

## Chapter 1

# Bounded $t$ -structures and finitistic dimensions - a survey

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This survey reports on work contained in the paper [6] which was a joint project with H. X. Chen, K. Manali Rahul, C. Parker, and J. Zheng where we established a categorical obstruction to the existence of bounded  $t$ -structures on abstract triangulated categories satisfying a small finiteness condition. In the process, we generalised Neeman's work on the existence of bounded  $t$ -structures on perfect complexes over noetherian schemes, and related results regarding equivalence classes of bounded  $t$ -structures on the bounded derived category. Our generalisations are in the language of completion theory of triangulated categories. Among our main tools were the introduction and use of a categorical notion of finitistic dimension, lifting  $t$ -structures along completions, etc. In this survey, a lot of emphasis has been placed on explaining the underlying ideas, intuitions, and motivations. Applications of our main results to algebraic geometry and representation theory have been discussed in detail.

## 1.1 Introduction

There were primarily two avenues of motivation for this work. One of them is from the world of classical representation theory of artin algebras, and the other is of a more categorical touch and essentially from the world of algebraic geometry.

We start with the first one. One of the most striking results in recent years where a property of one of the derived/triangulated categories associated to a fixed finite dimensional algebra has been tied to that algebra satisfying any of the longstanding open homological conjectures of the field was the 2019 result by Rickard which showed the following - for any finite dimensional algebra (over a field)  $A$ , if the smallest localising subcategory of the unbounded derived category of cochain complexes of  $A$ -modules containing all injective  $A$ -modules is the whole derived category (in other words, if the  $A$ -injectives generate  $D(A\text{-Mod})$  with coproducts), then the big finitistic dimension of  $A$  is finite (see [23, Theorem 4.3]). In Rickard's paper, there were several methods suggested that one could use to see if, for any given algebra, its unbounded derived category admitted the injective generation property but one could still look at the result and ask whether it might be possible to connect the small finitistic dimension of an algebra with a property of one of the essentially small triangulated categories, like the category of perfect complexes, that are usually associated with it. The category of

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perfect complexes (i.e. bounded complexes of finitely generated projectives) is often easier to work with than the unbounded derived category. *So, in a nutshell, the motivating question here was whether the small finitistic dimension of an algebra (or of a ring, more generally), or rather the finiteness of it, could be read off as an intrinsic property of the triangulated category of perfect complexes.*

Now we move to the second motivation. Antieau, Gepner, and Heller [4] conjectured that for any finite-dimensional noetherian scheme  $X$ , the category of perfect complexes,  $\text{Perf}(X)$ , has a bounded  $t$ -structure if and only if  $X$  is regular, i.e. if  $\text{Perf}(X) = \text{D}_{\text{coh}}^{\text{b}}(X)$ . The motivation they had behind this conjecture came out of their work on  $K$ -theoretic obstructions where they showed that the first negative  $K$ -group of any stable  $\infty$ -category that admits a bounded  $t$ -structure. As noted in their paper, there were already known examples of singular schemes that had vanishing negative  $K$ -groups. Still, there were no known examples of  $\text{Perf}(X)$  having a bounded  $t$ -structure without the standard bounded  $t$ -structure of  $\text{D}_{\text{coh}}^{\text{b}}(X)$  being itself a bounded  $t$ -structure on  $\text{Perf}(X)$ . So, their conjecture was a little ambitious, and it was subsequently settled by Neeman [19]. Our motivation came from the observation that, in the language of completion theory for triangulated categories as developed by Neeman,  $\text{D}_{\text{coh}}^{\text{b}}(X)$  can be realised as the completion of  $\text{Perf}(X)$  “with respect to the good metric generated by any classical generator of it”. On the back of this observation, one can ask *if any essentially small triangulated category satisfying some adequate finiteness property (one should require such a property because the scheme  $X$  in the above discussion had finite dimension and Neeman’s theorem, or rather the proof of it, crucially relied on this fact) has to be equal to its completion with respect to certain good metrics in the presence of a bounded  $t$ -structure.* Our work answers this question, and generalises several other related existing results from the literature including a new (and logically independent from Neeman’s) proof of the Antieau-Gepner-Heller conjecture and the setting up of a potentially more general conjecture/question. The finiteness condition on the starting category turned out to be a categorical notion of “finitistic dimension” which, at the same time, also manages to characterize small finitistic dimension as an intrinsic property of the derived category of perfect complexes thereby answering the question at the heart of the first motivation mentioned above.

## 1.2 Preliminaries and definitions

### 1.2.1 Finitistic dimensions

We start with the classical definitions for rings. Throughout this survey, all of our modules over rings will be by default assumed to be left modules.

**Definition 1.2.1.** *Let  $R$  be a ring.  $\text{Findim}(R)$ , the big finitistic dimension of  $R$ , is defined to be supremum over the projective dimension of all  $R$ -modules that have finite*

projective dimension. And,  $\text{findim}(R)$ , the small finitistic dimension of  $R$ , is defined to be the supremum over the projective dimension of all modules that have a finite-length projective resolution with finitely generated projective  $R$ -modules.

As finitistic dimensions are often studied in the representation theory of algebras, note that when  $R$  is an artin algebra,  $\text{findim}(R)$  can just be stated as the supremum over the projective dimension of all finitely generated  $R$ -modules that have finite projective dimension.

Throughout the paper, for any ring  $R$ ,  $R\text{-proj}$  denotes the category of finitely generated  $R$ -modules, and  $R\text{-mod}$  denotes the category of finitely presented  $R$ -modules.

For essentially small triangulated categories, the definition of the finitistic dimension that we use is a little bit involved. And although, as we shall see, it models the small finitistic dimension of rings when applied to the category of perfect complexes, we will not be using the adjective “small” when talking about the finitistic dimension of triangulated categories.

We need to first introduce some notations.

**Definition 1.2.2.** Take  $\mathcal{S}$  to be any essentially small triangulated category with the suspension functor denoted by  $[1]$ , and fix an object  $G \in \mathcal{S}$ .

- For any  $n \in \mathbb{Z}$ ,  $G(-\infty, n] = \{G[-i] : i \leq n\}$ .
- For any  $n \in \mathbb{Z}$ , let  $\langle G \rangle^{(-\infty, n]}$  denote the smallest full subcategory of  $\mathcal{S}$  that contains  $G[-n]$  and is closed under extensions, summands, and positive shifts. Similarly, let  $\langle G \rangle^{[n, \infty)}$  denote the smallest full subcategory of  $\mathcal{S}$  that contains  $G[-n]$  and is closed under extensions, summands, and negative shifts. Note that these are not triangulated subcategories.

Armed with these notations, we come to our promised definition of the finitistic dimension of a triangulated category.

**Definition 1.2.3.** For any object  $G$  in an essentially small triangulated category  $\mathcal{S}$ ,  $\text{findim}(\mathcal{S}, G) := \inf\{n \in \mathbb{N} : G(-\infty, -1]^\perp \subseteq \langle G \rangle^{[-n, \infty)}\}$ .

