1(A_j \cap A_k, x_0) \longrightarrow \pi_1(A_j, x_0)$ and $p_{kj}^* : \pi_1(A_j \cap A_k, x_0) \longrightarrow \pi_1(A_k, x_0)$ on the fundamental groups. Therefore, $\pi_1(x_0, X)$ is the free product of the $\pi_1(A_j, x_0)$ with amalgamation given by the p_{jk}^* and p_{kj}^*. In symbols,

$$\pi_1(X, x_0) \simeq \frac{\ast_{k \in J} \pi_1(A_k, x_0)}{N}$$

where $N = \langle p_{jk}^*(x) p_{kj}^*(x)^{-1} \mid j, k \in J, x \in \pi_1(A_j \cap A_k, x_0) \rangle$.

Proof. The proof is based on the free product's universal property, constructing the morphism

$$\varphi : \ast_{k \in J} \pi_1(A_k, x_0) \longrightarrow \pi_1(X, x_0)$$

by means of the standard morphisms $p_k^* : \pi_1(A_k, x_0) \longrightarrow \pi_1(X, x_0)$, then proving its surjectivity and showing finally that $\ker \varphi$ is exactly the amalgamation subgroup N.

Let us prove in a first stage that φ is surjective. Let $[\alpha] \in \pi_1(X, x_0)$ be a homotopy class on X, represented by a loop $\alpha : I \longrightarrow X$ with origin x_0. Now, the family $\{\alpha^{-1}(A_j)\}_{j \in J}$ is an open cover of the compact I, so its Lebesgue number $\delta > 0$ is well defined. Let $N \in \mathbf{N}$ a natural number such that $1/N < \delta$ (this exists as $\mathbf{R}$ is archimedean) and choose a partition $t_0 = 0 < t_1 < \ldots < t_{N-1} < t_N = 1$ such that $t_{k+1} - t_k = 1/N$. By the Lebesgue number Lemma, we know that

$$\alpha([t_k, t_{k+1}]) \subseteq A_k$$

up to renumber the indices of the A_k in the open cover. Let us call $\alpha_{k+1} : I \longrightarrow A_k$, for each $k = 0, \dots, N-1$, the paths

$$\alpha_{k+1}(s) = \alpha(st_{k+1} + (1-s)t_k)$$

which join $\alpha(t_k)$ to $\alpha(t_{k+1})$. Then let us consider the path $\rho_k : I \longrightarrow A_k \cap A_{k+1}$ joining x_0 to $\alpha(t_k)$, for each $k = 1, \dots, N-1$; note that this path exists as the 2-fold intersections are path connected. Therefore, we can decompose, up to homotopy

$$\alpha = \alpha_1 \dots \alpha_N \sim \alpha_1 \rho_1^{-1} \rho_1 \alpha_2 \dots \rho_{N-1}^{-1} \rho_{N-1} \alpha_N$$

Note that $\alpha_1 \rho_1^{-1}$ is a loop in $A_1 \cap A_2$ with origin $\alpha(t_0) = \alpha(0) = x_0$, that each $\rho_i \alpha_{i+1} \rho_{i+1}^{-1}$ is a loop in $A_i \cap A_{i+1}$ with origin x_0 and also that even $\rho_{N-1}\alpha_N$ is a loop in $A_{N-1} \cap A_N$ with origin x_0. Switching to homotopy classes, we then are allowed to say that

$$[\alpha] = [\alpha_1 \rho_1^{-1}] * [\rho_1 \alpha_2 \rho_2^{-1}] * \dots * [\rho_{N-1}\alpha_N]$$

where, for the sake of simplicity, we omit to indicate the inclusions on the right side: each term should actually be

$$p_k^*(p_{k,k+1}^*([\rho_k \alpha_{k+1} \rho_{k+1}^{-1}])) \in \pi(X, x_0)$$

but the point is that α can be decomposed as the juxtaposition of loops in $A_k \cap A_{k+1}$.

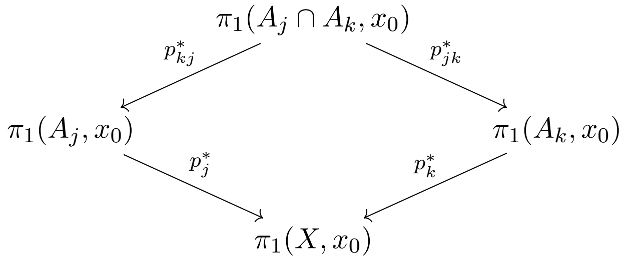
Hence, the universal property of the free product impose that $\varphi \circ i_k = p_k^*$ for each $k \in J$, where i_k is the canonical injection of $\pi_1(A_k, x_0)$ in the free product of all of them. Then, if with a slight abuse of notation we identify $i_k([x])$ with the class $[x] \in \pi_1(A_k, x_0)$, we have indeed proved that

$$\varphi([\alpha_1 \rho_1^{-1}] * \dots * [\rho_{N-1}\alpha_N]) = [\alpha]$$

namely, φ is surjective.

