EQUIVALENT FORMULATIONS OF REGULARITY IN TRIANGULATED CATEGORIES

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Abstract. We obtain some statements equivalent and related to the category of compacts in a triangulated category with a single compact generator being "regular" in the sense of [4]. As corollaries, we get some interesting statements equivalent to an artin algebra having finite global dimension and to a noetherian scheme being regular.

It is always useful to have equivalent formulations of an artin algebra having finite global dimension or a noetherian scheme being regular. It is often desirable that the equivalent statements be proved without any extra assumptions (like the satisfiability of longstanding conjectures: for eg., we have shown [4, Remark 1.10] that if $\mathrm{findim}(A^{\mathrm{op}}) < \infty$ for some artin algebra A, then the derived category of perfect complexes has a bounded t-structure iff A has finite global dimension, but it is not yet known if we can drop the "$\mathrm{findim}(A^{\mathrm{op}}) < \infty$" assumption). In this note, we obtain some such equivalent statements for the algebra case and the scheme case by first proving some abstract results about triangulated categories using Neeman's completion theory and the work contained in an earlier joint paper of ours [4]. In the algebra case in particular, it is noteworthy as to how many of our equivalent statements are quite close to the category of perfect complexes admitting a bounded t-structure.

1. Preliminaries from completion theory

We start with recollecting some definitions that are quite staple in Neeman's completion theory for triangulated categories as developed, for example, in papers like [11, 12].

Let $\mathcal{S}$ be an essentially small triangulated category, and denote by $\mathcal{S}$-Mod the abelian category of additive functors from $\mathcal{S}^{\mathrm{op}}$ to the category of abelian groups. Note that although $\mathcal{S}$ does not have arbitrary homotopy colimits, $\mathcal{S}$-Mod does have arbitrary colimits. Next, we recall the definitions of a good metric and Cauchy sequences with respect to a good metric.

Definition 1.1. *A good metric on $\mathcal{S}$ is a family of full subcategories $\mathcal{M} = \{\mathcal{M}_n\}_{n\in\mathbb{N}}$ such that, for each n, $0 \in \mathcal{M}_n$, $\mathcal{M}_n$ is closed under extensions, and $\mathcal{M}_{n+1}[-1] \cup \mathcal{M}_{n+1} \cup \mathcal{M}_{n+1}[1] \subseteq \mathcal{M}_n$.*

A good metric $\mathcal{M}$ is said to be finer than another good metric $\mathcal{N} := \{\mathcal{N}_n\}_{n\in\mathbb{N}}$ if for each n, there exists $m \in \mathbb{N}$ such that $\mathcal{M}_m \subseteq \mathcal{N}_n$. Two good metrics $\mathcal{M}$ and $\mathcal{N}$ are said to be equivalent *if simultaneously, $\mathcal{M}$ is finer than $\mathcal{N}$ and $\mathcal{N}$ is finer than $\mathcal{M}$.*

Example 1.2. *For any object $G \in \mathcal{S}$, denote by $\langle G\rangle^{(-\infty,-n]}$ the smallest full subcategory of $\mathcal{S}$ containing $G[n]$ and closed under extensions, summands, and positive shifts. It is easy to check that*

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$\{\langle G\rangle^{(-\infty,-n]}\}_{n\in\mathbb{N}}$ *is a good metric. We will be using the term "G-good metric" to refer to any good metric equivalent to this good metric.*

Definition 1.3. *A Cauchy sequence on $\mathcal{S}$ with respect to a good metric $\mathcal{M}$ is a sequence of morphisms* $\{X_\bullet, f_\bullet\}: \ X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3 \longrightarrow \cdots$ *in $\mathcal{S}$ where for every $i \geqslant 1$, there exists an $n_i \geqslant 1$ such that* $\mathrm{Cone}(f_j) \in \mathcal{M}_i$ *for all $j \geqslant n_i$.*

Consider the fully faithful Yoneda functor $\mathscr{Y}: \mathcal{S} \longrightarrow \mathcal{S}\text{-Mod}$ sending $X \mapsto \mathrm{Hom}_{\mathcal{S}}(-,X)$. From now on, we fix a good metric $\mathcal{M}$ on $\mathcal{S}$.