If no such  $n$  exists for any object  $G$ , we will declare the finitistic dimension of  $\mathcal{S}$  to be infinite, and similarly, we say  $\mathcal{S}$  has finite finitistic dimension if  $\text{findim}(\mathcal{S}, G) < \infty$  for some object  $G$ .

**Motivation behind Definition 1.2.3.** When  $\mathcal{S} = \text{K}^b(R\text{-proj})$ , for any ring  $R$ , and  $G$  is taken to be the regular module  $R$ , it turns out that  $\text{findim}(\text{K}^b(R\text{-proj}), R) = \text{findim}(R)$ . For general rings, the proof of this statement requires a little bit of work, so the reader can just refer to it in our paper (see [6, Lemma 4.1(5)]). Let me still provide a workable intuition for the case when  $R$  is an artin algebra - in this case,  $R(-\infty, -1]^\perp$  corresponds to the class of all finitely generated  $R$ -modules with finite

projective dimension, and  $\langle R \rangle^{[-n, \infty)}$  corresponds to the class of all finitely generated  $R$ -modules with projective dimension bounded above by  $n$ .

Often, in practical applications, our  $\mathcal{S}$  will be a classically generated triangulated category and  $G$  will be taken to be a classical generator. Recall that an object  $G \in \mathcal{S}$  is said to be its classical generator if the smallest thick subcategory of  $\mathcal{S}$  containing  $G$  is all of  $\mathcal{S}$ . The derived category of perfect complexes over any ring or any quasicompact quasiseparated scheme (the scheme does not even have to be noetherian here) are examples of such categories. So, one should definitely not have a situation where  $\text{findim}(\mathcal{S}, \text{one classical generator})$  is finite and  $\text{findim}(\mathcal{S}, \text{another classical generator})$  is not finite. This is indeed the case with our  $\text{findim}$ , and we record it as an observation below.

**Observation 1.2.4.** *(Consult [6, Lemma 4.1(1)] for a short proof) If we have two objects  $G, H \in \mathcal{S}$  such that  $G$  is in the smallest thick subcategory of  $\mathcal{S}$  containing  $H$ , then  $\text{findim}(\mathcal{S}, G)$  is finite iff  $\text{findim}(\mathcal{S}, H)$  is finite. This covers the case where  $G$  and  $H$  are both classical generators of  $\mathcal{S}$ .*

The following is an illustrative use of our  $\text{findim}$  concept but its importance in the wider story presented in this survey is minimal.

**Proposition 1.2.5.** *(surprisingly, not too standard - follows from [10, Theorem 1.1], was proved for coherent rings in [22]) Take two derived equivalent rings  $R_1$  and  $R_2$ . Then,  $\text{findim}(R_1)$  is finite iff  $\text{findim}(R_2)$  is finite.*

As the careful reader will have noticed, any well-behaved finitistic dimension concept for essentially small triangulated categories, that gives back the honest small finitistic dimension of a ring when applied to the category of its perfect complexes, would have achieved Proposition 1.2.5 as it should just be a one-line application, and this was indeed the case with a *different* concept of finitistic dimension for triangulated categories that was introduced by Henning Krause (see [16, Remark 3.3]). Although we will not discuss Krause's definition in this survey, there are two points that are worth bearing in mind - i) the two definitions differ quite clearly: for instance when, for any artin algebra  $A$ , one takes  $\mathcal{S} = D^b(A\text{-mod})$  and allows  $A$  to be of infinite global dimension, Krause's  $\text{findim}$  is not finite whereas ours is (see the discussion at the end of [6, Appendix B]), and ii) although Krause's  $\text{findim}$  achieves the goal set in our first motivation i.e. connecting the small finitistic dimension of a ring to its category of perfect complexes, it is ill-suited for the work set out in our second motivation, i.e. to be the correct abstract finiteness condition on a category to allow a generalisation of Neeman's work.

**Remark 1.2.6.** *For triangulated categories with a single compact generator (one can think of the unbounded derived category of any ring or of quasicohherent sheaves on a noetherian scheme), one can also work out a notion of big finitistic dimension. This is*

done in [6, Appendix B], but we will not be touching this concept in this survey due to it being not quite central to the main theme. I will just note that, for this big finitistic dimension, one has an analogous version of Observation 1.2.4 which in turn allows one to get the result that for any two derived equivalent rings  $R_1$  and  $R_2$ ,  $\text{Findim}(R_1) < \infty \iff \text{Findim}(R_2) < \infty$  - this result too follows from [10, Theorem 1.1]. Our approach provides the insight that Findim can be intrinsically characterised by the unbounded derived category.

## 1.2.2 t-structures

We start with the definition of  $t$ -structures. When our triangulated categories do not need to be essentially small, we will use the notation  $\mathcal{T}$  as opposed to  $\mathcal{S}$ .

**Definition 1.2.7.** A  $t$ -structure on a triangulated category  $\mathcal{T}$  is a pair of full subcategories  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  such that the following conditions are satisfied:

- $\mathcal{T}^{\leq -1} \subseteq \mathcal{T}^{\leq 0}$ , and  $\mathcal{T}^{\geq 0} \subseteq \mathcal{T}^{\geq -1}$ , where for any integer  $n$ ,  $\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n]$ , and  $\mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n]$ ,
- $\text{Hom}_{\mathcal{T}}(X, Y) = 0$ , for any  $X \in \mathcal{T}^{\leq -1}$  and any  $Y \in \mathcal{T}^{\geq 0}$ .
- For any object  $G \in \mathcal{T}$ , there is a distinguished triangle  $G^{\leq -1} \rightarrow G \rightarrow G^{\geq 0} \rightarrow G^{\leq -1}[1]$  with  $G^{\leq -1} \in \mathcal{T}^{\leq -1}$  and  $G^{\geq 0} \in \mathcal{T}^{\geq 0}$ .

The subcategories  $\mathcal{T}^{\leq 0}$ ,  $\mathcal{T}^{\geq 0}$  and  $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$  are respectively called the aisle, the coaisle, and the heart of the  $t$ -structure. And, we call the  $t$ -structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  bounded above if  $\mathcal{T} = \mathcal{T}^- := \bigcup_{n \in \mathbb{N}} \mathcal{T}^{\leq -n}$ , bounded below if  $\mathcal{T} = \mathcal{T}^+ := \bigcup_{n \in \mathbb{N}} \mathcal{T}^{\geq -n}$ , and bounded if  $\mathcal{T} = \mathcal{T}^b := \mathcal{T}^+ \cap \mathcal{T}^-$ .

An easy example of a  $t$ -structure is the standard  $t$ -structure on  $D(A\text{-Mod})$ , for any ring  $A$ , which has the subcategory of all complexes with cohomology concentrated in non-positive degrees as the aisle, and the subcategory of all complexes with cohomology concentrated in non-negative degrees as the coaisle. Similarly, the derived bounded and the unbounded category associated to any abelian category comes endowed with a standard  $t$ -structure.

**Warning 1.2.8.** If we start with an exact category instead of an abelian category, then the unbounded derived category or the bounded derived category does not come with a standard  $t$ -structure.

Next, we state a very key and important property of the heart of  $t$ -structures.

