ON GENERATION OPERATORS IN THE MODULE CATEGORY AND THEIR APPLICATIONS

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Abstract. Inspired from several notions of generation for triangulated categories (to check up on different notions of generation in triangulated categories, one can see [11]), we define a notion of generation in the module category from a given class of modules. We derive many interesting properties of this generation and show how they can be useful. We study the relations between generation and resolution and contrasting the two, we get two generation operators - we prove some interesting results on how these operators behave and interact with each other. In [2], using the very notions of generation discussed here, we proved some useful generation results in the module category and also derived and stable categories of a large class of infinite groups. The main motivation behind this article is to lay down an abstract framework for the generation concepts applied and utilised in [2].

Contents

1. Overview of the sections

As mentioned in the abstract, in this article, we provide an abstract framework for the concepts of generation that we used in [2] to derive many useful and interesting generation properties of

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the module category and derived categories and stable module categories for infinite groups like groups in Kropholler's hierarchy (for stable categories, we need the additional assumption that the group admits complete resolutions). So, unexpectedly, much of this article is quite abstract and technical.

In **Section** 2, we define our notion of generation and immediately compare it with another notion of generation that looks very similar but is actually different. We provide an example where these two notions of generation do not coincide. In the stable module categories, however, if one defines these two notions of generation in exactly the way they are defined in the module category here, they coincide. This section also contains some very useful properties of generation, we do not provide proofs of those that we have already proved in [2].

In **Section** 3, we look at resolutions of modules by a given class and see some relations between the properties of a module being generated in finite number of steps from a class and it admitting finite length resolutions with modules from that class.

In **Section** 4 and **Section** 5, we look at closure properties in generation - we look at two types of closure results, one where a class being closed under some operations or admitting certain properties implies that the class generated from it should also be closed under the same operation or have the same property, and the other type is where a class self-generates i.e. no module that is not already in the class is in the class generated by the original class. The motivation behind this comes from the fact that many standard well-behaved classes of modules that arise in module theory do not generate anything new (look at Section 5 for some examples).

In **Sections** 6-8, we look at our two generation operators (one from our definition of generation and the other from resolutions), which have been defined properly in Definition 6.1, and study their iterations and compositions as words of operators. We classify all the possibilities that such words of operators can achieve on any given class and prove some interesting results on the interactions between the two operators.

And, finally in **Section** 9, we look at some applications of our results - we provide some insight into what kind of generation results we obtained in the module and derived categories of groups in Kropholler's hierarchy in [2].

2. Introduction and definitions

Definition 2.1. *(also provided in [2]) Let R be a ring and $\mathscr{T}$ be a class of R-modules. We define generation of modules from $\mathscr{T}$ inductively - we say an R module is generated from $\mathscr{T}$ in n steps iff there exists a short exact sequence $0 \to M_1 \to M_2 \to M \to 0$ where M_1, M_2 are generated from $\mathscr{T}$ in a_1, a_2 steps respectively and $a_1 + a_2 \leq n-1$; to begin the induction, we say an R-module M is generated in 0 steps from $\mathscr{T}$ iff $M \in \mathscr{T}$. So, if we are given a short exact sequence $0 \to M_1 \to M_2 \to M \to 0$ and we know that each M_i is generated from $\mathscr{T}$ in a_i steps, then M is generated from $\mathscr{T}$ in $a_1 + a_2 + 1$ steps.*

We shall denote the class of all modules that can be generated in n steps from $\mathscr{T}$ by $\langle\mathscr{T}\rangle_n$ and the class of all modules that can be generated in finitely many steps from $\mathscr{T}$ by $\langle\mathscr{T}\rangle$.

For any R-module M, we define the $\mathscr{T}$-generation number, $\alpha_{\mathscr{T}}(M) := min\{n \in \mathbb{Z} : M \in \langle\mathscr{T}\rangle_n\}$. If $M \notin \langle\mathscr{T}\rangle_n$ for any finite n, we define $\alpha_{\mathscr{T}}(M)$ to be infinite.

The following lemma is clear from the definition of the number of steps of generation in Definition 2.1.

Lemma 2.2. *For any class of R-modules, $\mathscr{T}$, $\mathscr{T} = \langle\mathscr{T}\rangle_0 \subseteq \langle\mathscr{T}\rangle_1 \subseteq \langle\mathscr{T}\rangle_2 \subseteq ....$*

Looking at the position of our generated module in the short exact sequence used for generation in Definition 2.1, the following question arises naturally.

Question 2.3. *Why are we not putting the generated module in the middle in Definition 2.1?*

The main reason behind this is that when we put the new module on the right of the short exact sequence like we do in Definition 2.1, the generation number that we define coincides with the dimension (see Definition 3.1) when the class is well-behaved or "good" as in Definition 3.5. Later, in Section 6, we derive many interesting properties of these "good" classes, and many examples of such classes can be found, among other places, in Section 5.

In fact, if one does put the new generated module in the middle, then one can construct an infinite family of classes for each of which, the classes generated as per this definition (putting the new module in the middle) and Definition 2.1 differ. We provide one such example, the class of simple modules, in Lemma 2.4 of [2]. To see a family of classes satisfying this same property, one can take, for any n, the class of R-modules of length $\leq n$, where R is a fixed commutative ring - then, this class generates just itself as per Definition 2.1 but, putting the new module in the middle, one gets a strictly bigger class, and both of these results follow from the fact for any short exact sequence, the length of the module in the middle is the sum of the lengths of the other two modules.

However, if we are in a triangulated category and if we start with a class of modules that is closed under the suspension functor of the triangulated category then the two notions of generation coincide. To illustrate this, we replace R by RG where R is a commutative ring of finite global dimension and G is a group such that the trivial module admits a complete resolution over R. It was shown in [10], that for such groups which can be infinite, one can construct a stable module category, denoted $Stab(RG)$, with the inverse syzygy functor as the suspension functor. This stable module category coincides with the stable module category for finite groups when G is finite.

The following result is straightforward which we state with notations as above. While generating modules from a class $\mathscr{T}$, when we put the generated module in the middle of the generating short exact sequence, we call the generated class β-$\langle\mathscr{T}\rangle$, and denote the generation number for a module M by $\beta_{\mathscr{T}}(M)$ (see Definition 2.1 for the analogous definition with the module on the right).

Lemma 2.4. *Let $\mathscr{T}$ be a class of RG-modules in $Stab(RG)$ that contains the zero module and is closed under (both positive and negative) syzygies. Then, if $M \in \langle \mathscr{T} \rangle$ (respectively β-$\langle \mathscr{T} \rangle$), then $\Omega^n(M) \in \langle \mathscr{T} \rangle$ (respectively β-$\langle \mathscr{T} \rangle$) for all n.*

Note that in the stable category the zero module is stably isomorphic to projectives, so Lemma 2.4 is obvious.

Lemma 2.5. *In $Stab(RG)$, for any class of modules $\mathscr{T}$ containing the zero module and closed under both positive and negative syzygies, $\langle \mathscr{T} \rangle = \beta$-$\langle \mathscr{T} \rangle$.*

Proof. We can work with distinguished triangles instead of short exact sequences in $Stab(RG)$. If $\beta_{\mathscr{T}}(M) = 0$, then $M \in \mathscr{T} \subseteq \langle \mathscr{T} \rangle$. Assume as our induction hypothesis that for any RG-module M, if $\beta_{\mathscr{T}}(M) \leq n$, then $M \in \langle \mathscr{T} \rangle$. Now let M be an RG-module such that $\beta_{\mathscr{T}}(M) = n+1$. So, by Definition 2.1, we have a distinguished triangle $P \to M \to Q \to \Omega^{-1}(P)$ where $P, Q \in \beta$-$\langle \mathscr{T} \rangle$ and $\beta_{\mathscr{T}}(P), \beta_{\mathscr{T}}(Q) \leq n$. By our induction hypothesis, here $P, Q \in \langle \mathscr{T} \rangle$ and by Lemma 2.4, $\Omega^{-1}(P) \in \langle \mathscr{T} \rangle$. It now follows from the definition of triangulated categories that we have a distinguished triangle $M \to Q \to \Omega^{-1}(P) \to \Omega^{-1}(M)$, and again from the definition of triangulated categories, it follows that we have a distinguished triangle $Q \to \Omega^{-1}(P) \to \Omega^{-1}(M) \to \Omega^{-1}(Q)$, and as we have seen $Q, \Omega^{-1}(P) \in \langle \mathscr{T} \rangle$, it follows that $\Omega^{-1}(M) \in \langle \mathscr{T} \rangle$, and again by Lemma 2.4, we have that $M \in \langle \mathscr{T} \rangle$.

The other direction follows similarly. □

Coming back to our notion of generation as introduced in Definition 2.1, we now look at situations where we generate modules from a given class with every module of that class being generated in finitely many steps from another class. These situations arise later in Section 6 onward, and also it gives us a good idea of what we can say about the number of steps when we know that the every module in the first class can be generated in a fixed number of steps from the second class, every module in the second class can be generated in a fixed number of steps from a third class, etc.

Lemma 2.6. *(we have proved this in [2], Lemma 2.10) Let $\mathscr{T}$ and $\mathscr{U}$ be 2 classes of R-modules.*

a) If $\mathscr{T} \subseteq \langle \mathscr{U} \rangle$, then $\langle \mathscr{T} \rangle \subseteq \langle \mathscr{U} \rangle$. In other words, any module that can be generated in finitely many steps from $\mathscr{T}$ can also be generated in finitely many steps from $\mathscr{U}$ if every module in $\mathscr{T}$ is generated in finitely many steps from $\mathscr{U}$.

b) If $\mathscr{T} \subseteq \langle \mathscr{U} \rangle_m$, then $\langle \mathscr{T} \rangle_n \subseteq \langle \mathscr{U} \rangle_{mn+m+n}$.

The following results follow directly from Lemma 2.6 and they will be useful later in this article mainly Section 6 onward.

Corollary 2.7. *Let $\mathscr{T}$ be a class of R-modules. Then,*

a) $\langle \langle \mathscr{T} \rangle \rangle = \langle \mathscr{T} \rangle$.

b) Let $\mathscr{U}$ be a class of R-modules such that $\mathscr{T} \subseteq \mathscr{U}$, then $\langle \mathscr{T} \rangle \subseteq \langle \mathscr{U} \rangle$ and $\langle \mathscr{T} \rangle_n \subseteq \langle \mathscr{U} \rangle_n$ for all n.

Proof. a) $\langle \mathscr{T} \rangle \subseteq \langle \mathscr{T} \rangle$. So, by Lemma 2.6.a., $\langle \langle \mathscr{T} \rangle \rangle \subseteq \langle \mathscr{T} \rangle$. Note that by the definition of generation from the class $\langle \mathscr{T} \rangle$, $\langle \mathscr{T} \rangle \subseteq \langle \langle \mathscr{T} \rangle \rangle$. Therefore, $\langle \langle \mathscr{T} \rangle \rangle = \langle \mathscr{T} \rangle$.

b) If we take $m = 0$ in Lemma 2.6.b., we get $\langle \mathscr{T} \rangle_n \subseteq \langle \mathscr{U} \rangle_n$. And, $\langle \mathscr{T} \rangle = \bigcup_{n \geq 0} \langle \mathscr{T} \rangle_n \subseteq \bigcup_{n \geq 0} \langle \mathscr{U} \rangle_n = \langle \mathscr{U} \rangle$. □

The following result is quite important and it arises out of a natural query as to what can one say about the number of steps needed to generate a module from a given class when we have a finite-length resolution of that module by modules all of which can be generated in finitely many steps from the same class. We have also proved this in [2] where we developed this theory a little bit to derive results in generation of derived categories.