The above construction shows clearly that, in general, φ has no hope to be injective: indeed, the loop $\beta = \rho_k \alpha_{k+1} \rho_{k+1}^{-1}$ in $A_k \cap A_{k+1}$ can be regarded either as a loop in A_k (via $p_{k,k+1}$) or as a loop in A_{k+1} (via $p_{k+1,k}$). This leads to an unavoidable ambiguity when the corresponding homotopy classes are identified with their immersions in $\pi_1(X)$. In general, a class $[\beta] \in \pi_1(A_j \cap A_k, x_0)$ can be seen in A_k, identified via $p_{jk}^*([\beta]) \in \pi_1(A_k, x_0)$, and then in X identified via $p_k^*(p_{jk}^*([\beta])) \in \pi_1(X, x_0)$. But it can also be seen in A_j, firstly identified via $p_{kj}^*([\beta]) \in \pi_1(A_j, x_0)$ and then in X via $p_j^*(p_{kj}^*([\beta])) \in \pi_1(X, x_0)$. Basically, we need to impose that these two identification are the same or, in formal terms, that the following diagram commutes for each

$j, k \in J$:



Let $[\alpha] \in \pi_1(X, x_0)$; we define a *factorisation* of $[\alpha]$ every *formal* product $[\alpha_1] * \ldots * [\alpha_N]$ such that $[\alpha_j] \in \pi_1(A_j, x_0) \simeq p^*_j(\pi_1(A_j, x_0))$ and $\alpha \sim \alpha_1 \ldots \alpha_N$. In more precise terms, a factorisation of $[\alpha]$ is a word (not necessarily a reduced word) which lies in the free product of the $\pi_1(A_k, x_0)$ and which is mapped to $[\alpha]$ via φ. Since φ is surjective, we have already seen that each class in $\pi_1(X, x_0)$ admits a non trivial factorisation. We will say that two such factorisation of $[\alpha]$ are *equivalent* one can be turned in the other by means of a finite number of the following moves (and their reverses):

(1) if $[\alpha_i], [\alpha_{i+1}] \in \pi_1(A_j, x_0)$ are adjacent factors which lie in the same group, replace $[\alpha_i] * [\alpha_{i+1}]$ with $[\alpha_i \alpha_{i+1}]$;
(2) if α_i is a loop in $A_j \cap A_k$, identify $p^*_{ki}([\alpha_i]) \in \pi_1(A_k, x_0)$ with the class $p^*_{ik}([\alpha_i])$ in $\pi_1(A_j, x_0)$.

Indeed, the first action does not change the class $[\alpha]$ which is factorised, while the second one does not change the homomorphic image of the factorisation to the quotient modulo N (this because N is normal). If then we show that two factorisations of $[\alpha]$ are *always* equivalent, we have proved that N is the kernel of φ, hence

$$\frac{\mathop{\ast}\limits_{k \in J} \pi_1(A_k, x_0)}{N} \simeq \pi_1(X, x_0)$$

Suppose then we dispose of two factorisations of $[\alpha]$, namely:

$$[\alpha_1] * \ldots * [\alpha_N] = [\alpha] = [\alpha'_1] * \ldots * [\alpha'_M]$$

By definition, we know that $\alpha_1 \ldots \alpha_N \sim \alpha \sim \alpha'_1 \ldots \alpha'_M$, that is to say, there exists a homotopy F taking $\alpha_1 \ldots \alpha_n$ into $\alpha'_1 \ldots \alpha'_M$. Since $\{F^{-1}(A_j)\}_{j \in J}$ is an open over of the compact set $I \times I$, its Lebesgue number $\delta' > 0$ is well defined. Hence, there are two partitions $0 = s_0 < s_1 < \ldots < s_N = 1$ and $0 = t_0 < t_1 < \ldots < t_M = 1$ such that the rectangle $R_{ij} = [s_{i-1}, s_i] \times [t_{i-1}, t_i]$ is mapped, via F inside only one open set A_{i_j} of the cover. We ask moreover that these rectangles are sent in the right places through the homotopy, namely

$$F(t, 0)|_{[t_k, t_{k+m}]} = \alpha_k(t) \quad F(t, 1)|_{[t_k, t_{k+p}]} = \alpha'_k(t)$$

with m, p some integers appropriately chosen.

Now, with this construction, as $I \times I$ is divided in rectangles, we can see that F maps each open neighborhood of R_{ij} in the corresponding open set

A_{i_j}, respecting the factorisation as above; to avoid points in a 4-fold intersection of open (for instance, the common corners of the rectangles), we aim to slightly deform each edge of R_{ij}, in order that their extremes are not in correspondence with the extremes of those of the upper and lower rows. Actually, assuming that there are at least three rows of rectangles, this operation can be performed only on the intermediate sections, leaving unaltered the exterior rows. Note that this operation is allowed and it does not provoke any loss in the topological structure of the factorisation, because F is continuous; let us call the newly adjusted rectangles $R_1, R_2, \ldots, R_{MN}$ numbering them from left to right, starting from the bottom row and proceeding towards the roof.

Let γ_r be the *polygonal* path separating the first r rectangles $R_1, \ldots, R_r$ from the remaining, obtained by passing over the adjoining edges; γ_r has starting point in $\{0\} \times I$ and final point in $\{1\} \times I$. In this way, γ_0 is the segment $I \times \{0\}$ and γ_{MN} is the segment $I \times \{1\}$.