**Fact 1.2.9.** The heart of any  $t$ -structure on any triangulated category is an abelian category. If a triangulated category  $\mathcal{S}$  admits a bounded  $t$ -structure with heart  $\mathcal{H}$ ,

then the smallest thick subcategory of  $\mathcal{S}$  containing  $\mathcal{H}$  is the whole of  $\mathcal{S}$  - in other words, the heart of a bounded  $t$ -structure generates the whole category.

Although we will be discussing bounded  $t$ -structures and categorical obstructions to their existence in a later section where our new results will be elaborated on, it is worth noting that whenever bounded  $t$ -structures exist, there is an obvious Hom-vanishing property satisfied by any two objects of the category. More precisely,

**Observation 1.2.10.** (standard) *If  $\mathcal{T}$  has a bounded  $t$ -structure, then for any two objects  $M, N \in \mathcal{T}$ ,  $\text{Hom}_{\mathcal{T}}(M, N[i]) = 0$  for  $i \ll 0$ .*

This simple observation is easy to prove just by recalling the definition of a bounded  $t$ -structure and Definition 1.2.7.(b). As was pointed out to me by Xiao-Wu Chen, from this, it follows that the singularity category of an artin algebra  $A$  with infinite global dimension cannot have any bounded  $t$ -structures. This is because if one takes  $M = N = A/\text{rad}(A)$  with  $\mathcal{T} = \text{D}_{\text{sg}}(A)$  in Observation 1.2.10, then by [8, Example 2.3 and Theorem 2.8],  $\text{Hom}_{\mathcal{T}}(A/\text{rad}(A), A/\text{rad}(A)[i]) \neq 0$  for all  $i \in \mathbb{Z}$ . Of course, when  $A$  has finite global dimension,  $\text{D}_{\text{sg}}(A)$  is trivial so nothing interesting happens in that case. We see thus that the singularity category of an artin algebra is one example of a naturally occurring triangulated category that one can associate to an algebra that can never have a bounded  $t$ -structure.

In triangulated categories that admit arbitrary coproducts, the concept of *compactly generated  $t$ -structures* is often very useful. We have to first start with a deep theorem before getting to the definition.

**Theorem 1.2.11.** *Let  $\mathcal{T}$  be a triangulated category with arbitrary coproducts, and let  $\mathcal{A}$  be a shift-closed essentially small full subcategory of the compacts. Then,  $\text{Coproduct}(\mathcal{A})$ , the smallest full subcategory of  $\mathcal{T}$  containing  $\mathcal{A}$  and closed under all coproducts and extensions, is the aisle of a  $t$ -structure on  $\mathcal{T}$ . This  $t$ -structure is referred to as the  $t$ -structure generated by  $\mathcal{A}$ .*

Theorem 1.2.11 has an interesting history. It first appeared in [3, Theorem A.1 and Proposition A.2], then reproved in [2, Corollary 4.6.(a)], and reproved one more time in [7, Theorem 2.3.3] (this time with a mild warning that the original argument had a small but fixable gap, see [7, Remark 2.3.4] for this discussion).

**Notation 1.2.12.** *Let  $\mathcal{T}$  be a triangulated category with a single compact generator  $G$ . Then the  $t$ -structure on  $\mathcal{T}$  generated by the set  $\{G[i] : i \geq 0\}$ , in the sense of Theorem 1.2.11, is denoted  $(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0})$ .*

With this notation, it is natural to wonder how much the generated  $t$ -structure changes if  $G$  is replaced by a different compact generator  $H$ . It turns out that in this situation, the two generated  $t$ -structures will be equivalent in a sandwiching sense as made precise in the following definition.

**Definition 1.2.13.** Two  $t$ -structures  $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$  and  $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$  on a triangulated category  $\mathcal{T}$  are equivalent if there exists an integer  $n$  such that  $\mathcal{T}_1^{\leq -n} \subseteq \mathcal{T}_2^{\leq 0} \subseteq \mathcal{T}_1^{\leq n}$ .

If  $\mathcal{T}$  has a single compact generator  $G$ , then the equivalence class containing  $(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0})$  is called the preferred equivalence class of  $t$ -structures. As mentioned above, all  $t$ -structures of the form  $(\mathcal{T}_H^{\leq 0}, \mathcal{T}_H^{\geq 0})$ , for any other single compact generator  $H$  (this fact needs proving, see [19, Corollary 2.8]) are in this class.

The significance of the “preferred” in the term “preferred equivalence class of  $t$ -structures” is mainly that, as we will later see, when completing the compacts, the good metric that gets used is obtained from the aisle of a  $t$ -structure in the preferred equivalence class of  $t$ -structures in the big category after restriction to the compacts. It is known that in the derived unbounded category of a ring or the derived unbounded category of complexes with quasicohherent cohomology over a quasicompact quasi-separated scheme, the standard  $t$ -structure is in the preferred equivalence class (see [20, Example 4.1] and [19, Theorem 3.2]). So, in a way, the “preferred equivalence class” terminology also allows us to replace accidental standard  $t$ -structures that only occur in some examples by  $t$ -structures intrinsically characterized in the ambient triangulated category.

### 1.3 Neeman’s completion theory

**Main idea.** In short, we start with an essentially small triangulated category  $\mathcal{S}$ . Inside  $\mathcal{S}$ , one cannot take arbitrary homotopy colimits, so we take the Yoneda map into the category  $\mathcal{S}\text{-Mod}$  where there are arbitrary colimits. Then, inside  $\mathcal{S}\text{-Mod}$ , we look at the full subcategory of colimits of Cauchy sequences in  $\mathcal{S}$  with respect to a fixed good metric  $\mathcal{M}$  (one has to still clarify as to what the precise definition of a good metric on  $\mathcal{S}$  is, and what it means to consider Cauchy sequences with respect to it). Let us denote this subcategory by  $\mathcal{L}_{\mathcal{M}}(\mathcal{S})$ . This is not our completion output. For our end-result, we take a full subcategory of  $\mathcal{L}_{\mathcal{M}}(\mathcal{S})$ , written  $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$ , comprising of those functors  $F \in \mathcal{S}\text{-Mod}$  that kill off some high enough member of the good metric (see Definition 1.3.3), or in other words, that send “sufficiently small” morphisms to isomorphisms, and a fundamental result of Neeman’s completion theory is that this  $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$  is also a triangulated category, and this will be called the completion  $\mathcal{S}$  with respect to the good metric  $\mathcal{M}$ .

We will now substantiate the above idea with all the definitions fleshed out in full detail. We keep our (essentially small) triangulated category  $\mathcal{S}$  fixed.

**Definition 1.3.1.** A good metric on  $\mathcal{S}$  is a family of full subcategories  $\mathcal{M} = \{\mathcal{M}_n\}_{n \in \mathbb{N}}$  such that, for each  $n$ ,  $0 \in \mathcal{M}_n$ ,  $\mathcal{M}_n$  is closed under extensions, and  $\mathcal{M}_{n+1}[-1] \cup \mathcal{M}_{n+1} \cup \mathcal{M}_{n+1}[1] \subseteq \mathcal{M}_n$ .