Lemma 2.8. *(see Lemma 2.5 of [2]) Let $\mathscr{T}$ be a class of R-modules. If there exists an exact sequence $0 \to M_n \to ... \to M_1 \to M \to 0$, for some $n > 1$, where each M_i is generated in a_i steps from $\mathscr{T}$, then M can be generated from $\mathscr{T}$ in $n - 1 + \sum_{i=1}^{n} a_i$ steps.*

3. Resolutions of modules by a given class

Definition 3.1. *(also provided in [2]) For any class of R-modules, $\mathscr{T}$, we define the $\mathscr{T}$-dimension of a R-module M, denoted $\mathscr{T}$-$dim(M)$, to be $min\{i \in \mathbb{Z} : \exists$ a exact sequence $0 \to T_i \to T_{i-1} \to ... \to T_0 \to M \to 0$ where each $T_i \in \mathscr{T}\}$. If, for an R-module M, no such exact sequence exists for any i, we say $\mathscr{T}$-$dim(M)$ is infinite.*

$[\mathscr{T}]_n$ is the class of all R-modules M such that $\mathscr{T}$-$dim(M) \leq n\}$.

$[\mathscr{T}]$ is the class of all R-modules with finite $\mathscr{T}$-dimension.

The following result is obvious.

Lemma 3.2. *For any class of R-modules, $\mathscr{T}$,*

a) $[\mathscr{T}]_0 \subseteq [\mathscr{T}]_1 \subseteq [\mathscr{T}]_2 \subseteq ...$

b) $[\mathscr{T}]_n \subseteq \langle \mathscr{T} \rangle_n$ for any $n \in \mathbb{Z}_{\geq 0}$.

c) $[\mathscr{T}] \subseteq \langle \mathscr{T} \rangle$.

d) If $\mathscr{U}$ is a class of R-modules such that $\mathscr{T} \subseteq \mathscr{U}$, then $\mathscr{U}$-$dim(M) \leq \mathscr{T}$-$dim(M)$ for all R-modules M.

Proof. (a) follows from the definition of $[\mathscr{T}]_n$. (b) and (c) follow directly from Lemma 2.8. To prove (d), we can start with assuming that $\mathscr{T}$-$dim(M) = n < \infty$ and then note that if $0 \to T_n \to ... \to T_0 \to M \to 0$ is an exact sequence where all the T_i's are in $\mathscr{T}$, then they are also in $\mathscr{U}$ as $\mathscr{T} \subseteq \mathscr{U}$ and therefore $\mathscr{U}$-$dim(M) \leq n$. □

The following result is important, in light of what we said in answer to Question 2.3, to explain a major usefulness of putting the new module on the right of the generating exact sequence. Also,

the "generation number" invariant that we defined in Definition 2.1 comes into good use in this result. We proved a part of this result in [2].

Lemma 3.3. *Let $\mathscr{T}$ be a class of R-modules. Then, in the following statements, $(a) \Rightarrow (b) \Rightarrow (c)$.*

a) For any short exact sequence of R-modules, $0 \to M_2 \to M_1 \to M \to 0$, $\mathscr{T}$-$dim(M) \leq 1 + max\{\mathscr{T}\text{-}dim(M_1), \mathscr{T}\text{-}dim(M_2)\}$.

b) $\mathscr{T}$-$dim(M) = \alpha_{\mathscr{T}}(M)$, for all R-modules M.

c) For any short exact sequence of R-modules, $0 \to M_2 \to M_1 \to M \to 0$, $\mathscr{T}$-$dim(M) \leq 1 + \mathscr{T}$-$dim(M_1) + \mathscr{T}$-$dim(M_2)$.

Proof. We proved $(a) \Rightarrow (b)$ in Lemma 2.8 of [2].

To show $(b) \Rightarrow (c)$, note that if we have a short exact sequence $0 \to M_2 \to M_1 \to M \to 0$ where $\mathscr{T}$-$dim(M_1), \mathscr{T}$-$dim(M_2) < \infty$ and $\mathscr{T}$-$dim(M) > 1 + \mathscr{T}$-$dim(M_1) + \mathscr{T}$-$dim(M_2)$. M_1, M_2 can be generated from $\mathscr{T}$ in $\mathscr{T}$-$dim(M_1)$ and $\mathscr{T}$-$dim(M_2)$ steps respectively by Lemma 3.2.b. and therefore M can be generated from $\mathscr{T}$ in $1 + \mathscr{T}$-$dim(M_1) + \mathscr{T}$-$dim(M_2)$ steps which is strictly smaller than $\mathscr{T}$-$dim(M) = \alpha_{\mathscr{T}}(M)$ which is not possible. □

Remark 3.4. Many standard classes of modules like the projectives, the Gorenstein projectives, etc. satisfy (a) of Lemma 3.3, which means that for those classes, the generation number and the dimension coincide. We provided a short proof for the cases of Gorenstein projectives and proejctives in [2]. See Section 5 for other examples of classes satisfying (a) of Lemma 3.3 or variants of it.

Lemma 3.3 makes it clear why it makes sense to compare the class $[\mathscr{T}]$ with $\langle\mathscr{T}\rangle$ for any class $\mathscr{T}$. There are some results, like Lemma 2.6.b., which if true with [] instead of $\langle\rangle$, we end up with an extra condition on $\mathscr{T}$ - see Lemma 8.4. Still, in most cases, it is sensible to expect that the $\mathscr{T}$-dimension of a module be finite if its $\mathscr{T}$-generation number is finite. This motivates our next definition which we will revisit in Proposition 7.2 and Definition 7.3.

Definition 3.5. *For any ring R, a class of R-modules $\mathscr{T}$ is called good if $[\mathscr{T}] = \langle\mathscr{T}\rangle$, i.e. if $\mathscr{T}$-$dim(M) < \infty \Leftrightarrow \alpha_{\mathscr{T}}(M) < \infty$ for all R-modules M.*

Example: Note that if for a field k of characteristic p and a finite group G, $\mathscr{T}$ denotes the trivial module, then $[\mathscr{T}]$ is the class of the trivial module and the zero module whereas $\langle\mathscr{T}\rangle$ is the class of all finitely generated modules.

We also introduce a definition for two classes of modules to be equivalent in terms of how those classes behave under [].

Definition 3.6. *For any ring R, two classes of R-modules, $\mathscr{T}$ and $\mathscr{U}$, are said to be equivalent, written $\mathscr{T} \sim \mathscr{U}$, if $[\mathscr{T}] = [\mathscr{U}]$, i.e. $\mathscr{T}$-$dim(M) < \infty \Leftrightarrow \mathscr{U}$-$dim(M) < \infty$ for all R-modules M.*

The following lemma is easy to see.

Lemma 3.7. *Let $\mathscr{T}$ and $\mathscr{U}$ are two classes of R-modules that are equivalent. Then $\mathscr{T}$ is good iff $\mathscr{U}$ is good.*

Proof. Let $\mathscr{U}$ be a good class, then $\langle\mathscr{U}\rangle = [\mathscr{U}] = [\mathscr{T}]$ (the last equality follows from $\mathscr{T} \sim \mathscr{U}$). Now, $\langle[\mathscr{T}]\rangle = \langle\langle\mathscr{U}\rangle\rangle = \langle\mathscr{U}\rangle$ (the last equality follows from Corollary 2.7.a.). By Lemma 3.2.c., $[\mathscr{T}] \subseteq \langle\mathscr{T}\rangle$, so by Lemma 2.6.a., $\langle[\mathscr{T}]\rangle \subseteq \langle\mathscr{T}\rangle$. Since $\mathscr{T} \subseteq [\mathscr{T}]$, we have $\langle\mathscr{T}\rangle \subseteq \langle[\mathscr{T}]\rangle$ by Corollary 2.7.b. Thus, $\langle[\mathscr{T}]\rangle = \langle\mathscr{T}\rangle$. Thus, $\langle\mathscr{T}\rangle = \langle\mathscr{U}\rangle$. So, $\langle\mathscr{T}\rangle = [\mathscr{T}] = \langle\mathscr{U}\rangle$, so $\mathscr{T}$ is good. Similarly we can prove that $\mathscr{U}$ is good if $\mathscr{T}$ is good. □

Remark 3.8. Note that although for any ring R, the class of Gorenstein projective R-modules, $GProj(R)$, and the class of projective R-modules, $Proj(R)$, are good classes because they satisfy condition (a) of Lemma 3.3, they need not be equivalent. For an example, take R to be kG where k is a field of characteristic p and G is C_p, then the global dimension of kG is not finite. So there must exist a kG-module M such that $proj.dim_{kG}(M)$ is infinite as the finitistic dimension of kG is finite. But since all finite groups admit complete resolutions, the Gorenstein projective dimension of M as a kG-module must be finite, call it n. Then $\Omega^n(M)$ is Gorenstein projective (this follows from Theorem 2.20 of [6] noting that projectives are Gorenstein projective) but it cannot be projective and since Gorenstein projectives either are projective or have infinite projective dimension (Proposition 2.27 of [6]), we have that $GProj(kG) \nsim Proj(kG)$.

The two main examples of good classes that we have seen in this section are both classes that contain all projectives and are closed under syzygies. Below, we provide an example of a class of modules which generate all modules and is good but need not contain the class of all projectives.

Lemma 3.9. *Let $\mathscr{U}$ be a class of R-modules that satisfies the following conditions.*

a) $\mathscr{U}$ is closed under direct summands and finite direct sums.

b) For every R-module M, there is a surjective map of R-modules, $\phi : U_0 \twoheadrightarrow M$, where U_0 is a module in $\mathscr{U}$.

c) $\mathscr{U}$-$dim(M) \leq n \Leftrightarrow$ For every exact sequence $0 \to K \to U_{n-1} \to U_{n-2} \to ... \to U_0 \to M \to 0$, where all the $U_i's$ are in $\mathscr{U}$, K is in $\mathscr{U}$.

Then, for the class of all R-modules that do not have finite $\mathscr{U}$-dimension, $\mathscr{T}$, assuming $\mathscr{T}$ is non-empty, $[\mathscr{T}]$ and $\langle\mathscr{T}\rangle$ coincide with the class of all R-modules.