Note that $F \circ \gamma_r$ is a loop in X with origin in x_0: this descends from the properties of the homotopy F (recall that it preserves the base point). Let v a vertex of R_{ij} such that $F(v) \neq x_0$; therefore $F(v)$ belongs to the intersection of *at most* three open sets of the given cover of X, thanks to the effort put previously to avoid 4-fold intersections in the rectangles. As the 3-fold intersection if path connected, there exists a path $g_v : I \longrightarrow X$ such that $g_v(0) = x_0$ and $g_v(1) = F(v)$, in a way that $g_v(t)$ remains contained, for each t, in the intersection of *at most* three open sets of the given cover. Therefore, we are allowed to insert the loop $g_v g_v^{-1}$ inside each path passing through $F(v)$, obtaining then a factorisation of it by means of elements in $\Omega(A_{i_j}, x_0)$.

It is worth to notice that choosing different paths g_v (and even of different rectangles R_{ij}) does not affect the equivalence of the different factorisations obtained, as the paths $F \circ \gamma_r$ and $F \circ \gamma_{r+1}$ have equivalent factorisations[1]. Now we only have to choose appropriate loops such that the factorisation associated to γ_0 is equivalent to $[\alpha_1] * \ldots * [\alpha_N]$ and that the factorisation associated to γ_{MN} is equivalent to $[\alpha'_1] * \ldots * [\alpha'_M]$. As we pointed out above, all the factorisations associated to the γ_r are equivalent, so event the two chosen factorisations of $[\alpha]$ are equivalent.

This completes the proof. □

The simplest example of application is explained in the following.

Example 3.3. (Wedge sum) Let $\{(X_\alpha, x_\alpha)\}_{\alpha \in T}$ be a collection of pointed topological spaces (path connected). We define the *wedge sum* of the X_α as

[1] It should be checked, at this point, that γ_r and γ_{r+1} are homotopic but this is indeed very cumbersome. One could however avoid the chore to write it explicitly noting that $I \times I$ is simply connected, so each couple of paths are homotopic (if they have the same base points).

the quotient

$$\bigvee_{\alpha\in T} X_\alpha = \bigsqcup_{\alpha\in T} X_\alpha \Big/ \sim$$

where $\sim$ is the equivalence relation which identifies all the base-points x_α. Let us assume that every x_α is a deformation retract of some simply connected open set U_α in X_α. Choose

$$\Big\{A_\alpha = X_\alpha \bigvee_{\beta\neq\alpha} U_\beta\Big\}_{\alpha\in T}$$

as open cover of $\bigvee_\alpha X_\alpha$ and note that each A_α is a deformation retract of X_α; moreover, the intersection of two or more A_α's is the wedge sum of some opens U_β, which is still homotopic to a point. By Van Kampen's theorem, as no amalgamation is needed,

$$\pi_1\Big(\bigvee_{\alpha\in T} X_\alpha\Big) \simeq \mathop{\ast}_{\alpha\in T} \pi_1(X_\alpha)$$

In particular, if each X_j has the same homotopy type of $\mathbf{S}^1$ then

$$\pi_1\Big(\bigvee_{j=1}^{n} X_j\Big) \simeq \mathop{\ast}_{j=1}^{n} \pi_1(X_j) \simeq \mathop{\ast}_{j=1}^{n} \mathbf{Z}$$

In the remarkable case $n = 2$, let $X_1 = \mathbf{S}^1_\alpha \vee (\mathbf{S}^1_\beta\backslash\{p\})$ and $X_2 = \mathbf{S}^1_\beta \vee (\mathbf{S}^1_\alpha\backslash\{q\})$. Since $\mathbf{S}^1\backslash\{\text{point}\}$ in contractible, we have $\pi_1(X_i) = \pi_1(\mathbf{S}^1) \simeq \mathbf{Z}$ for $i = 1, 2$. The intersection is $\mathbf{S}^1_\alpha \vee \mathbf{S}^1_\beta\backslash\{p, q\}$ and it is simply connected. So in perfect coherence with what said in the general case, $\pi_1(\mathbf{S}^1 \vee \mathbf{S}^1) \simeq \mathbf{Z} * \mathbf{Z}$.

Remark 3.4. Note that, in general, the hypothesis on the path-connectedness of the 3-fold intersections can not be removed. Indeed, let be two triangles with a common edge and let A, B, C be, respectively, the external vertices and an interior point of the common edge. Clearly X has the same homotopy type of $\mathbf{S}^1 \vee \mathbf{S}^1$, hence its fundamental group is isomorphic to $\mathbf{Z} * \mathbf{Z}$. But if we try to apply Van Kampen's theorem with the open cover $\{A_\alpha, A_\beta, A_\gamma\}$ of X defined as $A_\alpha = X\backslash\{A\}$, $A_\beta = X\backslash\{B\}$ and $A_\gamma = X\backslash\{C\}$ we see that

$$\pi_1(X) \simeq \frac{\pi_1(A_\alpha) * \pi_1(A_\beta) * \pi_1(A_\gamma)}{N}$$

Since $\pi_1(A_\alpha) = \pi_1(A_\beta) = \pi_1(A_\gamma) \simeq \pi_1(\mathbf{S}^1) \simeq \mathbf{Z}$, we would find

$$\pi_1(X) \simeq \mathbf{Z} * \mathbf{Z} * \mathbf{Z}$$

as the 2-fold intersections are contractible and N is trivial. The contradiction arises from the fact that the above open cover is not admissible for Van Kampen's theorem, as the 3-fold intersection is not path connected (it is even disconnected).