A good metric  $\mathcal{M}$  is said to be finer than another good metric  $\mathcal{N} := \{\mathcal{N}_n\}_{n \in \mathbb{N}}$  if for each  $n$ , there exists  $m \in \mathbb{N}$  such that  $\mathcal{M}_m \subseteq \mathcal{N}_n$ . Two good metrics  $\mathcal{M}$  and  $\mathcal{N}$  are said to be equivalent if simultaneously,  $\mathcal{M}$  is finer than  $\mathcal{N}$  and  $\mathcal{N}$  is finer than  $\mathcal{M}$ .

It is easy to see that for any  $t$ -structure  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$ ,  $\{\mathcal{S}^{\leq -n}\}_{n \in \mathbb{N}}$  is a good metric. In fact, if some other triangulated category  $\mathcal{T}$  has a  $t$ -structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ , and  $\mathcal{S}$  is a triangulated subcategory of it, then  $\{\mathcal{S} \cap \mathcal{T}^{\leq -n}\}_{n \in \mathbb{N}}$  is a good metric on  $\mathcal{S}$ . We will see later that when we have a triangulated category  $\mathcal{T}$  that admits a single compact generator  $G$ , and we want to complete the category of compacts,  $\mathcal{T}^c$ , the good metric that will be of most use is  $\{\mathcal{T}^c \cap \mathcal{T}^{\leq -n}\}_{n \in \mathbb{N}}$ , with  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  being a  $t$ -structure from the preferred equivalence class of  $t$ -structures in the sense of Definition 1.2.13.

**Definition 1.3.2.** A Cauchy sequence on  $\mathcal{S}$  with respect to a good metric  $\mathcal{M}$  is a sequence of morphisms  $\{X_\bullet, f_\bullet\} : X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3 \longrightarrow \cdots$  in  $\mathcal{S}$  where for every  $i \geq 1$ , there exists an  $n_i \geq 1$  such that  $\text{Cone}(f_j) \in \mathcal{M}_i$  for all  $j \geq n_i$ .

Recall that  $\mathcal{S}\text{-Mod}$  is the standard notation for the abelian category of additive functors from  $\mathcal{S}^{\text{op}}$  to  $\text{Ab}$  (the category of abelian groups). Consider the fully faithful Yoneda functor  $\mathcal{Y} : \mathcal{S} \rightarrow \mathcal{S}\text{-Mod}$  sending  $X \mapsto \text{Hom}_{\mathcal{S}}(-, X)$ . From now on, we fix a good metric  $\mathcal{M}$  on  $\mathcal{S}$ .

**Definition 1.3.3.** Fix  $\mathcal{A}$  to be a full subcategory of  $\mathcal{S}$ .

- i.  $\mathcal{L}_{\mathcal{M}}(\mathcal{A}) := \{F \in \mathcal{S}\text{-Mod} : F \simeq \varinjlim \mathcal{Y}(X_n), \text{ where } \{X_\bullet\} \text{ is a Cauchy sequence in } \mathcal{S} \text{ with respect to } \mathcal{M} \text{ where all the terms } X_n \in \mathcal{A}\}$ .
- ii.  $\mathfrak{C}_{\mathcal{M}}(\mathcal{S}) := \{F \in \mathcal{S}\text{-Mod} : F(\mathcal{M}_j) = 0 \text{ for some } j \geq 0\}$ .
- iii.  $\mathfrak{S}_{\mathcal{M}}(\mathcal{A}) := \mathcal{L}_{\mathcal{M}}(\mathcal{A}) \cap \mathfrak{C}_{\mathcal{M}}(\mathcal{S})$ . When  $\mathcal{A} = \mathcal{S}$ ,  $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$  is called the completion of  $\mathcal{S}$  with respect to  $\mathcal{M}$ .

Note that if  $\mathcal{M} \simeq \mathcal{N}$  as good metrics in the sense of Definition 1.3.1, then  $\mathfrak{S}_{\mathcal{M}}(\mathcal{S}) = \mathfrak{S}_{\mathcal{N}}(\mathcal{S})$ .

*Remark 1.3.4.* Staying true to the sketched ‘‘Main idea’’ at the start of this section, we should explain as to why the subcategory  $\mathfrak{C}_{\mathcal{M}}(\mathcal{S})$  defined in Definition 1.3.3.(ii) consists of those functors ‘‘that send sufficiently small morphisms to isomorphisms’’. One can define the length of a morphism (in  $\mathcal{S}$ ) with respect to  $\mathcal{M}$  the following way:  $\text{length}_{\mathcal{M}, \mathcal{S}}(f) := \inf_{n \in \mathbb{N}} \{\frac{1}{n+1} : \text{Cone}(f) \in \mathcal{M}_n\}$ . Then, it turns out that for any  $F \in \mathcal{S}\text{-Mod}$ ,  $F \in \mathfrak{C}_{\mathcal{M}}(\mathcal{S})$  if and only if there exists an  $\epsilon > 0$  such that for any morphism  $f : a \rightarrow b$  in  $\mathcal{S}$  with  $\text{length}_{\mathcal{M}, \mathcal{S}}(f) < \epsilon$ ,  $F(f) : F(b) \rightarrow F(a)$  is an isomorphism.

As the reader can expect from the name,  $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$  will turn out to be a triangulated category. Before we state the precise theorem, we should say a few words about what will turn out to be the suspension functor in this triangulated structure.

$\mathcal{S}$  comes with an automorphism  $[1] : \mathcal{S} \rightarrow \mathcal{S}$  which gives way to an automorphism of abelian categories:  $\Sigma : \mathcal{S}\text{-Mod} \rightarrow \mathcal{S}\text{-Mod}$  sending  $F \mapsto [\Sigma(F) : X \mapsto F(X[-1])]$  for any  $F \in \mathcal{S}\text{-Mod}$  and  $X \in \mathcal{S}$ .

**Theorem 1.3.5.**  $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$  is a triangulated category with the suspension functor given by the above  $\Sigma$ . The distinguished triangles are given by sequences of the form  $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma(A)$  that are isomorphic to the colimit of the image under the Yoneda map  $\mathcal{Y}$  of Cauchy sequences (with respect to  $\mathcal{M}$ ) of triangles in  $\mathcal{S}$ :  $\{A_{\bullet}\} \xrightarrow{\alpha_{\bullet}} \{B_{\bullet}\} \xrightarrow{\beta_{\bullet}} \{C_{\bullet}\} \xrightarrow{\gamma_{\bullet}} \{A_{\bullet}\}[1]$ .