Proof. We start by noting that since $\mathscr{U}$ satisfies hypothesis (b), every R-module admits a resolution with modules in $\mathscr{U}$. This can be easily seen by taking an arbitrary R-module M and taking a surjective map onto M from some module in $\mathscr{U}$, say U_0. We call this map $\phi : U_0 \twoheadrightarrow M$. Then, (b) guarantees the existence of an onto map of R-modules between some module in $\mathscr{U}$, say U_1, and $Ker(\phi)$, $\phi_1 : U_1 \twoheadrightarrow Ker(\phi)$. We thus have two short exact sequences: $0 \to Ker(\phi_1) \hookrightarrow U_1 \twoheadrightarrow Ker(\phi) \to 0$ and $0 \to Ker(\phi) \hookrightarrow U_0 \twoheadrightarrow M \to 0$. We can stitch those exact sequences together and get $0 \to Ker(\phi_1) \hookrightarrow U_1 \to S_0 \to M \to 0$. Going on like this, we get a resolution of M with modules in $\mathscr{U}$ (of possibly infinite length).

Now, let P be a module in $\mathscr{T}$, i.e. $\mathscr{U}\text{-}dim(P) \not< \infty$. Take an arbitrary R-module Q and form the exact sequence $0 \to P \to P \oplus Q \to Q \to 0$. We now claim $\mathscr{U}\text{-}dim(P \oplus Q) \not< \infty$. To see this, assume to the contrary that $\mathscr{U}\text{-}dim(P \oplus Q) = t < \infty$. Take 2 exact sequences $0 \to K_t \to U_{t-1} \to U_{t-2} \to ... \to U_0 \to P \to 0$ and $0 \to K'_t \to U'_{t-1} \to U'_{t-2} \to ... \to U'_0 \to Q \to 0$ where each U_i and each U'_i is in $\mathscr{U}$ (we have explained in the previous paragraph why taking such exact sequences is always possible). We can form an exact sequence $0 \to K_t \oplus K'_t \to U_{t-1} \oplus U'_{t-1} \to ... \to U_0 \oplus U'_0 \to P \oplus Q \to 0$, where each $U_i \oplus U'_i$ is in $\mathscr{U}$ because $\mathscr{U}$ is closed under finite direct sums by (a). Now, as $\mathscr{U}\text{-}dim(P \oplus Q) = t$ and as $\mathscr{U}$ satisfies hypothesis (c), we have that $K_t \oplus K'_t$ is in $\mathscr{U}$. Since $\mathscr{U}$ is closed under direct summands by (a), it implies that K_t is in $\mathscr{U}$. So, $0 \to K_t \to S_{t-1} \to ... \to S_0 \to P \to 0$ is a resolution of P with modules in $\mathscr{U}$ of length t, which gives us that $\mathscr{U}\text{-}dim(P) \leq t < \infty$, which is absurd as $\mathscr{U}\text{-}dim(P) \not< \infty$. So, $P \oplus Q$ is in $\mathscr{T}$.

Thus, from the exact sequence $0 \to P \to P \oplus Q \to Q \to 0$, we can say that Q is in $[\mathscr{T}]$. Since Q was taken to be an arbitrary R-module, this implies that $[\mathscr{T}]$ contains all R-modules. From Lemma 3.2.c., we know that $[\mathscr{T}]$ is a subclass of $\langle\mathscr{T}\rangle$. So, $\langle\mathscr{T}\rangle$ contains all R-modules as well, and we are done.

□

Corollary 3.10. *Let R be a ring such that there exist R-modules of infinite projective dimension (see Remark 3.8.a. for an example) or such that there exist R-modules of infinite Gorenstein projective dimension (for this, we can take R to be the integral group ring of a group that does not admit complete resolutions, like a free abelian group of infinite rank for example). If $\mathscr{T}$ is the class of all R-modules of infinite projective dimension or the class of all R-modules of infinite Gorenstein projective dimension, then $[\mathscr{T}]$ and $\langle\mathscr{T}\rangle$ coincide with the class of all R-modules.*

Proof. This follows directly from Lemma 3.9 by noting that the class of projective R-modules satisfies hypotheses $(a), (b)$ and (c) of Lemma 3.9 and that the class of Gorenstein projective R-modules satisfies (a) by Theorem 2.5 of [6], (c) by Theorem 2.20 of [6] and (b) on account of the fact that projective modules are Gorenstein projective. □

4. Closure results in generation and resolution

As mentioned in the overview at the beginning, in this and the subsequent section, we look at two types of closures - in this section, we record some results where we show that if a given class is closed under certain operations, then the classes generated by them are closed under the same properties and in the next section we record some results where we show that certain special classes of modules that satisfy nice conditions are closed under $\langle\rangle$. We often have to close our classes under finite direct sums or arbitrary direct sums while using this theory in generating modules in Kropholler's hierarchy (see [2]).

Remark 4.1. Note that $Proj(R)$, the class of all projective R-modules, is closed under arbitrary direct sums. $\langle Proj(R)\rangle$ need not be closed under arbitrary direct sums however. For example, it is not when R is a ring of infinite finitistic dimension. This can be easily checked: if it is, then $M := \bigoplus_{n\in\mathbb{Z}_{\geq 1}} R_n$, where each R_n is so chosen that $n < proj.dim_R(R_n) = \alpha_{Proj(R)}(R_n) < \infty$ for each n (here clearly each $R_n \in \langle Proj(R)\rangle$) should satisfy $proj.dim_R(M) = \alpha_{Proj(R)}(M) < \infty$, which is not true. Note that here we have used that $proj.dim_R(L) = \alpha_{Proj(R)}(L)$ (see Remark 3.4 and Lemma 3.3)for all R-modules L.

If we move from closing our class of modules under arbitrary direct sums to finite direct sums, we get the following result.

Lemma 4.2. *If a class of R-modules, $\mathscr{T}$, is closed under finite direct sums, so is $\langle\mathscr{T}\rangle$.*

Proof. It suffices to prove that if two R-modules are in $\langle\mathscr{T}\rangle$, then so is their direct sum. We shall proceed by strong induction on the sums of the $\mathscr{T}$-generation numbers of the two modules. If this sum is 0, then each module is in $\mathscr{T}$, and so is their direct sum as $\mathscr{T}$ is closed under finite direct sums.

If $\alpha_{\mathscr{T}}(M)+\alpha_{\mathscr{T}}(N) = 1$, then both M and N admit $\mathscr{T}$-generation sequences where all the other modules are generated in 0 steps, i.e. are in $\mathscr{T}$. Taking a direct sum of these two sequences gives us a $\mathscr{T}$-generation sequence admitted by $M \oplus N$.

Now let us assume that our claim is true for all pairs of R-modules, M and N, if $\alpha_{\mathscr{T}}(M) + \alpha_{\mathscr{T}}(N) \leq t$. Now, let $\alpha_{\mathscr{T}}(M) + \alpha_{\mathscr{T}}(N) = t + 1$. M admits a $\mathscr{T}$-generation sequence $0 \to A \to B \to M \to 0$ where A, B are generated from $\mathscr{T}$ in a, b steps respectively and $a + b \leq \alpha_{\mathscr{T}}(M) - 1$. This means $a, b < \alpha_{\mathscr{T}}(M)$. Similarly, N admits a $\mathscr{T}$-generation sequence $0 \to C \to D \to N \to 0$ where C, D are generated from $\mathscr{T}$ in c, d steps respectively and $c + d \leq \alpha_{\mathscr{T}}(N) - 1$. This means $c, d < \alpha_{\mathscr{T}}(N)$. Taking a direct sum of these two sequences, we get the following short exact sequence

$$0 \to A \oplus C \to B \oplus D \to M \oplus N \to 0$$

As, $\alpha_{\mathscr{T}}(A)+\alpha_{\mathscr{T}}(C) \leq a+c < \alpha_{\mathscr{T}}(M)+\alpha_{\mathscr{T}}(N) = t+1$, by our induction hypothesis, $A\oplus C \in \langle\mathscr{T}\rangle$. Similarly, $B\oplus D \in \langle\mathscr{T}\rangle$. Thus, the above sequence is a $\mathscr{T}$-generation sequence admitted by $M\oplus N$, therefore $M \oplus N \in \langle\mathscr{T}\rangle$. That ends our induction. □

Lemma 4.3. *For any class of R-modules, $\mathscr{T}$, and any non-negative integer n, $[\mathscr{T}]_n$ is closed under arbitrary direct sums if $\mathscr{T}$ is closed under arbitrary direct sums.*

Proof. Let I be an indexing set such that $M_\alpha \in [\mathscr{T}]_n$ for all $\alpha \in I$. For each $\alpha \in I$, let $0 \to P_{\alpha,n} \to P_{\alpha,n-1} \to .. \to P_{\alpha,0} \to M_\alpha \to 0$ be an exact sequence where each $P_{\alpha,i}$ is in $\mathscr{T}$ or is the zero module (if $\mathscr{T}$-$dim(M_\alpha) = t < n$, take $P_{\alpha,i} = 0$ for $i = t + 1, t + 2, .., n.$). We now look at the exact sequence $0 \to \bigoplus_{\alpha\in I} P_{\alpha,n} \to ... \to \bigoplus_{\alpha\in I} P_{\alpha,0} \to \bigoplus_{\alpha\in I} M_\alpha \to 0$. For each i, $\bigoplus_{\alpha\in I} P_{\alpha,i} \in \mathscr{T}$ as $\mathscr{T}$ is closed under arbitrary direct sums. Thus, $\bigoplus_{\alpha\in I} M_\alpha \in [\mathscr{T}]_n$. □

Remark 4.4. There are other results of this sort that one can prove - for example, if $\mathscr{T}$ is a class of R-modules containing all projectives and closed under syzygies, then $\langle\mathscr{T}\rangle$ is closed under syzygies or that if $\mathscr{T}$ is closed under direct sums and summands, then $\langle\mathscr{T}\rangle$ is closed under direct summands. But we are choosing to not include these results as lemmas here because we will not making use of these results later.

We end this short section with the following easy formula on how far the generation numbers of two modules with respect to a given class differ; we shall not be making use of this lemma later but it is interesting in its own right.

Lemma 4.5. *Let $\mathscr{T}$ be a class of R-modules. Then, for any $M, N \in \langle\mathscr{T}\rangle$, $|\alpha_{\mathscr{T}}(M) - \alpha_{\mathscr{T}}(N)| \leq \alpha_{\mathscr{T}}(M \oplus N) + 1$.*

Proof. We can assume that $\alpha_{\mathscr{T}}(M \oplus N)$ is finite because if it is not we have nothing to prove. Looking at the short exact sequence $0 \to M \to M \oplus N \to N \to 0$, we can say that N can be generated from $\mathscr{T}$ in $\alpha_{\mathscr{T}}(M) + \alpha_{\mathscr{T}}(M \oplus N) + 1$ steps, so $\alpha_{\mathscr{T}}(N) \leq \alpha_{\mathscr{T}}(M) + \alpha_{\mathscr{T}}(M \oplus N) + 1$, i.e. $\alpha_{\mathscr{T}}(N) - \alpha_{\mathscr{T}}(M) \leq 1 + \alpha_{\mathscr{T}}(M \oplus N)$. Similarly, looking at the short exact sequence $0 \to N \to M \oplus N \to M \to 0$, we can say that M can be generated from $\mathscr{T}$ in $\alpha_{\mathscr{T}}(N) + \alpha_{\mathscr{T}}(M \oplus N) + 1$ steps, so $\alpha_{\mathscr{T}}(M) \leq \alpha_{\mathscr{T}}(N) + \alpha_{\mathscr{T}}(M \oplus N) + 1$, i.e. $\alpha_{\mathscr{T}}(M) - \alpha_{\mathscr{T}}(N) \leq \alpha_{\mathscr{T}}(M \oplus N) + 1$. Therefore, $|\alpha_{\mathscr{T}}(M) - \alpha_{\mathscr{T}}(N)| \leq \alpha_{\mathscr{T}}(M \oplus N) + 1$. □

5. Self-generating classes of modules

In this section, we give examples of many standard classes of R-modules for any ring R that self-generate, i.e. we do not get any new modules when we consider the classes generated by them. First, we record a general result about self-generating classes.