For the above reason, Van Kampen's theorem is most successfully applied to topological spaces which admit an admissible open cover made of two sets only. There is also a weaker version, which was precisely intended to apply the theorem to $\mathbf{S}^1$ (note that no open cover of $\mathbf{S}^1$ can satisfy the requirements of Van Kampen's theorem).

Theorem 3.5. *Let X be a path connected topological space such that $X = X_1 \cup X_2$ with X_1, X_2 path connected open sets such that $X_1 \cap X_2 = A \cup B$, where A, B are non empty path connected sets. If A, B, X_2 are simply connected, then for each $x_0 \in X_1 \cap X_2$,*

$$\pi_1(X, x_0) \simeq \pi_1(X_1, x_0) * \mathbf{Z}$$

4. Attachment of handles.

Let us formalise the notion of "attachment" with the following general scheme. Let X, Y be two disjoint topological spaces, $K \subseteq X$ a subset and $f : K \longrightarrow Y$ a continuous map. Let us endow $X \sqcup Y$ with the disjoint union topology[2] and let $\sim$ be the equivalence relation on $X \sqcup Y$ such that $x \sim f(x)$ for every $x \in K$. Therefore, we define the new topological space

$$X \cup_f Y := \frac{X \sqcup Y}{\sim}$$

which is said to be obtained *attaching X to Y* by means of f. It can be proved that this operation is well behaved under retraction.

Proposition 4.1. *Let $K \subseteq Z \subseteq X$ and Y be topological spaces. If Z is a deformation retract of X, then $Z \cup_f Y$ is a deformation retract of $X \cup_f Y$.*

Define the *n-cells* in $\mathbf{R}^n$ as the following spaces:

$$D^n = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_1^2 + \ldots + x_n^2 \leq 1\}$$
$$\mathbf{S}^{n-1} = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_1^2 + \ldots + x_n^2 = 1\}$$
$$\mathbf{e}_n = D^n \backslash \mathbf{S}^{n-1}$$

The attachment of a n-cell to a topological space Y by means a continuous map $f : \mathbf{S}^{n-1} \longrightarrow Y$ has a fundamental group which is easily comparable to the fundamental group of Y. To calculate the fundamental group of $Y \cup_f D^n$, let us choose a point p, in the interior of D^n, and let us consider

$$X_1 = (Y \cup_f D^n) \backslash \{p\}$$
$$X_2 = \mathbf{e}_n$$

Since $D^n \backslash \{p\}$ has the same homotopy type of $\mathbf{S}^{n-1}$ and attachment commutes with homotopy, one can deduce that X_1 has the same homotopy type of Y for $n \geq 3$. Moreover, $X_1 \cap X_2 = \mathbf{e}_n \backslash \{p\}$ has the same homotopy type of $\mathbf{S}^{n-1}$ so it is simply connected, as it is X_2. Hence, Van Kampen's theorem states that

$$\pi_1(Y \cup_f D^n) \simeq \pi_1(Y)$$

for $n \geq 3$.

[2]Namely, the coarsest topology such that the inclusions $X \hookrightarrow X \cup Y$ and $Y \hookrightarrow X \cup Y$ are continuous.

Now, if $n = 2$ then $X_1 \cap X_2 = \mathbf{e}_2 \setminus \{p\}$ has the same homotopy type of $\mathbf{S}^1$, which is not simply connected. Instead, $X_2 = \mathbf{e}_2$ is still contractible. Calling x any generator of $\pi_1(X_1 \cap X_2) \simeq \mathbf{Z}$, the amalgamation relations impose that $N = \langle f_*(x) \rangle$ where $f_* : \pi_1(X_1 \cap X_2) \longrightarrow \pi_1(Y)$ is the morphism induced by $f : X_1 \cap X_2 \longrightarrow Y$. Therefore, by Van Kampen's theorem

$$\pi_1(Y \cup_f D^2) \simeq \frac{\pi_1(Y)}{\langle f_*(x) \mid x \in \pi_1(\mathbf{S}^1) \rangle}$$

Finally, in the case $n = 1$, we need to attach $D^1 = \mathbf{S}^1$ to the space Y. It is immediate to prove that

$$\pi_1(Y \cup_f D^1) \simeq \pi_1(Y) * \mathbf{Z}$$

using the second version of Van Kampen's theorem, choosing $X_1 = Y \cup D^1 \setminus \{p\}$ and $X_2 = D^1 \setminus \{q\}$ with $q \neq p$; it is immediately found that $X_1 \cap X_2 = D^1 \setminus \{p, q\} = A \cup B$ with A and B simply connected open sets.

5. Exercises.

Exercise 5.1. (Infinite mug) Let C be the topological cylinder $\mathbf{S}^1 \times \mathbf{R}$ with a handle attached (the handle can be thought as a segment of a curve). Calculate $\pi_1(C)$.

Proof. We can proceed in three different ways.