Definition 1.4. *Fix $\mathcal{A}$ to be a full subcategory of $\mathcal{S}$.*

- i. $\mathcal{L}_{\mathcal{M}}(\mathcal{A}) := \{F \in \mathcal{S}\text{-Mod} : F \simeq \underrightarrow{\mathrm{colim}}\, \mathscr{Y}(X_n)$*, where $\{X_\bullet\}$ is a Cauchy sequence in $\mathscr{S}$ with respect to $\mathcal{M}$ where all the terms $X_n \in \mathcal{A}\}$.*
- ii. $\mathfrak{C}_{\mathcal{M}}(\mathscr{S}) := \{F \in \mathcal{S}\text{-Mod} : F(\mathcal{M}_j) = 0$ *for some $j \geqslant 0\}$.*
- iii. $\mathfrak{S}_{\mathcal{M}}(\mathcal{A}) := \mathcal{L}_{\mathcal{M}}(\mathcal{A}) \cap \mathfrak{C}_{\mathcal{M}}(\mathcal{S})$. *When $\mathcal{A} = \mathcal{S}$, $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$ is called the completion of $\mathcal{S}$ with respect to $\mathcal{M}$, and it is a triangulated category again* [11, Theorem 2.14].

Note that if $\mathcal{M} \simeq \mathcal{N}$ as good metrics in the sense of Definition 1.1, then $\mathfrak{S}_{\mathcal{M}}(\mathcal{S}) = \mathfrak{S}_{\mathcal{N}}(\mathcal{S})$. When $\mathcal{M}$ is a G-good metric, we write $\mathfrak{S}_G(-)$ instead of $\mathfrak{S}_{\mathcal{M}}(-)$.

Before stating the most relevant example of completion of a triangulated category with respect to some G-good metric for certain G, we need to take a small detour and discuss some facts related to t-structures and introduce some definitions of some intrinsic subcategories of triangulated categories with a single compact generator. We start with the following fundamental result about generation of t-structures.

Notation 1.5. *We will be writing our t-structures cohomologically or without any homological or cohomological notation - so, on $\mathcal{T}$, a t-structure will be written as $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ (or just as $(\mathcal{U}, \mathcal{V})$). $\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0}$, and $\mathcal{T}^{\leqslant 0} \bigcap \mathcal{T}^{\geqslant 0}$ are respectively called the aisle, the coaisle, and the heart of the t-structure $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$.*

Denote $\mathcal{T}^{\leqslant n} := \mathcal{T}^{\leqslant 0}[-n]$, and $\mathcal{T}^{\geqslant n} := \mathcal{T}^{\geqslant 0}[-n]$. Now, define $\mathcal{T}^- := \bigcup_{n\in\mathbb{N}} \mathcal{T}^{\leqslant n}$, $\mathcal{T}^+ := \bigcup_{n\in\mathbb{N}} \mathcal{T}^{\geqslant -n}$, and $\mathcal{T}^b := \mathcal{T}^- \bigcap \mathcal{T}^+$. The t-structure is said to be bounded above, bounded below, and bounded if, respectively, $\mathcal{T} = \mathcal{T}^-$, $\mathcal{T} = \mathcal{T}^+$, and $\mathcal{T} = \mathcal{T}^b$.

Fact 1.6. *Take a triangulated category $\mathcal{T}$ that has arbitrary coproducts, and let $\mathcal{C}$ be a set of compacts closed under non-negative shifts. Then, it is known that* $\mathrm{Coprod}(\mathcal{C})$*, i.e. the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{C}$ and closed under all coproducts and extensions, is the aisle of a t-structure on $\mathcal{T}$, and we refer to this t-structure as the t-structure generated by $\mathcal{C}$ - this result has been proved multiple times in the literature - see* [3, Theorem A.1 and Proposition A.2] *or* [2, Corollary 4.6.a].