Lemma 5.1. *Self-generating classes are good.*

Proof. This is easy to see as, if for some class of R-modules $\mathscr{T}$, $\mathscr{T} = \langle\mathscr{T}\rangle$, then $[\mathscr{T}] = [\langle\mathscr{T}\rangle] \subseteq \langle\langle\mathscr{T}\rangle\rangle = \langle\mathscr{T}\rangle = \mathscr{T}$. □

Our first example of a self-generating class is the class of all modules that admit projective resolutions of eventually finite type.

Lemma 5.2. *Let $\mathscr{T}$ be the class of all R-modules that admit projective resolutions which are of finite type after a finite number of steps. Then, $\langle\mathscr{T}\rangle = \mathscr{T}$.*

Proof. It is easy to see that $\mathscr{T} \subseteq \langle\mathscr{T}\rangle$. We now need to prove that if $M \in \langle\mathscr{T}\rangle$, then $M \in \mathscr{T}$. If $\alpha_{\mathscr{T}}(M) = 0$, then $M \in \mathscr{T}$, by definition. Let us assume that any module in $\langle\mathscr{T}\rangle$ that is of $\mathscr{T}$-generation number $\leq n$ is in $\mathscr{T}$, this is our induction hypothesis. If $\alpha_{\mathscr{T}}(M) = n + 1$, there exists a short exact sequence $0 \to C_2 \to C_1 \to M \to 0$, where $\alpha_{\mathscr{T}}(C_i) \leq n$ for $i = 1, 2$. By induction hypothesis, C_1 and C_2 admit projective resolutions that are of finite type after a k_1 and

k_2 steps respectively where $k_1, k_2 < \infty$. By Lemma 3.5 of [3], M admits a projective resolution that is of finite type after $max\{k_1, k_2 + 1\}$ steps. Thus, $M \in \mathscr{T}$. □

A class of modules that arises quite often in the cohomology theory of infinite groups is the class of modules of type FP_∞. Combining Theorem 2.5 [4] and Theorem 1.7 of [5], one can say that that if R is a commutative ring of finite global dimension and if G is a group of type $H\mathscr{F}$, then any RG-module of type FP_∞ admits a complete resolution, where $H\mathscr{F}$ denotes the class of groups in Kropholler's hierarchy starting with the class of all finite groups as the base class (see Definition 9.1). There was a related result in [4] that for any ring R, if M is an RG-module such that $M \otimes_R B(G, R)$ has finite projective dimension as an RG-module where $B(G, R)$ is the module of all $G \to R$ functions that are only allowed to take finitely many values, then M admits a complete resolution. The module $B(G, R)$ is known to be R-free (see Lemma 3.4 of [1]). So, we now consider three classes of modules for a given ring R and a given group G - the class of R-modules of type FP_∞, the class of all RG-modules M such that $proj.dim_{RG}(M \otimes_R B(G, R)) < \infty$, and the class of all modules that admit complete resolutions, and show that all these classes self-generate.

Lemma 5.3. *Let R be a ring and let G be a group. Take F to be a fixed R-flat RG-module. Then, the following classes self-generate.*

a) The class of all R-modules that are of type FP_∞.

b) The class of all RG-modules M such $proj.dim_{RG}(M \otimes F) < \infty$. The tensor product here is over R.

c) The class of all R-modules that admit complete resolutions

Proof. a) Let $\mathscr{T}$ denote the class of all R-modules of type FP_∞. Any module in $\mathscr{T}$ is generated from $\mathscr{T}$ in zero steps by definition, so $\mathscr{T} \subseteq \langle \mathscr{T} \rangle$. We shall now prove that if $M \in \langle \mathscr{T} \rangle$, then $M \in \mathscr{T}$. We proceed by strong induction on $\alpha_{\mathscr{T}}(M)$. If $\alpha_{\mathscr{T}}(M) = 0$, again by the same definition, $M \in \langle \mathscr{T} \rangle$. Let us assume that all modules of $\mathscr{T}$-generation number $\leq n$ are in $\mathscr{T}$, this is our induction hypothesis. Now let $\alpha_{\mathscr{T}}(M) = n + 1$. There exists a short exact sequence $0 \to A \to B \to M \to 0$ where $max\{\alpha_{\mathscr{T}}(A), \alpha_{\mathscr{T}}(B)\} \leq n$. By our induction hypothesis, A, B are modules of type FP_∞. We have the long exact sequence $... \to Ext^{i-1}_R(M, \_\_) \to Ext^{i-1}_R(B, \_\_) \to Ext^{i-1}_R(A, \_\_) \to Ext^i_R(M, \_\_) \to Ext^i_R(B, \_\_) \to Ext^i_R(A, \_\_) \to ...$ associated to the short exact sequence $0 \to A \to B \to M \to 0$, and as $Ext^j_R(A, \_\_), Ext^j_R(B, \_\_)$ commute with direct limits for all j (this is because A, B are modules of type FP_∞), $Ext^j_R(M, \_\_)$ commute with direct limits for all j as well, and that implies that M is of type FP_∞. Thus, $M \in \mathscr{T}$ and that ends our induction.

b) Let $\mathscr{T}_F := \{M \in Mod\text{-}R) : proj.dim_{RG}(M \otimes F) < \infty\}$. Clearly, $\mathscr{T}_F \subseteq \langle \mathscr{T}_F \rangle$. We shall now prove that if $M \in \langle \mathscr{T}_F \rangle$, then $M \in \mathscr{T}_F$. We proceed by strong induction on $\alpha_{\mathscr{T}_F}(M)$. If $\alpha_{\mathscr{T}_F}(M) = 0$, then $M \in \mathscr{T}_F \subseteq \langle \mathscr{T}_F \rangle$. Let us assume that all modules of $\mathscr{T}_F$-generation number $\leq n$ are in $\mathscr{T}_F$, this is our induction hypothesis. Now let $\alpha_{\mathscr{T}_F}(M) = n + 1$. There exists a short

exact sequence $0 \to A \to B \to M \to 0$ where $max\{\alpha_{\mathscr{T}_F}(A), \alpha_{\mathscr{T}_F}(B)\} \leq n$. As F is R-flat, we can tensor this short exact sequence with F to get the short exact sequence $0 \to A \otimes F \to B \otimes F \to M \otimes F \to 0$. By our induction hypothesis, $proj.dim_{RG}(A \otimes F), proj.dim_{RG}(B \otimes F) < \infty$. Therefore, $proj.dim_{RG}(M \otimes F) < \infty$, by Remark 3.4. Thus, $M \in \mathscr{T}_F$, and that ends our induction.

c) Denote the class of all R-modules that admit complete resolutions by $\mathscr{T}$. Again clearly, $\mathscr{T} \subseteq \langle \mathscr{T} \rangle$. We shall now prove that if $M \in \langle \mathscr{T} \rangle$, then $M \in \mathscr{T}$. We proceed by strong induction on $\alpha_{\mathscr{T}}(M)$. If $\alpha_{\mathscr{T}}(M) = 0$, again by the same definition, $M \in \langle \mathscr{T} \rangle$. Let us assume that all modules of $\mathscr{T}$-generation number $\leq n$ are in $\mathscr{T}$, this is our induction hypothesis. Now let $\alpha_{\mathscr{T}}(M) = n+1$. There exists a short exact sequence $0 \to A \to B \to M \to 0$ where $max\{\alpha_{\mathscr{T}}(A), \alpha_{\mathscr{T}}(B)\} \leq n$. By our induction hypothesis, A and B admit complete resolutions, and therefore $Gpd_R(A), Gpd_R(B) < \infty$. Therefore, $Gpd_R(M) < \infty$, since the class of Gorenstein projectives is good. This means M admits a complete resolution. That ends our induction. □

Following the proof of Lemma 3.5 in [3], the following result, which mirrors Lemma 5.2, follows.

Lemma 5.4. *Let $\mathscr{T}$ be the class of all R-modules that admit complete resolutions which are of eventually finite type. Then, $\langle \mathscr{T} \rangle = \mathscr{T}$.*

If we have a functor or a sequence of functors that translates short exact sequences of modules into long exact sequences, like the Ext-functor, then it becomes relatively easy to prove self-generation results on the classes of modules where this sequence of functors vanish eventually. We come across such functors in relative homological algebra a lot. Below, we provide a formal definition and show, in Lemma 5.6, what we mean by proving self-generation results on classes where our sequence of functors eventually vanish.

Definition 5.5. *(the first part is from Section 3 of [8]) Let R_1 and R_2 be two rings. A sequence of additive functors from the category of R_1-modules to the category of R_2-modules, $(S^i : i \in \mathbb{Z})$, denoted by S, is called a $(-\infty, \infty)$-cohomological functor from R_1-modules to R_2-modules if for any short exact sequence of R_1-modules $0 \to A \to B \to C \to 0$, there exists a long exact sequence of R_2-modules $... \to S^{n-1}(C) \to S^n(A) \to S^n(B) \to S^n(C) \to S^{n+1}(A) \to ....$*

We define $\underline{S}(M) := sup\{i \in \mathbb{Z} : S^i(M) \neq 0\}$.

The following result is noteworthy.

Lemma 5.6. *Let S be a $(-\infty, \infty)$-cohomological functor from R_1-modules to R_2-modules, and let $\mathscr{T}$ be the class of all modules M such that $\underline{S}(M) < \infty$. Then, $\langle \mathscr{T} \rangle = \mathscr{T}$.*

Proof. Any module that is in $\mathscr{T}$ is generated in 0 steps from $\mathscr{T}$ and is therefore in $\langle \mathscr{T} \rangle$. So, $\mathscr{T} \subseteq \langle \mathscr{T} \rangle$. Now to prove $\langle \mathscr{T} \rangle \subseteq \mathscr{T}$, we first note that modules with $\mathscr{T}$-generation number 0 are in $\mathscr{T}$ by definition. Let all modules of $\mathscr{T}$-generation number $\leq n$ are in $\mathscr{T}$, this is our induction hypothesis. Now let M be a module in $\langle \mathscr{T} \rangle$ such that $\alpha_{\mathscr{T}}(M) = n + 1$. There exists a short

exact sequence $0 \to A \to B \to M \to 0$ where $\alpha_{\mathscr{T}}(A), \alpha_{\mathscr{T}}(B) \leq n$. By our induction hypothesis and definition of $\mathscr{T}$, $\underline{S}(A), \underline{S}(B) < \infty$. Let $k = max\{\underline{S}(A), \underline{S}(B)\}$. In the long exact sequence $... \to S^{k-1}(M) \to S^k(A) \to S^k(B) \to S^k(M) \to S^{k+1}(A) \to S^{k+1}(B) \to ..., S^{k+l}(A), S^{k+l}(B) = 0$ for all $l \geq 0$. This implies that $S^{k+l}(M) = 0, \forall l \geq 0$, so M is in $\mathscr{T}$. This ends our induction. □

As noted in [8], the cohomology functors $(H^i(G, \_\_) : i \in \mathbb{Z})$, where $H^i(G, \_\_)$ is defined to be zero for negative i, and the Tate cohomology functors (when the group is such that Tate cohomology can be defined) $(\hat{H}^i(G, \_\_) : i \in \mathbb{Z})$ are examples of $(-\infty, \infty)$-cohomological functors from $\mathbb{Z}G$-modules to $\mathbb{Z}$-modules.