**Example 1.3.6.** (also contains some important notations that are used later)

- i. Take a triangulated category  $\mathcal{T}$  that admits a single compact generator  $G$ , and suppose that we are interested in completing the compacts with the use of an interesting metric. By interesting, we mean a metric that will, under some mild or no extra conditions, output a completion end-result that will be of interest in practical examples. This is where the good metric that we mentioned earlier, namely  $\{\mathcal{T}^c \cap \mathcal{T}^{\leq -n}\}_{n \in \mathbb{N}}$  with  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  any fixed t-structure in the preferred equivalence class (recall that such an equivalence class always exists when there is a single compact generator), comes in handy. Assume, additionally, that  $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$  for  $i \gg 0$ . Now, the completion with respect to our metric turns out to be an intrinsic thick subcategory of  $\mathcal{T}$  called the bounded closure of the compacts and denoted  $\mathcal{T}_c^b$ . More precisely, with respect to a fixed t-structure (which, in our case, is from the preferred equivalence class),  $\mathcal{T}_c^b$  is those bounded objects of  $\mathcal{T}$  that can be approximated arbitrarily well by the compacts, i.e.  $\mathcal{T}_c^b = \mathcal{T}^b \cap (\bigcap_{n \in \mathbb{N}} (\mathcal{T}^c * \mathcal{T}^{\leq -n}))$ . The notation  $\mathcal{T}_c^-$ , also sometimes referred to as the closure of the compacts, is often used for the triangulated subcategory given by  $\bigcap_{n \in \mathbb{N}} (\mathcal{T}^c * \mathcal{T}^{\leq -n})$ .
- ii. Putting the idea sketched in (i) to the test, one sees that if  $\mathcal{T}$  is taken to be  $\text{D}(R\text{-Mod})$ , for any ring  $R$ , then  $\mathcal{T}^c = \text{K}^b(R\text{-proj})$ ,  $\mathcal{T}_c^- = \text{K}^-(R\text{-proj})$ , the homotopy category of bounded above complexes of finitely generated projective  $R$ -modules, and  $\mathcal{T}_c^b = \text{K}^{-,b}(R\text{-proj})$ , the homotopy category of bounded above complexes of finitely generated projective  $R$ -modules with bounded cohomology (when  $R$  is left coherent, this category is equivalent to  $\text{D}^b(R\text{-mod})$ ; recall that  $R\text{-mod}$  is abelian here). Note that in this example,  $R$  is a single compact generator of  $\text{D}(R\text{-Mod})$ , and the Hom-vanishing condition on the generator is satisfied automatically.
- iii. Putting (i) to the test in an algebro-geometric setting, we see that if, for any quasicompact quasiseparated scheme  $X$ , we allow  $\mathcal{T} = \text{D}_{\text{qc}}(X)$ , the unbounded derived category of complexes of  $\mathcal{O}_X$ -modules with quasicohherent cohomology, then this category is known to admit a single compact generator which again satisfies the required Hom vanishing condition mentioned in (i), and  $\mathcal{T}^c = \text{Perf}(X)$ ,

the derived category of perfect complexes (as one might recall, a perfect complex in this setting is a bounded complex of  $\mathcal{O}_X$ -modules locally isomorphic to a bounded complex of finite rank vector bundles).  $\mathcal{T}_c^-$  becomes the derived category of pseudocoherent complexes. And, unsurprisingly,  $\mathcal{T}_c^b$  in this case turns out to be the derived category of bounded pseudocoherent complexes which, when  $X$  is noetherian, is exactly the bounded derived category of complexes with coherent cohomology or the bounded derived category of coherent sheaves on  $X$ , written as  $D_{\text{coh}}^b(X)$ .

- iv. In fact, just using  $G$ , and the notations introduced in Definition 1.2.2, if one writes down the family of full subcategories  $\{\langle G \rangle^{(-\infty, -n]}\}_{n \in \mathbb{N}}$ , one can check that this is a good metric itself, and with some extra work, one can show that this metric is actually equivalent to the metric mentioned earlier (this is proved in [21, Remark 4.7]). Note that if the category that we were interested in completing was not of the form  $\mathcal{T}^c$  for some  $\mathcal{T}$  with a single compact generator but was just an essentially small triangulated category  $\mathcal{S}$  with a single classical generator  $G$ , we would still have access to the good metric  $\{\langle G \rangle^{(-\infty, -n]}\}_{n \in \mathbb{N}}$  despite not seeing it as a metric equivalent to the restricted aisle metric of a  $t$ -structure in a preferred equivalence class upstairs. The starting categories in the examples from (ii) and (iii) both admit a single classical generator (due to them being the compacts in a triangulated category with a single compact generator). So, without appealing to the respective big unbounded derived categories, one can rewrite (ii) and (iii) as simply that if we complete  $\mathbf{K}^b(R\text{-proj})$  with the good metric  $\{\langle R \rangle^{(-\infty, -n]}\}_{n \in \mathbb{N}}$ , we get  $\mathbf{K}^{-,b}(R\text{-proj})$ , and that, if  $G$  is a single classical generator of  $\text{Perf}(X)$  (assume  $X$  is noetherian for notational simplicity), then the completion of  $\text{Perf}(X)$  with the metric  $\{\langle G \rangle^{(-\infty, -n]}\}_{n \in \mathbb{N}}$  is simply  $D_{\text{coh}}^b(X)$ .

**Definition 1.3.7.** (inspired by the point raised in Example 1.3.6.(iv)) Let  $\mathcal{S}$  be an essentially small triangulated category, and let  $G \in \mathcal{S}$ . Any good metric on  $\mathcal{S}$  that is equivalent to  $\{\langle G \rangle^{(-\infty, -n]}\}_{n \in \mathbb{N}}$  is called a  $G$ -good metric. When we are using a  $G$ -good metric, we denote the completion by  $\mathfrak{S}_G(\mathcal{S})$  (as opposed to  $\mathfrak{S}_{\{\langle G \rangle^{(-\infty, -n]}\}_{n \in \mathbb{N}}}(\mathcal{S})$ ).

Definition 1.3.7 makes sense even if  $\mathcal{S}$  does not have a single classical generator. If  $\mathcal{S} = \mathcal{T}^c$  with  $\mathcal{T}$  a triangulated category admitting a single compact generator  $G$ , the good metric  $\{\mathcal{T}^c \cap \mathcal{T}^{\leq -n}\}_{n \in \mathbb{N}}$ , with  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  a  $t$ -structure from the preferred equivalence class, is  $G$ -good - this is what Example 1.3.6.(iv) said.

The examples from algebraic geometry (still assuming  $X$  to be noetherian) can actually be relativized by, for any fixed closed subset  $Z \subseteq X$ , replacing  $\text{Perf}(X)$  and  $D_{\text{coh}}^b(X)$  with  $\text{Perf}_Z(X)$  and  $D_{\text{coh}, Z}^b(X)$  where the subscript “ $Z$ ” means we are only looking at those complexes in the respective category whose cohomology is supported on  $Z$ . We will see more of this in Section 1.5.

## 1.4 Starting with the new results

We will now slowly move towards the new results from our paper [6]. This section will have almost no examples, but the next two sections will be dedicated just to examples from algebraic geometry and representation theory.

One of our most useful new results arose from a question that can come to the mind of any reader who has just learned the definition of completion. And that question is - *if we start with an essentially small triangulated category  $\mathcal{S}$  that admits a certain structure or a certain nice property that we like, under what circumstances and using what metrics  $\mathcal{M}$ , can we ensure that  $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$  will also come with such niceties?*

If we take our attention to the existence of bounded  $t$ -structures as a property, one can immediately ask if such a property can be made invariant under the process of completion. To make the question slightly more tractable, one can ask if  $t$ -structures can be sent to  $t$ -structures in the completed category. It turns out that this is indeed possible. We start with a definition, and again, we keep an essentially small triangulated category  $\mathcal{S}$  fixed.