Definition 1.7. *Let $\mathcal{T}$ be any triangulated category.*

- i. *If we have two t-structures $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{W}, \mathcal{Z})$ on $\mathcal{T}$, then we say they are equivalent, or $(\mathcal{U}, \mathcal{V}) \simeq (\mathcal{W}, \mathcal{Z})$, if there exists an integer n such that $\mathcal{U}[n] \subseteq \mathcal{W} \subseteq \mathcal{U}[-n]$.*

ii. *Now, assume $\mathcal{T}$ has arbitrary coproducts and admits a single compact generator G. If we take $\mathcal{C} = \{G[i] : i \geqslant 0\}$ in Fact 1.6, then the t-structure generated on $\mathcal{T}$ generated by $\mathcal{C}$ is denoted by $(\mathcal{T}_G^{\leqslant 0}, \mathcal{T}_G^{\geqslant 0})$, and the equivalence class of t-structures containing this t-structure is called the preferred equivalence class (of t-structures on $\mathcal{T}$), denoted $\mathrm{pec}(\mathcal{T})$. This equivalence class does not change when G is replaced by a different single compact generator* [13, Corollary 2.8].

iii. *Again, assuming there exists a single compact generator of $\mathcal{T}$, let us fix an arbitrary t-structure $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ from the preferred equivalence class. Define $\mathcal{T}_c^- := \bigcap_{n\in\mathbb{N}}(\mathcal{T}^c * \mathcal{T}^{\leqslant -n})$, where $\mathcal{T}^c$ denotes the full subcategory of the compacts. This category is sometimes thought of as the closure of the compacts. The bounded closure of the compacts, written $\mathcal{T}_c^b$, is the full subcategory of $\mathcal{T}_c^-$ formed of those objects that are bounded with respect to our fixed t-structure $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$. In other words, $\mathcal{T}_c^b := \mathcal{T}_c^- \bigcap \mathcal{T}^b$.*

iv. *$\mathcal{T}$, as in (iii), is called coherent if it is weakly approximable in the sense of* [12, Definition 0.25] *and there is a t-structure in $\mathrm{pec}(\mathcal{T})$ that restricts to a t-structure on $\mathcal{T}_c^-$. In this case, the same t-structure restricts to a bounded t-structure on $\mathcal{T}_c^b$. This notion of coherence comes from* [11, Definition 5.1]*, but what we have presented here is a more workable definition (see* [11, Example 5.3]*).*

We will not be defining approximability or its "weak" cousin in this note, but it will help to keep in mind that the unbounded derived categories that are our main examples, i.e. of unbounded complexes of modules over a ring or of unbounded complexes of $\mathcal{O}_X$-modules with quasicoherent cohomology over a quasicompact quasiseparated scheme X, are always weakly approximable ([12, Example 4.1] *and* [13, Theorem 3.2.iv]*).*

Example 1.8. *When $\mathcal{T} = \mathrm{D}(R\text{-Mod})$ for any ring R, it is easy to check that R is a single compact generator. The standard t-structure is in $\mathrm{pec}(\mathrm{D}(R\text{-Mod}))$* [12, Example 4.1]. *We know that R is (left) coherent iff the standard t-structure restricts to $\mathcal{T}_c^-$ in this case.*

Also, for any quasicompact quasiseparated scheme X, $\mathrm{D}_{\mathrm{qc}}(X)$, the unbounded derived category of complexes of $\mathcal{O}_X$-modules with quasicoherent cohomology, has a single compact generator, but this is a nontrivial result which was first proved in [6, Theorem 3.1.1]*, and* [13, Theorem 3.2.ii.] *proves a relative version. The standard t-structure is again in $\mathrm{pec}(\mathrm{D}_{\mathrm{qc}}(X))$* [13, Theorem 3.2.iii.].

Now, we can state our main completion examples.