We end this section with the following question.

Question 5.7. *a) We have seen examples of classes $\mathscr{T}$ where $\alpha_{\mathscr{T}}$ and $\mathscr{T}$-dimension coincide for all modules. We have also seen an example of a class that is not good, i.e. where one of them is finite and the other one is not for at least one module. But is there a class $\mathscr{T}$ of R-modules that is good but $\alpha_{\mathscr{T}}(M) > \mathscr{T}\text{-}dim(M)$ for some R-module M?*

b) Given a class of R-modules $\mathscr{T}$ and an R-module M such that $\alpha_{\mathscr{T}}(M) = m$, with what precision can we find a module N such that $\alpha_{\mathscr{T}}(N) = n$ for a given $n < m$?

6. Words of generation operators in the module category

Sections 6-8 are perhaps the most technical sections of this article. Note that if we take a class of R-modules and apply any of the brackets [] or $\langle\rangle$ on it, and then apply some other bracket again and so on, what we get is a string of brackets applied to the class, so it makes sense to study these brackets as operators denoted by a symbol and their iterations as concatenated words. This motivates us to introduce the following definition.

For Sections 6-8, we introduce two new notations for the brackets [] and $\langle\rangle$.

Definition 6.1. *For any class of R-modules $\mathscr{T}$, we denote $B(\mathscr{T}) = [\mathscr{T}]$ and $C(\mathscr{T}) = \langle\mathscr{T}\rangle$. For any non-empty word in B and C, $W = B_1B_2..B_n$ for some n where each B_i is either B or C, we define $W(\mathscr{T}) := B_1(B_2(..(B_n(\mathscr{T}))..)$. The empty word is denoted by B^0 or C^0, and, as operators, $B^0(\mathscr{T}) = C^0(\mathscr{T}) := \mathscr{T}$.*

For the rest of this article, we shall denote the set of all finite words in B and C by $W(B, C)$. Since one can form the operators B^n, i.e., $BB...n$ times, it seems a natural question to ask whether defining the operator B^α where α is a limit ordinal as $B^\alpha(\mathscr{T}) := \bigcup_{\beta<\alpha} B^\beta(\mathscr{T})$ for any class $\mathscr{T}$, gives us new operators. Unfortunately, that is not the case as our next result shows.

Theorem 6.2. *For any class of R-modules, $\mathscr{T}$, $C(\mathscr{T}) = B^\omega(\mathscr{T})$ where ω is the first infinite ordinal.*

Proof. First we show that $B^n(\mathscr{T}) \subseteq \langle\mathscr{T}\rangle$ for all n. This can be shown easily by induction on n. If $n = 1$, it follows from Lemma 2.8. Let $B^k(\mathscr{T}) \subseteq \langle\mathscr{T}\rangle = C(\mathscr{T})$ for all $k \leq n$-this is our induction

hypothesis. Then, if we take $M \in B^{n+1}(\mathscr{T}) = [B^n(\mathscr{T})]$, we see that we have an exact sequence $0 \to M_t \to M_{t-1} \to ... \to M_0 \to M \to 0$ for some t where each $M_i \in B^n(\mathscr{T})$ and therefore by our induction hypothesis, each M_i is generated from $\mathscr{T}$ in, say, a_i steps where a_i is a finite integer. Thus, by Lemma 2.8 again, M can be generated from $\mathscr{T}$ in $t + \sum_{i=0}^{t} a_i$ steps and is therefore in $\langle \mathscr{T} \rangle$.

Now to prove that if $M \in \langle \mathscr{T} \rangle$, then $M \in \bigcup_{n\in\mathbb{N}} B^n(\mathscr{T})$, we proceed by induction on $\alpha_{\mathscr{T}}(M)$. If $\alpha_{\mathscr{T}}(M) = 0$, then $M \in \mathscr{T} \subseteq [\mathscr{T}] = B(\mathscr{T})$. Let us assume that for all $k \leq n$, if $\alpha_{\mathscr{T}}(M) = k$, then $M \in \bigcup_{n\in\mathbb{N}} B^n(\mathscr{T})$ - this is our induction hypothesis. Now let $\alpha_{\mathscr{T}}(M) = n+1$, then by definition, we have an exact sequence $0 \to A \to B \to M \to 0$ for some A and B where $\alpha_{\mathscr{T}}(A), \alpha_{\mathscr{T}}(B) < n$. Therefore, $A, B \in \bigcup_{n\in\mathbb{N}} B^n(\mathscr{T})$, by our induction hypothesis. Let $A \in B^a(\mathscr{T})$, and $B \in B^b(\mathscr{T})$. Then, for any $d > max\{a, b\}$, both $A, B \in B^d(\mathscr{T})$, and that implies $M \in [B^d(\mathscr{T})] = B^{d+1}(\mathscr{T})$. This ends our induction.

□

In the remaining results of this section, we investigate how the operators B and C interact with each other and we achieve a full classification of all words in $W(B, C)$ as operators on any given class. It is also noteworthy, as Corollary 6.4 will show, that the operator C is quite strong and if it is in a word the word becomes equal to C as an operator.

It makes sense to wonder if choosing W to be a large word in $W(B, C)$ gives rise to a long expression for $W(\mathscr{T})$. In our next result, we see how things get simplified if we have C in our word.

Lemma 6.3. *Let $W \in W(B, C)$ be a non-empty word.*

a) If W is of the form UC for some $U \in W(B, C)$, then $W(\mathscr{T}) = C(\mathscr{T})$ for all classes of modules $\mathscr{T}$.

b) If W is of the form CV for some $V \in W(B, C)$, then $W(\mathscr{T}) = C(\mathscr{T})$ for all classes of modules $\mathscr{T}$.

Proof. a) It suffices to prove that $C^2(\mathscr{T}) = C(\mathscr{T})$ and $BC(\mathscr{T}) = C(\mathscr{T})$. $C^2(\mathscr{T}) = C(\mathscr{T})$, by Corollary 2.7. Now, it is easy to see that $C(\mathscr{T}) \subseteq BC(\mathscr{T})$. If $M \in BC(\mathscr{T})$, by definition of B, there exists an exact sequence $0 \to C_n \to ... \to C_1 \to M \to 0$, for some n, where each $C_i \in C(\mathscr{T})$. By Lemma 2.8, therefore, $M \in C(\mathscr{T})$. So, $BC(\mathscr{T}) = C(\mathscr{T})$.

b) We first prove that $C(\mathscr{T}) = CB(\mathscr{T})$. By Lemma 3.2.c., $B(\mathscr{T}) \subseteq C(\mathscr{T})$. Therefore, by Lemma 2.6, $CB(\mathscr{T}) \subseteq C(\mathscr{T})$. Also, as $\mathscr{T} \subseteq B(\mathscr{T})$, $C(\mathscr{T}) \subseteq CB(\mathscr{T})$ by Corollary 2.7. Thus, $C(\mathscr{T}) = CB(\mathscr{T})$, for all $\mathscr{T}$. Putting $B(\mathscr{T})$ in place of $\mathscr{T}$ in $C(\mathscr{T}) = CB(\mathscr{T})$, we get $CB^2(\mathscr{T}) = CB(\mathscr{T}) = C(\mathscr{T})$. Thus, we have shown that for any non-negative integer n, $CB^n(\mathscr{T}) = C(\mathscr{T})$ for all classes of modules $\mathscr{T}$. Now, for any $V \in W(B, C)$, it follows that $CV = C$ (as operators) because $CC = C$ and $CB^n = C$ for any n (as operators).

□

Corollary 6.4. *For any word $W \in W(B,C)$, if W contains C, then $W(\mathscr{T}) = C(\mathscr{T})$, for all classes of modules $\mathscr{T}$.*

Proof. If C is in a word W, then W is of the form UCV for some $U, V \in W(B,C)$. Now, by Lemma 6.3, as operators, $W = (UC)V = CV = C$. So, we are done. □

The following result, which follows as a corollary to Lemma 6.3, shows that for any word in $W(B,C)$ and for any class of modules $\mathscr{T}$, $W(\mathscr{T})$ can be one of two things.

Corollary 6.5. *Take an arbitrary non-empty $W \in W(B,C)$. Then, for any class of modules $\mathscr{T}$, $W(\mathscr{T})$ is one of the following*

a) $C(\mathscr{T})$.

b) Of the form $B^n(\mathscr{T})$, for some positive integer n.

Proof. If W contains C, then, by Lemma 6.3, $W = C$ as operators. If there is no C in W, then $W = B^n$, for some n. □

Corollary 6.6. *$BC = CB$ as operators, i.e. the brackets $[]$ and $\langle\rangle$ commute with each other.*

Proof. This is obvious from Corollary 6.4 □

7. Characterisation of good classes through word relations

Proposition 7.2 is the main result of this section. We show that as operators on a given class $\mathscr{T}$, $B^2 = B \Leftrightarrow B = C$, and that $\mathscr{T}$ satisfies many other nice conditions if and only if either of these conditions hold. In this case, any module that is generated from $\mathscr{T}$ in finitely many steps admits a finite-length resolution of modules from $\mathscr{T}$ which means the class is "good" in the sense of Definition 3.5. We start this section with a basic lemma which will be used in the proof of Proposition 7.2.

Lemma 7.1. *Let $\mathscr{T}$ be a class of R-modules. Then, the following two statements are equivalent.*

a) For any short exact sequence of R-modules $0 \to A \to B \to C \to 0$, if $A, B \in [\mathscr{T}]$, then $C \in [\mathscr{T}]$.

b) For any exact sequence of R-modules $0 \to C_n \to C_{n-1} \to ... \to C_1 \to C \to 0$, for any $n > 1$, if each $C_i \in [\mathscr{T}]$, then $C \in [\mathscr{T}]$.

Proof. $(b) \Rightarrow (a)$ is obvious.

$(a) \Rightarrow (b)$: we shall proceed by induction on n. If $n = 2$, (b) holds true, by (a). Let the statement of (b) hold true for $n = k$, this is our induction hypothesis. Now let $0 \to C_{k+1} \to C_k \to ... \to C_1 \to C \to 0$ be an exact sequence where each C_i is in $[\mathscr{T}]$. We split this into two exact sequences.