**Definition 1.4.1.** [6, Definition 3.2] *A  $t$ -structure  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$  is called extendable with respect to a good metric  $\mathcal{M}$  if there exists  $n \in \mathbb{N}$ ,  $\mathcal{M}_n \subseteq \mathcal{S}^{\leq 0}$ .*

Recall from the definition of one good metric being finer than another one (Definition 1.3.1) and note that a  $t$ -structure being extendable with respect to a good metric is the same as saying that that metric is finer than the aisle metric coming from the  $t$ -structure. The only reason we are using the term “extendable” is the justification provided to us by the upcoming result.

**Theorem 1.4.2.** [6, Theorem 3.5] *Consider a good metric  $\mathcal{M}$  on an essentially small triangulated category  $\mathcal{S}$ , and let us take a  $t$ -structure  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$  that is extendable with respect to this metric. Then,*

- i.  $(\mathfrak{S}_{\mathcal{M}}(\mathcal{S}^{\leq 0}), \mathfrak{S}_{\mathcal{M}}(\mathcal{S}^{\geq 0}))$  is a  $t$ -structure on  $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$  whose heart, as an abelian category, is equivalent to the heart of  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$ .
- ii. Let us elaborate the part in (i) about the heart not changing a little bit more. For any full subcategory  $\mathcal{A} \subseteq \mathcal{S}^{\geq 0}$ , the Yoneda functor  $\mathcal{Y} : \mathcal{S} \rightarrow \mathcal{S}\text{-Mod}$  restricts to an equivalence of additive categories,  $\mathcal{Y}_{\mathcal{A}} : \mathcal{A} \xrightarrow{\cong} \mathfrak{S}_{\mathcal{M}}(\mathcal{A})$ .
- iii. If  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$  is bounded above,  $(\mathfrak{S}_{\mathcal{M}}(\mathcal{S}^{\leq 0}), \mathfrak{S}_{\mathcal{M}}(\mathcal{S}^{\geq 0}))$  is as well.
- iv. If  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$  is bounded below, then  $\mathcal{Y} : \mathcal{S} \rightarrow \mathcal{S}\text{-Mod}$  restricts to a fully faithful triangle functor  $\mathcal{Y} : \mathcal{S} \rightarrow \mathfrak{S}_{\mathcal{M}}(\mathcal{S})$  such that  $\mathcal{Y}(\mathcal{S}^{\leq 0}) \subseteq \mathfrak{S}_{\mathcal{M}}(\mathcal{S}^{\leq 0})$  and  $\mathcal{Y}(\mathcal{S}^{\geq 0}) = \mathfrak{S}_{\mathcal{M}}(\mathcal{S}^{\geq 0})$ .

**Observation 1.4.3.** *Any  $t$ -structure is clearly equivalent with respect to the aisle metric induced by itself. Also, if  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$  is a bounded above  $t$ -structure on  $\mathcal{S}$ , then, for*

any object  $G \in \mathcal{S}$ , it is extendable with respect to the  $G$ -good metric  $\{\langle G \rangle^{(-\infty, -n)}\}_{n \in \mathbb{N}}$ . This is recorded in [6, Lemma 3.1.(3)] but is quite straightforward to check because as  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$  is bounded above, there has to be a positive integer  $m$  such that  $G \in \mathcal{S}^{\leq m}$  and just from this, using the fact that  $\mathcal{S}^{\leq m}$  is closed under extensions, summands, and positive shifts, we get that  $\langle G \rangle^{(-\infty, -m]} \subseteq \mathcal{S}^{\leq 0}$ . One can also check quite directly that if we have two equivalent  $t$ -structures, then one of them is extendable with respect to a fixed good metric iff the other one is - see [6, Lemma 3.1.(1)]. This means that any bounded above  $t$ -structure is extendable with respect to any  $G$ -good metric. So, bounded above  $t$ -structures are excellent candidates to apply the machinery of Theorem 1.4.2 to.

Theorem 1.4.2 is a key ingredient in proving the following theorem which, as briefly mentioned in the introduction, establishes an abstract categorical generalisation of Neeman's work [19] where he had settled a conjecture due to Antiau, Gepner, and Heller.

**Theorem 1.4.4.** [6, Theorem 3.11.(2)] *Let  $\mathcal{S}$  be an essentially small triangulated category with a bounded  $t$ -structure. Assume that  $\text{findim}(\mathcal{S}^{\text{op}}, G) < \infty$  for some object  $G \in \mathcal{S}$ . Then,  $\mathcal{S} = \mathfrak{S}_G(\mathcal{S})$ .*

Throughout this survey, even when we have to consider  $G \in \mathcal{S}$  as an object of  $\mathcal{S}^{\text{op}}$ , we still write  $G$  as opposed to something like  $G^{\text{op}}$ .

**Warning 1.4.5.** *Although in the statement of Theorem 1.4.4, we are not requiring our  $G$  to be a classical generator of  $\mathcal{S}$ , actually whenever  $\text{findim}(\mathcal{S}, G)$  or  $\text{findim}(\mathcal{S}^{\text{op}}, G)$  is finite and  $\mathcal{S}$  has a bounded  $t$ -structure,  $G$  is automatically a classical generator of  $\mathcal{S}$  - for a proof of this fact, see [6, Lemma 4.1.(2)].*

**One sketch for proving Theorem 1.4.4 directly using Theorem 1.4.2.** Let us take a bounded  $t$ -structure  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$  on  $\mathcal{S}$ . One first notes that  $\mathcal{S} \subseteq \mathfrak{S}_G(\mathcal{S})$  (this can be seen most intuitively with the language of *embeddable metrics* but as we have not introduced this concept in this survey, one can see this just from Theorem 1.4.2.(iv)). Observation 1.4.3 tells us that the  $t$ -structure  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$  has to be extendable with respect to any  $G$ -good metric which means, according to Theorem 1.4.2.(iii), that the “lifted/completed”  $t$ -structure  $(\mathfrak{S}_G(\mathcal{S}^{\leq 0}), \mathfrak{S}_G(\mathcal{S}^{\geq 0}))$  on  $\mathfrak{S}_G(\mathcal{S})$  is also bounded above. Now, we make use of the assumption that  $\text{findim}(\mathcal{S}^{\text{op}}, G) < \infty$ . It turns out, and this is quite a nontrivial part that requires a lot of work, that when  $\text{findim}(\mathcal{S}^{\text{op}}, G) < \infty$ ,  $(\mathfrak{S}_G(\mathcal{S}^{\leq 0}), \mathfrak{S}_G(\mathcal{S}^{\geq 0}))$  is also bounded below in addition to being bounded above, i.e. it is a bounded  $t$ -structure. Now, recall that  $(\mathfrak{S}_G(\mathcal{S}^{\leq 0}), \mathfrak{S}_G(\mathcal{S}^{\geq 0}))$  has the same heart as  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$  by Theorem 1.4.2.(i). We know, thanks to Fact 1.2.9, that the heart of a bounded  $t$ -structure generates the whole category. So, we are done. ■

We will be providing another sketch for proving Theorem 1.4.4. This second sketch, or rather the second method, will make use of a neat corollary of Theorem 1.4.2. This corollary is about applying Theorem 1.4.2 to any  $t$ -structure  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$  and its induced aisle metric  $\{\mathcal{S}^{\leq -n}\}_{n \in \mathbb{N}}$ .