Example 1.9. *Take $\mathcal{T}$ to be a triangulated category with a single compact generator G and assume additionally that $\mathrm{Hom}_{\mathcal{T}}(G, G[i]) = 0$ for $i \gg 0$. Then, $\mathfrak{S}_G(\mathcal{T}^c) = \mathcal{T}_c^b$. If we put another single compact generator H in place of G, we get the same output under $\mathfrak{S}_H(-)$ because any good metric that is H-good is also G-good and vice versa.*

When $\mathcal{T} = \mathrm{D}(R\text{-Mod})$, for any ring R with $G = R$, $\mathcal{T}^c = \mathrm{K}^{\mathrm{b}}(R\text{-proj})$, and $\mathcal{T}_c^b = \mathrm{K}^{-,b}(R\text{-proj})$, the homotopy category of bounded above cochain complexes of finitely generated R-projectives. When we take R to be an artin algebra A, then $\mathcal{T}_c^b = \mathrm{D}^{\mathrm{b}}(A\text{-mod})$. Similarly if $\mathcal{T} = \mathrm{D}_{\mathrm{qc}}(X)$, for a noetherian scheme X, $\mathcal{T}^c = \mathrm{Perf}(X)$ and $\mathcal{T}_c^b = \mathrm{D}^b_{\mathrm{coh}}(X)$.

We end this section recalling a lifting result for t-structures that will be useful in stating one of the statements in Theorem 2.3. First, we state a definition.

Definition 1.10. [4, Definition 3.2] *Take a good metric* $\mathcal{M}$ *on* $\mathcal{S}$. *A* t*-structure* $(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 0})$ *is said to be extendable with respect to* $\mathcal{M}$ *if* $\mathcal{M}_n \subseteq \mathcal{S}^{\leqslant 0}$ *for some* n.

Example 1.11. [4, Lemma 3.3.(3)] *Any bounded above* t*-structure* $(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 0})$ *is extendable with respect to any* G*-good metric.*

Theorem 1.12. (*part of* [4, Theorem 3.5]) *Let* $(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 0})$ *be a* t*-structure on* $\mathcal{S}$ *that is extendable with respect to a good metric* $\mathcal{M}$. *Then,* $(\mathfrak{S}_{\mathcal{M}}(\mathcal{S}^{\leqslant 0}), \mathfrak{S}_{\mathcal{M}}(\mathcal{S}^{\geqslant 0}))$ *is a* t*-structure on* $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$ *whose heart and coaisle are equivalent to the heart and coaisle of* $(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 0})$ *respectively. Moreover, if* $(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 0})$ *is bounded above,* $(\mathfrak{S}_{\mathcal{M}}(\mathcal{S}^{\leqslant 0}), \mathfrak{S}_{\mathcal{M}}(\mathcal{S}^{\geqslant 0}))$ *is also bounded above.*

In the next section, whenever we will talk about a "lifted t-structure", we will mean lifted as per Theorem 1.12.

2. Statements equivalent to finite global dimension and regularity

We have seen in Fact 1.6 that a t-structure on the compacts can be used to generate a t-structure on the big category, and Theorem 1.12 tells us that, under some conditions, we can send t-structures from the compacts to their bounded closure. So, it can be a natural question to ask whether we would get anything different if instead of taking the first route, we first sent the t-structure to the completion, and then applied $\mathrm{Coprod}(-)$ to the new aisle. The next result tells us that the answer is no.

Proposition 2.1. *Let* $\mathcal{T}$ *be a triangulated category with a single compact generator* G *and assume that* $\mathrm{Hom}_{\mathcal{T}}(G, G[i]) = 0$ *for* $|i| \gg 0$. *Fix a* G*-good metric, and let* $(\mathcal{U}, \mathcal{V})$ *be a* t*-structure on* $\mathcal{T}^c$ *that is extendable with respect to this metric (in this case, it turns out it is equivalent to* $(\mathcal{U}, \mathcal{V})$ *being bounded above - see* [4, Remark 3.4]*). Then,* $\mathrm{Coprod}(\mathcal{U}) = \mathrm{Coprod}(\mathfrak{S}_G(\mathcal{U}))$.