S0) $0 \to C_{k+1} \to C_k \to Im(C_k \to C_{k-1}) \to 0$.

S1) $0 \to Im(C_k \to C_{k-1}) \hookrightarrow C_{k-1} \to ... \to C_1 \to C \to 0$.

From (a), it follows that in $(S0)$, $Im(C_k \to C_{k-1}) \in [\mathscr{T}]$. Therefore, in $(S1)$, every module other than C is in $[\mathscr{T}]$. So, by our induction hypothesis, $C \in [\mathscr{T}]$. □

The following proposition gives us some equivalent properties of a class being good in terms of words of operators from $W(B, C)$ acting on it.

Proposition 7.2. *For any class of R-modules, $\mathscr{T}$, the following statements are equivalent.*

a^*) For any short exact sequence of R-modules, $0 \to A \to B \to C \to 0$, if $\mathscr{T}$-$dim(A), \mathscr{T}$-$dim(B) < \infty$, then $\mathscr{T}$-$dim(C) < \infty$.

a) $B^2(\mathscr{T}) = B(\mathscr{T})$.

b^*) For all R-modules M, $\mathscr{T}$-$dim(M) < \infty$ iff $\alpha_{\mathscr{T}}(M) < \infty$.

b) $B(\mathscr{T}) = C(\mathscr{T})$.

c) $C(\mathscr{T}) \sim \mathscr{T}$.

d) $\mathscr{T} \sim B(\mathscr{T})$.

e) For any class of R-modules, $\mathscr{U}$, if $\mathscr{U} \subseteq B(\mathscr{T})$, then $B(\mathscr{U}) \subseteq B(\mathscr{T})$.

f) For any non-empty word $W \in W(B, C)$, $W(\mathscr{T}) = B(\mathscr{T})$.

g) For any non-empty word $W \in W(B, C)$, $W(\mathscr{T}) = C(\mathscr{T})$.

Proof. We start by noting that $(a^*) \Leftrightarrow B^2(\mathscr{T}) = B(\mathscr{T})$ (by Lemma 7.1). Also, as, by definition, $B(\mathscr{T}) = \{M \in Mod(R) : \mathscr{T}\text{-}dim(M) < \infty\}$ and $C(\mathscr{T}) = \{M \in Mod(R) : \alpha_{\mathscr{T}}(M) < \infty\}$, $(b^*) \Leftrightarrow B(\mathscr{T}) = C(\mathscr{T})$.

Now, for any two classes of modules, $\mathscr{U}$ and $\mathscr{V}$, $\mathscr{U} \sim \mathscr{V} \Leftrightarrow B(U) = B(V)$. So, $(c) \Leftrightarrow BC(\mathscr{T}) = B(\mathscr{T}) \Leftrightarrow C(\mathscr{T}) = B(\mathscr{T}) \Leftrightarrow (b)$ (here we used Corollary 6.4 to say that $BC(\mathscr{T}) = C(\mathscr{T})$). And, $(d) \Leftrightarrow B(\mathscr{T}) = B^2(\mathscr{T}) \Leftrightarrow (a)$. We thus have $(b) \Leftrightarrow (c)$, $(a) \Leftrightarrow (d)$.

$(b) \Rightarrow (a)$: $B(\mathscr{T}) = C(\mathscr{T}) \Rightarrow B^2(\mathscr{T}) = BC(\mathscr{T}) = C(\mathscr{T}) = B(\mathscr{T})$ ($BC = C$, by Corollary 6.4).

$(a) \Rightarrow (b)$: As $B(\mathscr{T}) \subseteq C(\mathscr{T})$ by Lemma 3.2.c, for all R-modules M, $\alpha_{\mathscr{T}}(M) < \infty$ if $\mathscr{T}$-$dim(M) < \infty$. We now proceed by strong induction on $\alpha_{\mathscr{T}}(M)$ to show that $\mathscr{T}$-$dim(M) < \infty$ if $\alpha_{\mathscr{T}}(M) < \infty$. If $\alpha_{\mathscr{T}}(M) = 0$, $M \in \mathscr{T}$, and $\mathscr{T}$-$dim(M) < \infty$. Let us assume that any module with $\mathscr{T}$-generation number $\leq n$ has finite $\mathscr{T}$-dimension, this is our induction hypothesis. Now if $\alpha_{\mathscr{T}}(M) = n + 1$, we have a short exact sequence $0 \to C_2 \to C_1 \to M \to 0$ such that $\alpha_{\mathscr{T}}(C_1), \alpha_{\mathscr{T}}(C_2) \leq n$. By the induction hypothesis, $\mathscr{T}$-$dim(C_1), \mathscr{T}$-$dim(C_2) < \infty$. By (a), $\mathscr{T}$-$dim(M) < \infty$. So, if $\alpha_{\mathscr{T}}(M) < \infty$, $\mathscr{T}$-$dim(M) < \infty$.

We have thus shown $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$.

$(d) \Rightarrow (e)$: Let $\mathscr{U} \subseteq B(\mathscr{T})$. Take $M \in B(\mathscr{U})$. There exists an exact sequence $0 \to C_n \to C_{n-1} \to ... \to C_1 \to M \to 0$, for some n, where each $C_i \in \mathscr{U} \subseteq B(\mathscr{T})$. Thus, $B(\mathscr{T})$-$dim(M) < \infty$, which implies that $\mathscr{T}$-$dim(M) < \infty$ as $\mathscr{T} \sim B(\mathscr{T})$. Thus, $M \in B(\mathscr{T})$. So, $B(\mathscr{U}) \subseteq B(\mathscr{T})$.

$(e) \Rightarrow (a)$: Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules where $\mathscr{T}$-$dim(A), \mathscr{T}$-$dim(B) < \infty$, i.e., $A, B \in B(\mathscr{T})$. Let $\mathscr{U}$ be a class of R-modules that consists of just

A and B. Then, $C \in B(\mathscr{U})$ and $\mathscr{U} \subseteq B(\mathscr{T})$. From (e), it follows that, $C \in B(\mathscr{U}) \subseteq B(\mathscr{T})$. Thus, $\mathscr{T}\text{-}dim(C) < \infty$.

Thus, $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$.

$(f) \Rightarrow (a)$: Take $W = BB$, then $W(\mathscr{T}) = B^2(\mathscr{T}) = B(\mathscr{T}) \Leftrightarrow (a)$.

$(a) \Rightarrow (f)$: Let $W \in W(B, C)$ be an arbitrary non-empty word. Then $W(\mathscr{T})$ is either $C(\mathscr{T})$ or $B^n(\mathscr{T})$ for some n by Corollary 6.5. But $(a) \Rightarrow B^2(\mathscr{T}) = B(\mathscr{T}) \Rightarrow B^n(\mathscr{T}) = B(\mathscr{T})$, for all n, and $(a) \Rightarrow (b) \Rightarrow B(\mathscr{T}) = C(\mathscr{T})$. So, we are done.

$(b) \Rightarrow (g)$: Let $W \in W(B, C)$ be an arbitrary non-empty word. Then $(b) \Rightarrow B(\mathscr{T}) = C(\mathscr{T}) \Rightarrow W(\mathscr{T}) = C(\mathscr{T})$ (by Corollary 6.4, if W contains C, then $W(\mathscr{T}) = C(\mathscr{T})$).

$(g) \Rightarrow (b)$. Take $W = C$. Then, $W(\mathscr{T}) = B(\mathscr{T}) \Rightarrow C(\mathscr{T}) = B(\mathscr{T}) \Rightarrow (b)$. □

We defined good classes in Definition 3.5, below we provide an equivalent definition of good classes in light of Proposition 7.2.

Definition 7.3. *For any ring R, a class of modules $\mathscr{T}$ is called good iff $\mathscr{T}$ satisfies the equivalent conditions of Proposition 7.2.*

Note that a class of R-modules $\mathscr{T}$ is good iff a module being generated in a finite number of steps from $\mathscr{T}$ is equivalent to the same module admitting finite resolutions by modules in $\mathscr{T}$. This is a nice property to have and the main motivation behind choosing to call such classes "good". Also, from Proposition 7.2, we see that on good classes all words from $W(B, C)$ as operators give the same result.

Corollary 7.4. *If $\mathscr{T}$ is a good class, then so is $B^n(\mathscr{T})$, $\forall n \in \mathbb{N}$.*

Proof. $\mathscr{T}$ is a good class $\Leftrightarrow B^2(\mathscr{T}) = B(\mathscr{T}) \Rightarrow B^{n+2}(\mathscr{T}) = B^{n+1}(\mathscr{T}) \Leftrightarrow B^2(B^n(\mathscr{T})) = B(B^n(\mathscr{T}))$, for any $n \in \mathbb{N}$. Thus condition (a) of Proposition 7.2 holds for $B^n(\mathscr{T})$. □

Corollary 7.5. *For any class of modules $\mathscr{T}$, $B(\mathscr{T})$ is a good class iff $B(\mathscr{T}) \sim C(\mathscr{T})$.*

Proof. $B(\mathscr{T})$ satisfies the condition (b) of Proposition 7.2
$\Leftrightarrow B^2(\mathscr{T}) = CB(\mathscr{T}) = BC(\mathscr{T})$ (By Corollary 6.4, $BC(\mathscr{T}) = CB(\mathscr{T}) = C(\mathscr{T})$.)
$\Leftrightarrow B(\mathscr{T}) \sim C(\mathscr{T})$.

□

Remark 7.6. We have the conditions "$C(\mathscr{T}) \sim \mathscr{T}$" and "$\mathscr{T} \sim B(\mathscr{T})$" in Proposition 7.2. Whether "$B(\mathscr{T}) \sim C(\mathscr{T})$" can be added as an equivalent condition in Proposition 7.2 is a natural question to ask. From Corollary 7.5, "$B(\mathscr{T}) \sim C(\mathscr{T})$" can be added as an equivalent condition in Proposition 7.2 iff $\mathscr{T}$ being a good class and $[\mathscr{T}]$ being a good class are equivalent.

Corollary 7.4 and Corollary 7.5 deal with cases where $B^n(\mathscr{T})$ is a good class for some n as a result of some conditions on $\mathscr{T}$. Also, note that if $B^n(\mathscr{T})$ is good for some n, then words with

more than n many B's collapse to simpler expressions as operators on $\mathscr{T}$ - Theorem 8.2, which we prove later, details this. This motivates our next definition.

Definition 7.7. *For any class of R-modules, $\mathscr{T}$, $g(\mathscr{T}) := min\{n \in \mathbb{Z} : B^n(\mathscr{T})$ is a good class $\}$.*

Lemma 7.8. *For any class of R-modules $\mathscr{T}$, $g(\mathscr{T}) > n$ iff $B^n(\mathscr{T})$ is not a good class.*

Proof. The "only if" part is easy to see. If $g(\mathscr{T}) > n$, then by definition of $g(\mathscr{T})$, $B^n(\mathscr{T})$ cannot be a good class.