**Corollary 1.4.6.** [6, Corollary 3.6] *Let  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$  be a  $t$ -structure on  $\mathcal{S}$ , and let its induced aisle metric be denoted by  $\mathcal{M}$ , i.e.  $\mathcal{M}_n = \mathcal{S}^{\leq -n}$  for all  $n$ . Then, the Yoneda functor  $\mathcal{Y} : \mathcal{S} \rightarrow \mathcal{S}\text{-Mod}$ , when restricted to  $\mathcal{S}^+$ , gives a triangulated equivalence between  $\mathcal{S}^+$  and  $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$ .*

**Proof sketch for Corollary 1.4.6.** First, we need to state an abstract completion fact (and just for this general fact, let the good metric under consideration be denoted by  $\mathcal{N}$  and not  $\mathcal{M}$ ): the Yoneda functor  $\mathcal{Y} : \mathcal{S} \rightarrow \mathcal{S}\text{-Mod}$  always restricts to a fully faithful triangle functor  $\bigcup_{m \in \mathbb{N}} \mathcal{N}_m^{\perp} \rightarrow \mathfrak{S}_{\mathcal{N}}(\mathcal{S})$ . Now, let us take  $\mathcal{N}$  to be the aisle metric  $\mathcal{M}$ . Then,  $\bigcup_{n \in \mathbb{N}} \mathcal{M}_n^{\perp} = \bigcup_{n \in \mathbb{N}} (\mathcal{S}^{\leq -n})^{\perp} = \bigcup_{n \in \mathbb{N}} \mathcal{S}^{\geq -n+1} = \mathcal{S}^+$ . So, the Yoneda functor restricted to  $\mathcal{S}^+$  is a fully faithful triangle functor  $\mathcal{Y}_{\mathcal{S}^+} : \mathcal{S}^+ \rightarrow \mathfrak{S}_{\mathcal{M}}(\mathcal{S})$ . The proof then goes on to show that  $\mathcal{Y}_{\mathcal{S}^+}$  is dense and that is enough. On the way to prove the denseness of  $\mathcal{Y}_{\mathcal{S}^+}$ , one needs to show that the lifted  $t$ -structure  $(\mathfrak{S}_{\mathcal{M}}(\mathcal{S}^{\leq 0}), \mathfrak{S}_{\mathcal{M}}(\mathcal{S}^{\geq 0}))$  is a bounded below  $t$ -structure on  $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$ ; and on the way to show this in turn, one makes use of the fact that the restricted Yoneda functor  $\mathcal{Y}_{\mathcal{S}^{\geq 0}} : \mathcal{S}^{\geq 0} \rightarrow \mathfrak{S}_{\mathcal{M}}(\mathcal{S}^{\geq 0})$  is an additive equivalence as is guaranteed by Theorem 1.4.2.(ii). We would like to refer the reader to the proof of this in our paper. ■

Now, how does one connect Corollary 1.4.6 with a second proof of Theorem 1.4.4 or a second way of seeing why Theorem 1.4.4 is true? Well, first note that Corollary 1.4.6 tells us that if  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$  is a bounded  $t$ -structure (in this case,  $\mathcal{S} = \mathcal{S}^b = \mathcal{S}^- = \mathcal{S}^+$ ), then the completion end-result when one uses the aisle metric induced by this  $t$ -structure is  $\mathcal{S}$  again.

**The second sketch for proving Theorem 1.4.4.** As the reader can expect, we will have to use the findim finite hypothesis at some point. It turns out, and this requires a substantial amount of nontrivial work (and that is precisely why what we are providing here is just a sketch and not the full proof), that when  $\text{findim}(\mathcal{S}^{\text{op}}, G)$  is finite for some  $G \in \mathcal{S}$ , the aisle metric  $\{\mathcal{S}^{\leq -n}\}_{n \in \mathbb{N}}$  coming from the bounded  $t$ -structure  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$  is actually  $G$ -good itself. In other words, completing with the aisle metric and completing with any  $G$ -good metric will give the same result, which means that  $\mathfrak{S}_G(\mathcal{S}) = \mathcal{S}$ . There is still a tiny bit of subtlety in how the proof works here. What we prove is the following - keep intact the assumption that  $\text{findim}(\mathcal{S}^{\text{op}}, G) < \infty$  but instead of directly assuming that  $\mathcal{S}$  has a bounded  $t$ -structure, assume that there is a bounded  $t$ -structure on a triangulated category  $\mathcal{X}$  that contains  $\mathcal{S}$  and is a triangulated subcategory of  $\mathfrak{S}_G(\mathcal{S})$  (note that here the assumption that  $\mathcal{S} \subseteq \mathfrak{S}_G(\mathcal{S})$  is implicit which, as we have seen before, is always satisfied when  $\mathcal{S}$  has a bounded  $t$ -structure), then, as

good metrics on  $\mathcal{S}$ ,  $\{\langle G \rangle^{(-\infty, -n]}\}_{n \in \mathbb{N}} \simeq \{\mathcal{S} \cap \mathcal{X}^{\leq -n}\}_{n \in \mathbb{N}}$  (we will have to revisit this fact in the next subsection in Theorem 1.4.7, so please see there for a precise reference to our paper); and then you take  $\mathcal{X} = \mathcal{S}$  in this statement. The point to note here is that although from the surface it seems like the “ $\mathcal{S} \subseteq \mathfrak{S}_G(\mathcal{S})$ ” fact was not being used in this second sketch, it is actually being used, and also instead of just using the fact about the heart staying the same in the lifting of the  $t$ -structures in Theorem 1.4.2, this second sketch uses the related fact from Theorem 1.4.2 about the coaisle staying the same additively as mentioned earlier in the Proof sketch for Corollary 1.4.6. ■

**The actual use of the finiteness condition.** In both the sketches of proving Theorem 1.4.4 as mentioned here, the assumption that  $\text{findim}(\mathcal{S}^{\text{op}}, G)$  is finite gets used only technically, i.e. only to establish a technical result or a chain of inclusions or equivalences. We have to transfer the information from  $\mathcal{S}^{\text{op}}$  to  $\mathcal{S}$ , so the same technical help will not be available if we just naively replace this assumption with, say, the finiteness of  $\text{findim}(\mathcal{S}, G)$ . The reader is invited to look at the proofs of [6, Lemma 3.9 and Theorem 3.11] from the current (i.e. second) version of the preprint or the proof of Theorem 5.1.(2) from the first version of the same preprint.