Proof. We will be using some good extension facts as a blackbox. The embedding $\mathcal{T}^c \hookrightarrow \mathcal{T}$ is a good extension in the sense of [11, Notation 0.14], and because of this, instead of dealing with colimits in $\mathcal{T}^c$-Mod, we are allowed to deal with honest homotopy colimits in $\mathcal{T}$ in the sense of [5]. From the definition of homotopy colimits in triangulated categories, it now follows that $\mathfrak{S}_G(\mathcal{U}) \subseteq \mathrm{Coprod}(\mathcal{U})$, and therefore $\mathrm{Coprod}(\mathfrak{S}_G(\mathcal{U})) \subseteq \mathrm{Coprod}(\mathcal{U})$. As $\mathrm{Hom}_{\mathcal{T}}(G, G[i]) = 0$ for $i \gg 0$, we have that $\mathfrak{S}_G(\mathcal{T}^c) = \mathcal{T}^b_c$ by Example 1.9. So, Theorem 1.12 tells us that $(\mathfrak{S}_G(\mathcal{U}), \mathfrak{S}_G(\mathcal{V}))$ is a t-structure on $\mathcal{T}^b_c$ with an additive equivalence $\mathfrak{S}_G(\mathcal{V}) \simeq \mathcal{V}$. So, $\mathfrak{S}_G(\mathcal{U}) = ({}^{\perp_{\mathcal{T}^b_c}}\mathcal{V})[-1]$, and $\mathcal{U} = ({}^{\perp_{\mathcal{T}^c}}\mathcal{V})[-1]$. As $\mathrm{Hom}_{\mathcal{T}}(G, G[i]) = 0$ for $i \ll 0$, $\mathcal{T}^c \subseteq \mathcal{T}^b_c$ [4, Lemma 3.12], and that implies $\mathcal{U} \subseteq \mathfrak{S}_G(\mathcal{U})$. Therefore, $\mathrm{Coprod}(\mathcal{U}) \subseteq \mathrm{Coprod}(\mathfrak{S}_G(\mathcal{U}))$.

□

Proposition 2.2. *Again, we are assuming that* $\mathcal{T}$ *has a single compact generator. If* $\mathcal{T}^c$ *has a* t*-structure* $(\mathcal{U}, \mathcal{V})$ *that generates, on* $\mathcal{T}$*, a* t*-structure in* $\mathrm{pec}(\mathcal{T})$*, then* $\mathcal{T}^b_c \subseteq \mathcal{T}^c$.

Proof. This proof can be deduced from a careful reading of [13, *Proof that Theorem 0.1 follows from Lemma 3.4*].

Let $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ be a t-structure on $\mathcal{T}$ in the preferred equivalence class. By Definition 1.7.ii., we have that as t-structures on $\mathcal{T}$, $(\operatorname{Coprod}(\mathcal{U}), (\operatorname{Coprod}(\mathcal{U})[1])^{\perp_{\mathcal{T}}}) \simeq (\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$. There must be an integer n such that

$$\mathcal{T}^{\leqslant -n} \subseteq \operatorname{Coprod}(\mathcal{U}) \subseteq \mathcal{T}^{\leqslant n}.$$

Now, take an object $F \in \mathcal{T}^b_c$ and assume without loss of generality that $F \in \mathcal{T}^{\geqslant 0}$. By Definition 1.7.iii., $F \in \mathcal{T}^-_c$ and for any integer M, there is a distinguished triangle $D \to E \to F$ with $E \in \mathcal{T}^c$ and $D \in \mathcal{T}^{\leqslant -M}$.

Put $M = 2n + 1$. Now,

a. $\mathcal{T}^{\leqslant -n} \subseteq \operatorname{Coprod}(\mathcal{U}) \implies \mathcal{T}^{\leqslant -2n-1} \subseteq (\operatorname{Coprod}(\mathcal{U}))[n+1]$,
b. $\operatorname{Coprod}(\mathcal{U}) \subseteq \mathcal{T}^{\leqslant n} \implies \mathcal{T}^{\geqslant 0} \subseteq (\operatorname{Coprod}(\mathcal{U})[n+1])^{\perp_{\mathcal{T}}}$.