The "if" part follows from the fact that if $B^n(\mathscr{T})$ is not a good class, then $B^{\leq n}(\mathscr{T})$ cannot be good - this is because if $B^k(\mathscr{T})$ is good for some $k < n$, then, as operators on $\mathscr{T}$, $B^{k+2} = B^{k+1}$, and so $B^2.B^n = B^{n-k}.B^{k+2} = B^{n-k}.B^{k+1} = B.B^n$, which implies that $\mathscr{U} = B^n(\mathscr{T})$ satisfies $B^2(\mathscr{U}) = B(\mathscr{U})$, and so $B^n(\mathscr{T})$ is good. □

We end this section with the following couple of interesting questions that we have not settled yet.

Question 7.9. *a) For any arbitrary $n > 0$, can we get an example of a class $\mathscr{T}$ of R-modules, for some R, such that $g(\mathscr{T}) \leq n$? We need $B^m(\mathscr{T})$ to be not good for all $m < n$.*

b) We saw in Theorem 6.2 that $C(\mathscr{T}) = B^\omega(\mathscr{T})$, for all $\mathscr{T}$. If $B^\omega(\mathscr{T}_1) = B^\omega(\mathscr{T}_2)$(i.e. the operator C gives the same output on both classes, which is quite easy to check), for two classes of R-modules $\mathscr{T}_1$ and $\mathscr{T}_2$, for some R, then does it imply that $g(\mathscr{T}_1)$ is finite iff $g(\mathscr{T}_2)$ is finite? Intuitively, it seems from the involvement of the B operators in Definition 7.7 that the answer should be yes, but we do not have a proof yet.

8. On sequences of words and other results

In this section, we look at sequences of words in $W(B, C)$. We prove some results on the eventual stability of sequences words in B and C as operators on a given class $\mathscr{T}$. We find what precise word such sequences stabilise to and what conclusions we can then make about $\mathscr{T}$ in terms of its 'good'ness.

We start this section with the following lemma which later helps is determining some crucial properties about the behaviour of sequences of words in $W(B, C)$ as operators using Proposition 7.2.

Lemma 8.1. *Let W_1,$W_2 \in W(B, C)$, not necessarily non-empty such that $W_1(\mathscr{T}) = W_2(\mathscr{T})$, for some class of R-modules, $\mathscr{T}$. Then, one of the following is true:*

a) $W_1 = W_2$, as operators.

b) $B^n(\mathscr{T})$ is a good class for some integer n, i.e. $g(\mathscr{T}) < \infty$.

Proof. If $W_1 = W_2$ as words in $W(B, C)$, then (a) holds true, so we can assume that $W_1 \neq W_2$ in $W(B, C)$.

Case 1 : One of W_1 and W_2 is empty. Say, W_1 is empty. Then, by Corollary 6.5, one of the following is true:

Case 1.*i* : $\mathscr{T} = B^n(\mathscr{T})$, for some n. Note that $\mathscr{T} \subseteq B(\mathscr{T}) \subseteq B^2(\mathscr{T}) \subseteq ....$ So, $B^2(\mathscr{T}) = B(\mathscr{T})$, i.e. $\mathscr{T}$ is a good class by (a) of Proposition 7.2.

Case 1.*ii* : $\mathscr{T} = C(\mathscr{T})$. Then, $B(\mathscr{T}) = BC(\mathscr{T}) = C(\mathscr{T})$ (by Lemma 6.3).

Case 2 : Neither of W_1 and W_2 is empty. There are three possibilities here.

Case 2.*i* : Neither of W_1 and W_2 contain C. If both the words contain the same number of B's, then $W_1 = W_2$ as words in $W(B, C)$.

If W_1 is comprised of n B's and W_2 is comprised of $n + t$ B's for some $t > 0$. Again, as $B^n(\mathscr{T}) \subseteq B(B^n(\mathscr{T})) \subseteq B^2(B^n(\mathscr{T})) \subset ...$, this means $B^2(B^n(\mathscr{T})) = B(B^n(\mathscr{T}))$, thus $B^n(\mathscr{T})$ is a good class by (a) of Proposition 7.2.

Case 3.*ii* : One of the sequences, say W_1, contains C, but W_2 does not. Then, $C(\mathscr{T}) = B^n(\mathscr{T})$, for some n. Then, $C.B^{n-1} = B.B^{n-1}$, as operators on $\mathscr{T}$, because $C.B^{n-1} = C$ as operators on $\mathscr{T}$, by Corollary 6.4. Thus, $B^{n-1}(\mathscr{T})$ is a good class.

Case 3.*iii* : Both W_1 and W_2 contain C. Then, by Corollary 6.4, $W_1 = W_2 = C$ as operators. □

We are now in a position to prove the following theorem about sequences of words in $W(B, C)$ as operators on a given class of modules.

Theorem 8.2. *For any class of R-modules, $\mathscr{T}$, the following are equivalent.*

a^*) Any sequence $\{W_i(\mathscr{T})\}_{i\in\mathbb{N}}$ where, for each i, $W_i \in W(B, C)$, and as words in $W(B, C)$, $W_i \neq W_j$ when $i \neq j$, eventually stabilises.

a) Any sequence $\{W_i(\mathscr{T})\}_{i\in\mathbb{N}}$ where, for each i, $W_i \in W(B, C)$, and as words in $W(B, C)$, $W_i \neq W_j$ when $i \neq j$, eventually stabilises to $C(\mathscr{T})$.

b) The sequence $B^0(\mathscr{T})$, $B(\mathscr{T})$, $B^2(\mathscr{T})$, $B^3(\mathscr{T})$, . . . eventually stabilises.

c) The sequence in (b) stabilises to $C(\mathscr{T})$.

d) $g(\mathscr{T}) < \infty$.

Proof. Note that $(c) \Rightarrow (b)$ trivially and also $(a) \Rightarrow (b)$ trivially because we can choose W_i to be B^i in the statement of (a).

$(b) \Rightarrow (c)$, $(b) \Rightarrow (d)$: Let n be the smallest integer such that $B^n(\mathscr{T}) = B^m(\mathscr{T})$ for all integers $m \geq n$. We then have $B(B^n(\mathscr{T})) = B^2(B^n(\mathscr{T}))$. So, $B^n(\mathscr{T})$ satisfies the condition on $\mathscr{T}$ in (a) of Proposition 7.2, and is therefore good. This proves that $g(\mathscr{T}) \leq n$, which establishes that $(b) \Rightarrow (d)$. $\mathscr{U} := B^n(\mathscr{T})$ should also satisfy (b) of Proposition 7.2, so $B(\mathscr{U}) = C(\mathscr{U})$. We therefore have $B(B^n(\mathscr{T})) = C(B^n(\mathscr{T})) = C(\mathscr{T})$ (as operators on $\mathscr{T}$, $CB^n = C$, by Corollary 6.4). Thus, $B^{\geq n}(\mathscr{T}) = B^{n+1}(\mathscr{T}) = C(\mathscr{T})$. This establishes that $(b) \Rightarrow (c)$.

$(d) \Rightarrow (b)$: If $g(\mathscr{T}) = k < \infty$, then $B^k(\mathscr{T})$ is a good class. $B^k(\mathscr{T})$ satisfies the conditions on $\mathscr{T}$ in Proposition 7.2. Condition (f) of Proposition 7.2 says that $B^{\geq 0}(B^k(\mathscr{T})) = B^{k+1}(\mathscr{T})$, for all $t \geq 0$. Thus, (b) holds.

$(c) \Rightarrow (a)$: Let n be the smallest integer such that $C(\mathscr{T}) = B^m(\mathscr{T})$ for all integers $m \geq n$. We can assume that $n > 0$ because if $n = 0$, then $C(\mathscr{T}) = \mathscr{T}$, which means that $C(\mathscr{T}) = BC(\mathscr{T}) = B(\mathscr{T})$ ($C = BC$, as operators, follows from Corollary 6.4), which in turn means that every term in any sequence of the form described in (a) will be equal to $C(\mathscr{T})$.

For any $W \neq B^i$, where $i \leq n-1$, $W(\mathscr{T}) = C(\mathscr{T})$. This is because if we take a $W \neq B^i$ for some $i \leq n-1$, then either W contains C, in which case $W(\mathscr{T}) = C(\mathscr{T})$ by Corollary 6.4, or W does not contain C, in which case it must have at least n B's, and then $W(\mathscr{T}) = C(\mathscr{T})$ by our definition of n. Let us now take an arbitrary sequence $\{W_i(\mathscr{T})\}_{i\in\mathbb{N}}$ where, as words in $W(B, C)$, $W_i \neq W_j$ if $i \neq j$. Any term in this sequence that is not of the form $B^m(\mathscr{T})$ for some $m \leq n-1$ is equal to $C(\mathscr{T})$. Thus, the sequence $\{W_i(\mathscr{T})\}_{i\in\mathbb{N}}$ eventually stabilises to $C(\mathscr{T})$ because all of its terms that are not equal to $C(\mathscr{T})$ can occur only finitely many times (this is where we are using the fact that the exact same word cannot occur more than once which is guaranteed by the assumption that $W_i \neq W_j$ when $i \neq j$).

We thus have $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$. Now note that $(a^*) \Rightarrow (a)$ trivially. Now assume that (a) holds. Then, there exists some n such that $W_n(\mathscr{T}) = W_{n+1}(\mathscr{T})$, and from the proof of Lemma 8.1, we have that either $W_n = W_{n+1} = C$ as operators on $\mathscr{T}$ or $W_n = W_{n+1}$ as words in $W(B, C)$ which is not possible as $W_i \neq W_j$ as words in $W(B, C)$ when $i \neq j$, or $g(\mathscr{T}) < \infty$. If, for all $m \geq n$, $W_m = C$ as operators on $\mathscr{T}$, then (a) follows. Otherwise, from the proof of Lemma 8.1 as mentioned, (d) follows.

□

Remark 8.3. If we take A to be a commutative ring of finite non-zero global dimension and G to be a group, then we can have AG-modules that are of finite projective dimension but are not projective. So, if $\mathscr{T}$ is the class of all projective AG-modules, then $B(\mathscr{T}) \neq \mathscr{T}$, and so if $\mathscr{U}$ is a class of AG-modules such that $\mathscr{T} = B(\mathscr{U})$, then $B^2(\mathscr{U}) \neq B(\mathscr{U})$, and from the $(a) - (b)$ equivalence of Proposition 7.2, it follows that $\mathscr{U}$ does not satisfy the equivalent statements of Proposition 7.2, i.e. $\exists M \in B(\mathscr{U})$ such that $\mathscr{U}\text{-}dim(M)$ is not finite.

In Lemma 2.6.b., we saw that for any two classes of R-modules $\mathscr{T}$ and $\mathscr{U}$, if $\mathscr{U} \subseteq \langle \mathscr{T} \rangle_m$, for some m, then $\langle \mathscr{U} \rangle_n \subseteq \langle \mathscr{T} \rangle_{mn+m+n}$ for any n. We mentioned in Section 3 that a similar result involving a different kind of brackets implies stronger conditions on $\mathscr{T}$. The following lemma details that comment.

Lemma 8.4. *Let $\mathscr{T}$ be a class of R-modules. In the following sequence of statements, $(a) \Rightarrow (b)$.*

(a) There exists a function $f : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ such that, for any class of R-modules, $\mathscr{U}$, if $\mathscr{U} \subseteq [\mathscr{T}]_m$ for some m, then, for any n, $[\mathscr{U}]_n \subseteq [\mathscr{T}]_{f(m,n)}$.