(1) C retracts into $\mathbf{S}^1 \cup M$ by deformation, where M is a segment of curve. Up to homotopy, this can be thought as $\mathbf{S}^1 \vee \mathbf{S}^1$, so

$$\pi_1(C) \simeq \pi_1(\mathbf{S}^1 \vee \mathbf{S}^1) \simeq \mathbf{Z} * \mathbf{Z}$$

(2) We can use the first Van Kampen's theorem, choosing X_1 as C minus an interior point of M and X_2 as an open neighborhood of the same point, such that it is completely contained in M and $X_1 \cap X_2$ is the union of two open curves A, B. All the hypotheses are satisfied, as X_2, A and B are simply connected, so

$$\pi_1(C) = \pi_1(X_1) * \mathbf{Z}$$

Let us note that $\pi_1(X_1) \simeq \pi_1(\mathbf{S}^1 \times \mathbf{R})$ as M minus an interior point can be retracted to the cylinder. Hence we find $\pi_1(C) \simeq \mathbf{Z} * \mathbf{Z}$.

(3) C can be viewed as $\mathbf{S}^1 \times \mathbf{R}$ with an 1-cell attached to it (the handle of the mug). We then know that $\pi_1(\mathbf{S}^1 \times \mathbf{R}) \simeq \pi_1(\mathbf{S}^1) * \mathbf{Z} \simeq \mathbf{Z} * \mathbf{Z}$.

□

Exercise 5.2. Compute the fundamental groups of $\mathbf{RP}^1$ and $\mathbf{RP}^2$.

Proof. We know that $\mathbf{RP}^1$ is the Alexandroff compactification of $\mathbf{R}$, hence it is homeomorphic to $\mathbf{S}^1$. Therefore $\pi_1(\mathbf{RP}^1) \simeq \pi_1(\mathbf{S}^1) \simeq \mathbf{Z}$.

Instead, $\mathbf{RP}^2$ can be seen as the quotient of a solid plane disk D^2 modulo the relation which identifies antipodal points on the boundary. We use the first Van Kampen's theorem to calculate its fundamental group. Let us consider $X_1 = \mathbf{RP}^2 \setminus \{(0,0)\}$ and $X_2 = \mathbf{RP}^2 \setminus \{a\} \simeq B(0,1)$. Note that

$X_1 \cap X_2 = B(0,1) \backslash \{(0,0)\}$ is not simply connected, while X_2 is. Now, X_1 is deformation retract of a circle $\mathbf{S}^1$ modulo the relation identifying antipodal points; but $\mathbf{S}^1$ modulo this relation is homeomrphic to $\mathbf{S}^1$ itself, so $\pi_1(X_1) \simeq \mathbf{Z}$. X_2 has trivial fundamental group. It remains only to determine the amalgamation. Let f a nontrivial loop in $X_1 \cap X_2$, namely such that $[f] \neq [e]$, and let d any path joining a point of f to the boundary a. Therefore, in terms of homotopy equivalence, inside X_1 we see that

$$f \sim daad^{-1} = dad^{-1}dad^{-1} = x^2$$

where $x = d^{-1}ad$ is a generator of $\pi_1(X_1) \simeq \mathbf{Z}$. On the other hand, f is trivial in X_2 as it is simply connected. The amalgamation then is given by the subgroup $N = \langle x^2 \rangle \subseteq \langle x \rangle \simeq \mathbf{Z}$ so that

$$\pi_1(\mathbf{RP}^2) \simeq \frac{\pi_1(X_1) * \pi_1(X_2)}{N} = \frac{\langle x \rangle}{\langle x^2 \rangle} \simeq \frac{\mathbf{Z}}{2\mathbf{Z}}$$

□

Exercise 5.3. Calculate the fundamental group of the Klein bottle $\mathbf{K}$.

Proof. $\mathbf{K}$ is obtained identifying the edges a, b of a square two by two, in a way that each couple of edge are indentified with opposing orientation. The result is a non orientable surface which can not be immersed in $\mathbf{R}^3$ (it is indeed homeomorphic to $\mathbf{RP}^2 \# \mathbf{RP}^2$).

Let us use the Van Kampen's theorem to calculate $\pi_1(\mathbf{K})$: as before, we choose $X_1 = \mathbf{K} \backslash \{(0,0)\}$ and $X_2 = \mathbf{K} \backslash \{a, b\}$ where a, b are the edges of the square. We see that $X_2 \simeq B(0,1)$ so it is simply connected; instead X_1 retracts to the boundary of $\mathbf{K}$, which is exactly $\mathbf{S}^1 \vee \mathbf{S}^1$[3]. Hence, $\pi_1(X_1) \simeq \pi_1(\mathbf{S}^1 \vee \mathbf{S}^1) \simeq \mathbf{Z} * \mathbf{Z}$ and $\pi_1(X_2) = \mathbf{1}$. Let us find the amalgamation: pick any non trivial loop f inside $X_1 \cap X_2 \simeq B(0,1) \backslash \{(0,0)\}$ and let d path joining f to the boundary of $\mathbf{K}$ as before. Therefore, on X_1 the homotopy of f can be read as

$$\begin{aligned} f \sim daba^{-1}bd^{-1} &= (dad^{-1})(dbd^{-1})(da^{-1}d^{-1})(dbd^{-1}) = \\ &= xyx^{-1}y \end{aligned}$$

where $x = dad^{-1}$ and $y = dbd^{-1}$ are two generators for $\pi_1(X_1)$. Instead, inside X_2 the loop f becomes trivial thanks to simple connectedness. Eventually one finds

$$\pi_1(\mathbf{K}) \simeq \frac{\langle x, y \rangle}{\langle xyx^{-1}y \rangle}$$

□

Exercise 5.4. Calculate the fundamental group of $\mathbf{R}^3$ minus a circle $\mathscr{C}$.