We can truncate E with respect to the t-structure $(\mathcal{U}[n], \mathcal{V}[n])$ on $\mathcal{T}^c$. So we have a distinguished triangle $E^{\leqslant -n-1} \to E \to E^{\geqslant -n}$ where $E^{\leqslant -1-n} \in \mathcal{U}[n+1] \subseteq (\operatorname{Coprod}(\mathcal{U}))[n+1]$ and $E^{\geqslant -n} \in \mathcal{V}[n] = (\mathcal{U}[n+1])^{\perp_{\mathcal{T}^c}} \subseteq (\mathcal{U}[n+1])^{\perp_{\mathcal{T}}} = (\operatorname{Coprod}(\mathcal{U})[n+1])^{\perp_{\mathcal{T}}}$. So, both $D \to E \to F$ and $E^{\leqslant -n-1} \to E \to E^{\geqslant -n}$ are truncation triangles for E with respect to the same t-structure $(\operatorname{Coprod}(\mathcal{U})[n], (\operatorname{Coprod}(\mathcal{U})[n+1])^{\perp_{\mathcal{T}}})$ on $\mathcal{T}$, which implies that $F \simeq E^{\geqslant -n} \in \mathcal{T}^c$, and we are done. ☐

We are now ready to state the main result of this section. Note that we are not apriori putting any finiteness conditions on $\mathcal{T}$ or $\mathcal{T}^c$. It might be worth recalling at this point from [4] that if $\operatorname{Hom}_{\mathcal{T}}(G, G[i]) = 0$ for $i \gg 0$ with G denoting a single compact generator of $\mathcal{T}$, then $\mathcal{T}^c$ is regular if $\mathcal{T}^c = \mathcal{T}^b_c$. Clearly, this notion of regularity does not depend on the choice of G.

Theorem 2.3. *Let $\mathcal{T}$ be a triangulated category with a single compact generator G such that $\operatorname{Hom}_{\mathcal{T}}(G, G[i]) = 0$ for $|i| \gg 0$ (this means $\mathfrak{S}_G(\mathcal{T}^c) = \mathcal{T}^b_c$ and $\mathcal{T}^c \subseteq \mathcal{T}^b_c$). Consider the following statements.*

i. *$\mathcal{T}^c = \mathcal{T}^b_c$, i.e. $\mathcal{T}^c$ is regular.*
ii. *$\mathcal{T}^c$ has a t-structure whose aisle generates on $\mathcal{T}$ a t-structure $\in \operatorname{pec}(\mathcal{T})$.*
iii. *There is a t-structure $\in \operatorname{pec}(\mathcal{T})$ such that it restricts to a t-structure on $\mathcal{T}^c$.*
iv. *$\mathcal{T}^c$ has an algebraic t-structure. Recall that an algebraic t-structure is a bounded t-structure whose heart is a length category with only finitely many isomorphism classes of simple objects.*
v. *$\mathcal{T}^c$ has a bounded above t-structure such that the lifted t-structure on $\mathcal{T}^b_c$ is bounded below.*

Then,

a. *(iii) $\implies$ (ii) $\implies$ (i), and (iv) $\implies$ (i) unconditionally,*
b. *(v) $\implies$ (i) if $\mathcal{T}^b_c$ has exactly one equivalence class of bounded t-structures and one bounded t-structure whose aisle generates a t-structure $\in \operatorname{pec}(\mathcal{T})$,*
c. *if $\mathcal{T}$ is coherent, (i) $\implies$ (iii), and (i) $\implies$ (ii),*
d. *(i) $\implies$ (iv) if $\mathcal{T}^b_c$ has an algebraic t-structure,*
e. *(i) $\implies$ (v) if $\mathcal{T}^b_c$ has at least one bounded t-structure.*

Proof.