(b) $\mathscr{T}$ is a good class.

Proof. Take $\mathscr{X} \subseteq [\mathscr{T}]$. Pick an arbitrary module M in $[\mathscr{X}]$. Let $\mathscr{X}\text{-}dim(M) = k$, this means we have an exact sequence $0 \to X_k \to X_{k-1} \to ... \to X_0 \to M \to 0$ where $X_i \in \mathscr{X} \subseteq [\mathscr{T}]$, for all $i \in \{0, 1, .., k\}$.

Let $p := max\{\mathscr{T}\text{-}dim(X_i) : 0 \leq i \leq k\}$. Now, if $\mathscr{M} := \{X_0, X_1, ..., X_k\}$, then $\mathscr{M} \subseteq [\mathscr{T}]_p$. Note that $M \in [\mathscr{M}]_k$, and therefore by (a), $M \in [\mathscr{M}]_k \subseteq [\mathscr{T}]_{f(p,k)} \subseteq [\mathscr{T}]$. Thus, $[\mathscr{X}] \subseteq [\mathscr{T}]$, and by (e) of Proposition 7.2, $\mathscr{T}$ is a good class. □

In Lemma 8.1, we investigated the question of the different words as operators yielding the same result upon being applied to a given class of modules. It is an interesting question to ponder as to what one can tell about one operator giving the same result upon being applied to two different classes if we have a different operator that gives the same result upon being applied to the same two classes. From Corollary 6.4 it follows that if for any two classes of R-modules, $\mathscr{T}$ and $\mathscr{U}$, $B(\mathscr{T}) = B(\mathscr{U})$, then $C(\mathscr{T}) = C(\mathscr{U})$. Note that whether it can be concluded from $C(\mathscr{T}) = C(\mathscr{U})$ that $B(\mathscr{T}) = B(\mathscr{U})$ is not clear. The following result deals with the case when that implication is true.

Lemma 8.5. *Let Λ be a collection of classes of R-modules that is closed under B or under C. Then, the following statements are equivalent.*

a) $B(\mathscr{T}) = B(\mathscr{U})$ iff $C(\mathscr{T}) = C(\mathscr{U})$, for all $\mathscr{T}, \mathscr{U}$ in Λ.

b) All classes in Λ are good classes.

Proof. $(b) \Rightarrow (a)$ is easy to see - if all classes in Λ are good classes, then $B(\mathscr{T}) = CB(\mathscr{T})$ for all $\mathscr{T}$ in Λ, and therefore, for any $\mathscr{T}$ and $\mathscr{U}$ in Λ, the statements $B(\mathscr{T}) = B(\mathscr{U})$ and $C(\mathscr{T}) = C(\mathscr{U})$ are equivalent.

$(a) \Rightarrow (b)$. First let us assume that Λ is closed under the operation B. Let $B(\mathscr{T}) = B(\mathscr{U})$ iff $C(\mathscr{T}) = C(\mathscr{U})$, for all $\mathscr{T}, \mathscr{U}$ in Λ. Note that it is always true that $C(\mathscr{T}) = C(\mathscr{U})$ if $B(\mathscr{T}) = B(\mathscr{U})$ (by Corollary 6.4). As Λ is closed under B, $B(\mathscr{T})$ is in Λ if $\mathscr{T}$ is in Λ. Now note that $\mathscr{U} = B(\mathscr{T})$ satisfies $C(\mathscr{T}) = C(\mathscr{U})$ for all $\mathscr{T}$ in Λ (as $CB = C$ as operators on any class, by Lemma 6.3). Therefore, $B(\mathscr{T}) = B^2(\mathscr{T})$ for all $\mathscr{T}$ in Λ. By condition (a) of Proposition 7.2, it now follows that all $\mathscr{T}$ in Λ are good.

Now let us assume that Λ is closed under the operation C. Again, assume $B(\mathscr{T}) = B(\mathscr{U})$ iff $C(\mathscr{T}) = C(\mathscr{U})$, for all $\mathscr{T}, \mathscr{U}$ in Λ. Note that, for any $\mathscr{T}$, $\mathscr{U} = C(\mathscr{T})$ satisfies $C(\mathscr{T}) = C(\mathscr{U})$ because $C^2 = C$ as operations on any class by Corollary 6.4. Therefore, $B(\mathscr{T}) = BC(\mathscr{T}) = C(\mathscr{T})$ (by Lemma 6.3), so $\mathscr{T}$ is good. □

9. Applications of our generation results in Kropholler's hierarchy

We said at the beginning that in [2], we have used the concepts of generation discussed in this article to prove useful results in generating module an derived categories for a large class of infinite

groups. We end this article with highlighting some of the results achieved in [2] with sufficient context and detail.

The groups that we deal with in [2] come from a hierarchy which was first introduced by Peter Kropholler (see [8]). We provide a definition below.

Definition 9.1. *([8]) Let $\mathscr{X}$ be a class of groups. We define $H_0\mathscr{X} := \mathscr{X}$, and for any positive integer n, a group G is said to be in $H_n\mathscr{X}$ iff there exists a finite dimensional contractible CW-complex on which G acts by permuting the cells with cell stabilizers in $H_{n-1}\mathscr{X}$. If α is a limit ordinal, then we define $H_\alpha\mathscr{X}$ as $\bigcup_{\beta<\alpha} H_\beta\mathscr{X}$. G is said to be in $H\mathscr{X}$ iff G is in $H_\alpha\mathscr{X}$ for some ordinal α (note that α need not be a limit ordinal here).*

For any two ordinals α and β such that $\alpha < \beta$, it is easy to show that $H_\alpha\mathscr{X} \subseteq H_\beta\mathscr{X}$. If we take $\mathscr{X}$ to be the class of all finite groups $\mathscr{F}$, then for every iteration of H, we get a strictly bigger class of groups. This is quite a strong result, see [7]. To just give a small example of how one iteration can dramatically increase the size of a class, note that it can be shown that if $H_1\mathscr{F}$ contains all groups of finite virtual cohomological dimension, all groups of finite Bredon cohomological dimension, etc.

One of the main motivations behind introducing this hierarchy was to find a simple algebraic characteristic of groups that could imply the existence of a finite-dimensional model for the proper classifying space. It was proved in [9] that $H\mathscr{F}$-groups of type FP_∞ admit finite dimensional models for their proper classifying spaces.

For the rest of this section, for any group G and any class of groups $\mathscr{X}$ and a fixed commutative ring R, we denote by $\Lambda_\alpha(G, \mathscr{X})$ the class of all RG-modules induced up from $H_\alpha\mathscr{X}$-subgroups of G. One of the most useful generation results in the module category of groups in Kropholler's hierarchy that we proved in [2] is the following:

Lemma 9.2. *(Lemma 3.3 of [2]) Let G be a group and let $\mathscr{X}$ be a class of groups. Then, $\Lambda_{\alpha+1}(G, \mathscr{X}) \subseteq [\Lambda_\alpha(G, \mathscr{X})^\oplus]$ for any successor ordinal α. Here, the symbol $^\oplus$ means that we are closing the class under arbitrary direct sums.*

Making use of Lemma 9.2, we were able to prove, under an extra finiteness condition, that the number of generation steps for modules in $\Lambda_n(G, \mathscr{X})^\oplus$ from $\Lambda_m(G, \mathscr{X})^\oplus$ for $m \leq n$, for fixed G and $\mathscr{X}$, can be taken to be a polynomial of degree $n - m$. Before we state this result, we make the following definition.

Definition 9.3. *Take a class of RG-modules $\mathscr{T}$. We define the finitistic $\mathscr{T}$-dimension of RG to be the supremum over the $\mathscr{T}$-dimension of all RG-modules with finite $\mathscr{T}$-dimension.*

It is straightforward that the finitistic $\mathscr{U}$-dimension provides an upper bound for the finitistic $\mathscr{T}$-dimension for two classes $\mathscr{U}$ and $\mathscr{T}$ whenever $\mathscr{U}$ contains $\mathscr{T}$. Thus, the finitistic $\Lambda_n(G, \mathscr{X})^\oplus$-dimension is an upper bound for the finitistic $\Lambda_m(G, \mathscr{X})^\oplus$-dimension for all $m < n$. We mention

this now because this is made use of, although this is not the main step, in proving Theorem 9.4 in [2].

Theorem 9.4. *(Theorem 3.5 and Remark 3.7 in [2]) Using notations as above, let the finitistic $\Lambda_{n-1}(G,\mathscr{X})^{\oplus}$-dimension of RG be finite. Denote it by t. Then for any $m < n$, any module in $\Lambda_n(G,\mathscr{X})^{\oplus}$ can be generated from $\Lambda_m(G,\mathscr{X})^{\oplus}$ in $(1+t)^{n-m}-1$ steps.*

From Theorem 9.4, we get the following corollary for the case where G is in $H_n\mathscr{X}$ itself.

Corollary 9.5. *Use notations as above and make the extra assumption that $G \in H_n\mathscr{X}$, then every RG-module can be generated from $\Lambda_0(G,\mathscr{X})^{\oplus}$ in $(1+t)^n-1$ steps.*

Corollary 9.5 is quite a useful result because it tells us that for $H_n\mathscr{F}$-groups, where $\mathscr{F}$ again is the class of all finite groups, we can look at and study all modules over the group algebra by generating modules from the class of modules induced up from finite subgroups. And since finite groups are in general easier to handle than infinite groups, this result is very helpful.

We end this section and the article with a couple of generation results for the derived categories that we proved in [2] making use of the above results.

Theorem 9.6. *(part of Theorem 4.5 of [2]) Let $G \in H_n\mathscr{X}$, for some class of groups $\mathscr{X}$ and some integer n. Then,*

a) In the derived bounded category of finite length chain complexes of RG-modules $\mathcal{D}^b(Mod\text{-}RG)$, the smallest triangulated subcategory of $\mathcal{D}^b(Mod\text{-}RG)$ containing the class $\Lambda_{n-1}(G,\mathscr{X})^{\oplus}$ (we treat this class of modules as a class of chain complexes concentrated in degree zero) is all of $\mathcal{D}^b(Mod\text{-}RG)$. Using standard parlance from the generation of triangulated categories, one can say here that $\Lambda_{n-1}(G,\mathscr{X})^{\oplus}$ "strongly" generates $\mathcal{D}^b(Mod\text{-}RG)$.

b) In the derived unbounded category of complexes of RG-modules $\mathcal{D}(Mod\text{-}RG)$, the smallest localising (and also colocalising) subcategory of $\mathcal{D}(Mod\text{-}RG)$ containing the class $\Lambda_{n-1}(G,\mathscr{X})^{\oplus}$ is all of $\mathcal{D}(Mod\text{-}RG)$. And again, using generation language from triangulated categories with arbitrary products and coproducts, one can say that $\Lambda_{n-1}(G,\mathscr{X})^{\oplus}$ both "generates" and "cogenerates" $\mathcal{D}(Mod\text{-}RG)$.

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