Proof. Up to a homeomorphism (more precisely, an affine transformation), we may assume $\mathscr{C} = \{(x,y,z) \in \mathbf{R}^3 \mid x^2 + y^2 = 1, z = 0\}$; we want to use the second version of Van Kampen's theorem. Let $X_1 = \mathbf{R}^3 \backslash D$, where

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[3] A good eye is necessary to see this identification.

$D = \{(x,y,z) \in \mathbf{R}^3 \mid x^2+y^2 \leq 1, z=0\}$ is a closed disk, and let $X_2 = \{(x,y,z) \in \mathbf{R}^3 \mid x^2+y^2 < 1, |z| < 1\}$ is an open (solid) cylinder. Clearly, X_2 is simply connected and the intersection $X_1 \cap X_2$ is the union of two similar disjoint cylinders A, B, with the same centre, the same radius and cut in half by D itself. Since even A are B both simply connected, the theorem can be applied, to get

$$\pi_1(\mathbf{R}^3 \backslash \mathscr{C}) \simeq \pi_1(X_1) * \mathbf{Z}$$

Now, X_1 retracts by deformation to $\mathbf{R}^3 \backslash p$ since D is contractible; hence, being X_1 simply connected, we finally conclude $\pi_1(\mathbf{R}^3 \backslash \mathscr{C}) \simeq \mathbf{Z}$.

Alternatively, one could use the first Van Kampen's theorem by considering the open sets

$$X_1 = \operatorname{int}(\mathscr{C} \times D), \quad X_2 = \mathbf{R}^3 \setminus D$$

noting that X_2 retracts by deformations onto $\mathbf{S}^2$, while X_1 retracts by deformation onto a circle and $X_1 \cap X_2$ is simply connected. □

Exercise 5.5. Calculate the fundamental group of $\mathbf{R}^3$ minus a line r and a circle $\mathscr{C}$.

Proof. We need to distinguish various cases, depdending on the mutual positions of r and $\mathscr{C}$.

(1) Suppose that r and $\mathscr{C}$ are disjoint and *separated*, namely assume that r does not pass through $\mathscr{C}$; so there is a plane between r and $\mathscr{C}$. In this case, let P_1 and P_2 be the two half spaces meeting in a small open neighborhood of this plane, respectively containing r amd $\mathscr{C}$, covering the entire $\mathbf{R}^3$ and then let $X_1 := P_1 \backslash r$ and $X_2 := P_2 \backslash \mathscr{C}$. The intersection $X_1 \cap X_2$ is therefore simply connected, being only an infinite solid strip. Instead, $\pi_1(X_1) = \pi_1(\mathbf{R}^3 \backslash r) \simeq \mathbf{Z}$ while $\pi_1(X_2) = \pi_1(\mathbf{R}^3 \backslash \mathscr{C}) \simeq \mathbf{Z}$ as before. By Van Kampen's theorem, it follows that

$$\pi_1(\mathbf{R}^3 \backslash (r \cup \mathscr{C})) \simeq \mathbf{Z} * \mathbf{Z}$$

(2) Suppose that r is tangent to $\mathscr{C}$; we want to use the second Van Kampen's theorem. Let $X_1 = \mathbf{R}^3 \backslash (r \cup D)$ where D is a closed disk havng $\mathscr{C}$ as boundary, and let X_2 the open (solid) cylinder with basis D; note that X_2 is simply connected. And it clear that $X_1 \cap X_2$ is the union of two open cylinders, disjoint in correspondence of D, both simply connected. Therefore, it follows that

$$\pi_1(\mathbf{R}^3 \backslash (r \cup \mathscr{C})) \simeq \pi_1(X_1) * \mathbf{Z}$$

But $X_1 = \mathbf{R}^3 \backslash (r \cup D)$ is deformation retract of $\mathbf{R}^3 \backslash r$, dato che D è contrattile, quindi $\pi_1(X_1) \simeq \pi_1(\mathbf{R}^3 \backslash \{r\}) \simeq \mathbf{Z}$. Adding up everything, proves that $\pi_1(\mathbf{R}^3 \backslash \{r \cup \mathscr{C}\}) \simeq \mathbf{Z} * \mathbf{Z}$.

(3) Suppose that r and $\mathscr{C}$ are disjoint, but not separated; hence, r pass through $\mathscr{C}$. This is surprisingly the simplest case, and does not require any application of Van Kampen. Up to topological equivalence

(translating and rotating $\mathscr{C}$ and r), note that $\mathbf{R}^3\backslash(r\cup\mathscr{C})$ can be obtained rotating the punctured open half plane $\{(x,y,z)\in\mathbf{R}^3 \mid z = 0, x>0\}\backslash\{p=(1,0,0)\} = Y$. In other words, $\mathbf{R}^3\backslash r\cup\mathscr{C} = \mathbf{S}^1\times Y$, so

$$\pi_1(\mathbf{R}^3\backslash\{r\cup\mathscr{C}\}) = \pi_1(\mathbf{S}^1\times Y)\simeq \mathbf{Z}\times\pi_1(Y)$$

But Y is homeomorphic to the punctured (whole) plane, so $\pi_1(Y) = \pi_1(\mathbf{R}^2\backslash\{p\})\simeq\mathbf{Z}$, and finally $\pi_1(\mathbf{R}^3\backslash(r\cup\mathscr{C})\simeq\mathbf{Z}\times\mathbf{Z}$.