a. It is obvious that $(iii) \implies (ii)$ because if we take a t-structure from $\operatorname{pec}(\mathcal{T})$ that restricts to $\mathcal{T}^c$, then the aisle of the restricted t-structure on $\mathcal{T}^c$ generates, on $\mathcal{T}$, the t-structure that we had started with. $(ii) \implies (i)$ follows from Proposition 2.2 and the assumption that $\mathcal{T}^c \subseteq \mathcal{T}^b_c$. And, $(iv) \implies (i)$ because when $\mathcal{T}^c$ has an algebraic t-structure, $\operatorname{findim}((\mathcal{T}^c)^{\mathrm{op}}, G) < \infty$ [4, Lemmas 4.1.(1) and 4.1.(3)] - this "findim" notion for triangulated categories is from [4, Definition 1.3]. Now, we can just directly use [4, Corollary 4.14.(1)].

b. Let $(\mathcal{U}, \mathcal{V})$ be a bounded above t-structure on $\mathcal{T}^c$. Theorem 1.12 says $(\mathfrak{S}_G(\mathcal{U}), \mathfrak{S}_G(\mathcal{V}))$ is a bounded above t-structure on $\mathcal{T}^b_c$, and as it is also bounded below by hypothesis, it is bounded. We know from [4, Theorem 3.13.(1)] that $(\operatorname{Coprod}(\mathfrak{S}_G(\mathcal{U})), -)$ is a t-structure on $\mathcal{T}$, and as aisles of equivalent bounded t-structures on $\mathcal{T}^b_c$ generate equivalent t-structures on $\mathcal{T}$ this way, our hypothesis tells us that $(\operatorname{Coprod}(\mathfrak{S}_G(\mathcal{U})), -) \in \operatorname{pec}(\mathcal{T})$. Now, Proposition 2.1 implies that, as t-structures on $\mathcal{T}$, $(\operatorname{Coprod}(\mathcal{U}), -) = (\operatorname{Coprod}(\mathfrak{S}_G(\mathcal{U})), -)$, which means we have reached Statement (ii), and we are done by Part (a).

c. Both claims follow directly from the definition of coherence.

d. This is trivial.

e. Assuming (i) and the hypothesis, we get a bounded t-structure on $\mathcal{T}^c$. Applying $\mathfrak{S}_G(-)$ to this t-structure, we get the same t-structure on $\mathcal{T}^b_c = \mathcal{T}^c$. The fact that the t-structure lifts to itself is checked directly by noting that the coaisle remains invariant under this lift by Theorem 1.12.

□

When $\mathcal{T} = \mathrm{D}(A\text{-Mod})$ for some artin algebra A, and we take $G = A$, $\mathcal{T}^b_c = \mathrm{D}^{\mathrm{b}}(A\text{-mod})$ has the standard bounded t-structure which is algebraic, and has only one equivalence class of bounded t-structures (this follows from [1, Lemma 3.22] but can also be directly deduced from [4, Proposition 4.5 and Corollary 4.14.(2)] as $\mathrm{D}^{\mathrm{b}}(A\text{-mod})$ admits a strong generator). The aisle of this standard t-structure generates the standard t-structure on $\mathrm{D}(A\text{-Mod})$ which, in turn, is known to be in $\operatorname{pec}(\mathrm{D}(A\text{-Mod}))$ by Example 1.8. So, in this setup, Theorem 2.3 gives way to the following consequence.

Corollary 2.4. *For any artin algebra A, the following statements are equivalent.*

i. *A has finite global dimension.*
ii. *$\mathrm{K}^{\mathrm{b}}(A\text{-proj})$ has a t-structure whose aisle generates, on $\mathrm{D}(A\text{-Mod})$, a t-structure that is equivalent to the standard t-structure of $\mathrm{D}(A\text{-Mod})$.*
iii. *The standard t-structure of $\mathrm{D}(A\text{-Mod})$ restricts to a t-structure on $\mathrm{K}^{\mathrm{b}}(A\text{-proj})$.*
iv. *$\mathrm{K}^{\mathrm{b}}(A\text{-proj})$ has an algebraic t-structure.*
v. *$\mathrm{K}^{\mathrm{b}}(A\text{-proj})$ has a bounded above t-structure such that the lifted t-structure on $\mathrm{D}^{\mathrm{b}}(A\text{-mod})$ is bounded below.*