(4) Suppose finally that r is secant to $\mathscr{C}$, so it intersects $\mathscr{C}$ in two (distinct) points. Up to homotopies and retractions, $r\cup\mathscr{C}$ can then be seen as a wedge sum $\mathbf{S}^1\vee\mathbf{S}^1$ together with a line r passing only by the tangency point. For the sake of simplicity, let us call α the right circle and β the left circle (respect to a fixed, but absolutely arbitrary, reference). In order to use the second version of Van Kampen's theorem, let us define $X_1 = \mathbf{R}^3\backslash(r\cup\beta\cup D_\alpha)$ where D_α is a closed disk having α as boundary, and let X_2 be an open (solid) cylinder with basis D_α. Therefore, X_2 is simply connected, while $X_1\cap X_2$ is union of two (solid) cylinders, so it is simply connected. It follows that

$$\pi_1(\mathbf{R}^3\backslash\{r\cup\mathscr{C}\})\simeq\pi_1(X_1)*\mathbf{Z}$$

But since D_α is contractible, X_1 is a deformation retract of $\mathbf{R}^3\backslash(r\cup\alpha)$. We have already studied this configuration, as it consists of a circle and a tangent line; hence quindi $\pi_1(X_1)\simeq\mathbf{Z}*\mathbf{Z}$. Putting everything together,

$$\pi_1(\mathbf{R}^3\backslash(r\cup\mathscr{C}))\simeq\mathbf{Z}*\mathbf{Z}*\mathbf{Z}$$

□

Exercise 5.6. Calculate the fundamental group of $\mathbf{R}^3\backslash(r\cup s)$, where $r\neq s$ are two distinct space lines.

Proof. Two cases need to be addressed separately.

(1) Suppose r and s are skew or parallel. In this setting, let simply X_1, X_2 be, respectively, the two open half-spaces containing r and s minus the lines itselves; we can apply Van Kampen's theorem. The intersection $X_1\cap X_2$ is simply connected, so there is no amalgamation. Therefore

$$\pi_1(\mathbf{R}^3\backslash(r\cup s))\simeq\pi_1(\mathbf{R}^3\backslash r)*\pi_1(\mathbf{R}^3\backslash s)\simeq\mathbf{Z}*\mathbf{Z}$$

(2) Suppose r and s are secant in a point. Up to homeomorphism, we can assume $r=\{y=z=0\}$ and $s=\{x=z=0\}$ and with some efforts it can be seen that $\mathbf{R}^3\backslash(r\cup s)$ is a deformation retract of $\mathbf{S}^2\backslash\{4 \text{ points}\}$. Moreover, $\mathbf{S}^2\backslash\{4 \text{ points}\}$ is a deformation retract of $\mathbf{S}^1\vee\mathbf{S}^1\vee\mathbf{S}^1$, whose fundamental group is $\mathbf{Z}*\mathbf{Z}*\mathbf{Z}$. In general, $X=\mathbf{S}^2\backslash\{p_1,\dots,p_n\}$ with p_i distinct points on $\mathbf{S}^2$ is a deformation retract of the wedge sum of $n-1$ circles.

□

Exercise 5.7. Calculate the fundamental group of $\mathbf{R}^3$ minus two circles $\mathscr{C}_1, \mathscr{C}_2$.

Proof. We need to consider various cases.

(1) Suppose the two circles are distinct and separated; therefore we can choose two half spaces P_1, P_2 containing $\mathscr{C}_1$ and $\mathscr{C}_2$ respectively and such that $P_1 \cap P_2$ is a solid strip. Then define $X_1 := P_1 \backslash \mathscr{C}_1$ e $X_2 := P_2 \backslash \mathscr{C}_2$; clearly $X_1 \cap X_2 = P_1 \cap P_2$ is simply connected, so by Van Kampen's theorem:

$$\pi_1(\mathbf{R}^3 \backslash (\mathscr{C}_1 \cup \mathscr{C}_2)) \simeq \pi_1(\mathbf{R}^3 \backslash \mathscr{C}_1) * \pi_1(\mathbf{R}^3 \backslash \mathscr{C}_2) \simeq \mathbf{Z} * \mathbf{Z}$$

(2) Suppose that $\mathscr{C}_1 \cup \mathscr{C}_2 \simeq \mathbf{S}^1 \vee \mathbf{S}^1$, namely the two circles are tangent in a point; let us use the second Van Kampen's theorem. Let $X_1 = \mathbf{R}^3 \backslash (\mathscr{C}_1 \cup D_2)$ where D_2 is a closed disk having $\mathscr{C}_2$ as boundary, and let X_2 be the open (solid) cylinder with basis D_2. Then clearly X_2 is simply connected and the intersection $X_1 \cap X_2$ is union of two open (solid) cylinders disjoint in correspondence of D_2. It follows that $\pi_1(\mathbf{R}^3 \backslash (\mathscr{C}_1, \mathscr{C}_2)) \simeq \pi_1(X_1) * \mathbf{Z}$. Now, since D_2 is contractible, X_1 retracts by deformation on $\mathbf{R}^3 \backslash \mathscr{C}_1$, hence $\pi_1(X_1) \simeq \mathbf{Z}$. Therefore,

$$\pi_1(\mathbf{R}^3 \backslash (\mathscr{C}_1 \cup \mathscr{C}_2)) \simeq \mathbf{Z} * \mathbf{Z}$$