Remark 2.5. *In Corollary 2.4, "$(iii) \implies (i)$" is not very difficult to establish, and it has been observed before* [9, Non-example 7.8]. *Also, for finite dimensional algebras,* [1, Proposition 4.12] *shows using very different methods that $(iv) \implies (i)$. For connective $\mathbb{E}_1$-rings, more generally, there* *is a very interesting notion of regularity due to Burklund and Levy* [7, Section 2.1] *that defines regularity for the ring by the property that the standard t-structure on the stable ∞-category of all (left) modules restricts to a t-structure on the perfect modules, and this regularity notion is equivalent to all coherent modules being perfect as long as the $\mathbb{E}_1$-ring itself is coherent.*

When $\mathcal{T} = \mathrm{D}_{\mathrm{qc}}(X)$, for some noetherian scheme X, we know that $\mathcal{T}^c = \mathcal{T}^b_c$ if and only if X is regular. Here again, $\mathrm{D}^b_{\mathrm{coh}}(X)$ has a standard bounded t-structure whose aisle generates the standard t-structure on $\mathrm{D}_{\mathrm{qc}}(X)$ which, in turn, is in $\mathrm{pec}(\mathrm{D}_{\mathrm{qc}}(X))$ by Example 1.8. If we additionally assume that X is of finite dimension or that $\mathrm{D}^b_{\mathrm{coh}}(X)^{\mathrm{op}}$ has finite finitistic dimension in the sense of [4, Definition 1.3], then there is only one equivalence class of bounded t-structures on $\mathrm{D}^b_{\mathrm{coh}}(X)$ [4, Corollary 4.14.(2)].

Now, Theorem 2.3 gives us the following corollary.

Corollary 2.6. *For any noetherian scheme X, consider the following statements.*

i. *X is regular.*
ii. $\mathrm{Perf}(X)$ *has a t-structure whose aisle generates a t-structure on* $\mathrm{D}_{\mathrm{qc}}(X)$ *that is equivalent to the standard one.*
iii. *The standard t-structure of* $\mathrm{D}_{\mathrm{qc}}(X)$ *restricts to a t-structure on* $\mathrm{Perf}(X)$.
iv. $\mathrm{Perf}(X)$ *has a bounded above t-structure that lifts to a bounded below t-structure on* $\mathrm{D}^b_{\mathrm{coh}}(X)$.

Here, $(i) \iff (ii) \iff (iii) \impliedby (iv)$, and if X is finite dimensional or $\mathrm{D}^b_{\mathrm{coh}}(X)^{\mathrm{op}}$ *has finite finitistic dimension, then (i) - (iv) are equivalent.*

It is noteworthy that X having finite dimension need not imply that $\mathrm{D}^b_{\mathrm{coh}}(X)^{\mathrm{op}}$ has finite finitistic dimension - see [4, Lemma 4.1.(2) and Remark 4.9]. Similarly, the opposite direction need not be true.

Remark 2.7. Note that there are already several results in the literature providing statements equivalent to an artin algebra being of finite global dimension or a finite-dimensional noetherian scheme being regular. We have here focused entirely on statements involving t-structures. Let us still mention some of these other equivalent statements. For an artin algebra A, it is known that A has finite global dimension $\iff$ $\mathrm{D}^{\mathrm{b}}(A\text{-mod})$ has a silting object [2, Example 2.5.a] $\iff$ $\mathrm{D}^{\mathrm{b}}(A\text{-mod})$ has a Serre functor [10, Theorem 6.4.13]. There are more such statements in [10, Theorem 6.4.13] and [1, Proposition 4.12]. For (not necessarily finite dimensional) noetherian schemes, one of the most recently discovered equivalent formulations of regularity is that the scheme is regular iff the blowup along any closed point is a quasi-perfect morphism [8, Theorem 3.3]. Of course, for finite dimensional noetherian schemes, it is already known that X is regular iff $\mathrm{Perf}(X)$ has a bounded t-structure [13, 4], but one of the main objectives of Corollary 2.6 is to not require X to be of finite dimension for all of the equivalences.

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