(3) Let us suppose the two circles are secant, namely they intersect in two different points; for each $i = 1, 2$ let us call D_i the closed disk having $\mathscr{C}_i$ as boundary and let $X_1 = \mathbf{R}^3 \backslash (\mathscr{C}_1 \cup \mathscr{C}_2 \cup (D_2 \backslash D_1))$, and $X_2 = (-1, 1) \times (D_2 \backslash D_1)$, namely a kind of half-moon shaped cylinder. Then it is clear that X_2 is simply connected and $X_1 \cap X_2$ is the union of two simply connected pieces. Hence by the second Van Kampen's theorem,

$$\pi_1(\mathbf{R}^3 \backslash (\mathscr{C}_1 \cup \mathscr{C}_2)) \simeq \pi_1(X_1) * \mathbf{Z}$$

Now it is easy to see that $\mathscr{C}_1 \cup \mathscr{C}_2 \cup D_2 \backslash D_1$ retracts by deformation on $\mathbf{S}^1 \vee \mathbf{S}^1$, since $D_2 \backslash D_1$ is contractible; therefore $\pi_1(X_1) \simeq \pi_1(\mathbf{R}^3 \backslash (\mathbf{S}^1 \vee \mathbf{S}^1)) \simeq \mathbf{Z} * \mathbf{Z}$. Putting everything together, we get

$$\pi_1(\mathbf{R}^3 \backslash (\mathscr{C}_1 \cup \mathscr{C}_2)) \simeq \mathbf{Z} * \mathbf{Z} * \mathbf{Z}$$

(4) Suppose, finally, that $\mathscr{C}_1$ and $\mathscr{C}_2$ are *chained* (or *linked*) in the sense that $\mathscr{C}_1$ does not intersect $\mathscr{C}_2$ but passes through the disk having it as boundary; up to homeomorphism we can assume $\mathscr{C}_1 = \{x^2 + z^2 = 1, y = 0\}$ and $\mathscr{C}_2 = \{(x-1)^2 + y^2 = 1, z = 0\}$. For each $i = 1, 2$, call D_i the open disk having $\mathscr{C}_i$ as boundary and let us define

$$X_1 := \operatorname{int}(\mathscr{C}_1 \times (D_1^*))$$

where D_1^* is D_1 with the centre removed. Clearly X_1 is an open (solid) torus with the circle $\mathscr{C}_2$ "carved" from the interior. Moreover, choose

$$X_2 = \mathbf{R}^2 \setminus \overline{D}_1 \cup \mathscr{C}_2$$

Hence, $X_1 \cap X_2$ is a solid torus with a circular hole inside and with a whole slice removed. So it retracts by deformation to the topological finite cylinder $\mathbf{S}^1 \times I$ (this can be seen in two steps: first, we retract the interior of the torus to its boundary, as the central points have been removed; second, we note that this amounts to a surface torus without $\mathscr{C}_1$ - essentially, the torus is not closed anymore - and this is actually homeomorphic to the cylinder). By Van Kampen's theorem, we finally get

$$\pi_1(\mathbf{R}^3 \backslash (\mathscr{C}_1 \cup \mathscr{C}_2)) \simeq \frac{\pi_1(X_1) * \pi_1(X_2)}{N}$$

Now, X_2 retracts by deformation on $\mathbf{R}^3 \backslash \mathscr{C}_2$ (by contracting D_1 to its centre, which belongs to $\mathscr{C}_2$) so has fundamental group isomorphic to $\mathbf{Z} := \langle \beta \rangle$; moreover X_1 retracts by deformation on $\mathbf{Z} \times \mathbf{Z} := \langle \delta \rangle \times \langle \alpha \rangle$, so it only remains to determine the amalgamation; recall that $\pi_1(X_1 \cap X_2) \simeq \mathbf{Z} = \langle \gamma \rangle$. Therefore, γ will look, inside X_1, as homotopy equivalento to the longitudinal loop α, the second generating loop of the torus, while inside X_2 it will look as the only non trivial generator β of $\pi_1(X_2) \simeq \mathbf{Z}$, as in can not be trivial. We can then conclude that

$$\begin{aligned} \pi_1(\mathbf{R}^3 \backslash (\mathscr{C}_1 \cup \mathscr{C}_2)) &\simeq \frac{\langle \delta \rangle \times \langle \alpha \rangle * \langle \beta \rangle}{\langle \alpha \beta^{-1} \rangle} = \\ &= \langle \delta \rangle \times \langle \alpha, \beta \mid \alpha \beta^{-1} = 1 \rangle \simeq \\ &\simeq \langle \delta \rangle \times \langle z \rangle \simeq \mathbf{Z} \times \mathbf{Z} \end{aligned}$$

□