### 1.4.1 On the number of bounded $t$ -structures upto equivalence

Although Theorem 1.4.4 is about an abstract categorical obstruction to the existence of bounded  $t$ -structures in the presence of a finiteness condition, to appreciate how one can use this result in examples, it is important to realize first of all when said finiteness condition is satisfied and when it is not. We postpone this discussion till Section 1.6.

We turn our attention to triangulated categories where although the existence of a bounded  $t$ -structure is never a surprise, it could still be worthwhile to figure out how many bounded  $t$ -structures there can be (up to equivalence). An easy example of such categories would be bounded derived categories of abelian categories. Let us take  $D^b(R\text{-mod})$  where  $R$  is a commutative noetherian ring (usually we have stuck to putting the properties on the left side, like “left coherent”, but to help the reader with their visualisation, we are taking  $R$  to be a genuine commutative noetherian ring). And say we want to put a standard finiteness condition on  $R$ , so we can ask for it to have finite Krull dimension or finite small  $\text{findim}$  (note that in this case,  $\text{findim}(R) \leq$  the Krull dimension of  $R$ ). Now, of course,  $D^b(R\text{-mod})$  has the standard  $t$ -structure which is bounded. What can we say about the other bounded  $t$ -structures? There is no result in the existing literature that says whether all bounded  $t$ -structures on  $D^b(R\text{-mod})$  are equivalent to the standard  $t$ -structure, and as the reader can expect, this is precisely the kind of question our next “abstract” result (Theorem 1.4.8) will answer explicitly in the positive (Corollary 1.5.7).

**The rough idea.** As we saw in Example 1.3.6.(ii)-(iii), many naturally occurring derived bounded categories can be seen as the completion of the compacts under good

metrics generated by the classical generators of the compacts. So when we are trying to establish a property for such bounded derived categories, an adequate abstractification would be to prove our results for such completions using the language of completion theory if need be. It turns out we can actually do a bit better by doing it for all triangulated categories lying between the starting category (often the compacts of a big category with a single compact generator, as we have seen) and their completion. The result below is a manifestation of this philosophy as it shows that the aisle of any bounded  $t$ -structure (in fact, we only need “bounded above”) on any intermediate triangulated category lying between  $\mathcal{S}$  and  $\mathfrak{S}_G(\mathcal{S})$  can be recovered by first restricting it to  $\mathcal{S}$  and then lifting it along  $\mathfrak{S}_G(-)$ .

**Theorem 1.4.7.** [6, Lemma 3.9.(1) and Lemma 3.9.(3)] *Fix an object  $G \in \mathcal{S}$ . Take  $\mathcal{X}$  to be a triangulated subcategory of  $\mathfrak{S}_G(\mathcal{S})$  that contains  $\mathcal{S}$ , and let  $(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0})$  be a bounded above  $t$ -structure on  $\mathcal{X}$ . Then,*

- i.  $\mathcal{X}^{\leq 0} = \mathcal{X} \cap \mathfrak{S}_G(\mathcal{S} \cap \mathcal{X}^{\leq 0})$ .
- ii. *If  $(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0})$  is bounded and  $\text{findim}(\mathcal{S}^{\text{op}}, G)$  is finite, then there exist nonnegative integers  $m, n$  such that  $\langle G \rangle^{(-\infty, -m]} \subseteq \mathcal{S} \cap \mathcal{X}^{\leq 0} \subseteq \langle G \rangle^{(-\infty, n]}$ .*

Armed with Theorem 1.4.7, we can prove the following:

**Theorem 1.4.8.** *Fix an object  $G \in \mathcal{S}$  such that  $\text{findim}(\mathcal{S}^{\text{op}}, G)$  is finite. Any triangulated subcategory  $\mathcal{X}$  of  $\mathfrak{S}_G(\mathcal{S})$  that contains  $\mathcal{S}$  has only one equivalence class of bounded  $t$ -structures - this is covered in [6, Theorem 3.11.(3)].*

**Warning 1.4.9.** *To avoid confusion, please note that there is no guarantee that any such  $\mathcal{X}$  (as in, any  $\mathcal{X}$  satisfying the hypothesis of Theorem 1.4.8) should have a bounded  $t$ -structure at all. When there are zero bounded  $t$ -structures, we are still saying that there is only one equivalence class.*

*Proof of Theorem 1.4.8.* Take two bounded  $t$ -structures on  $\mathcal{X}$ , namely  $(\mathcal{X}_1^{\leq 0}, \mathcal{X}_1^{\geq 0})$  and  $(\mathcal{X}_2^{\leq 0}, \mathcal{X}_2^{\geq 0})$ . Let the  $(m, n)$  pair promised by Theorem 1.4.7.(ii) for  $(\mathcal{X}_1^{\leq 0}, \mathcal{X}_1^{\geq 0})$  be  $(m_1, n_1)$  and the one for  $(\mathcal{X}_2^{\leq 0}, \mathcal{X}_2^{\geq 0})$  be  $(m_2, n_2)$ . So,

- a.  $\langle G \rangle^{(-\infty, -m_1]} \subseteq \mathcal{S} \cap \mathcal{X}_1^{\leq 0} \subseteq \langle G \rangle^{(-\infty, n_1]}$ .
- b.  $\langle G \rangle^{(-\infty, -m_2]} \subseteq \mathcal{S} \cap \mathcal{X}_2^{\leq 0} \subseteq \langle G \rangle^{(-\infty, n_2]}$ .

It is not hard to see (in fact, one has to just take adequate shifts) that (a) and (b) together imply that  $\mathcal{S} \cap \mathcal{X}_2^{\leq -(n_1+m_2)} \subseteq \mathcal{S} \cap \mathcal{X}_1^{\leq 0} \subseteq \mathcal{S} \cap \mathcal{X}_2^{\leq -(n_2+m_1)}$ . From this, it follows that if we denote  $k$  to be the maximum of  $m_1 + n_2$  and  $m_2 + n_1$ , then

$$\mathcal{S} \cap \mathcal{X}_1^{\leq -k} \subseteq \mathcal{S} \cap \mathcal{X}_2^{\leq 0} \subseteq \mathcal{S} \cap \mathcal{X}_1^{\leq k}.$$

We can also write this as

$$(\mathcal{S} \cap \mathcal{X}_1^{\leq 0})[k] \subseteq \mathcal{S} \cap \mathcal{X}_2^{\leq 0} \subseteq (\mathcal{S} \cap \mathcal{X}_1^{\leq 0})[-k].$$

The above chain in turn allows us to deduce that  $\Sigma^k(\mathfrak{S}_G(\mathcal{S} \cap \mathcal{X}_1^{\leq 0})) \subseteq \mathfrak{S}_G(\mathcal{S} \cap \mathcal{X}_2^{\leq 0}) \subseteq \Sigma^{-k}(\mathfrak{S}_G(\mathcal{S} \cap \mathcal{X}_1^{\leq 0}))$ . Now let us intersect everywhere with  $\mathcal{X}$  and note that as  $\mathcal{X}$  is a triangulated subcategory of  $\mathfrak{S}_G(\mathcal{S})$ ,  $\mathcal{X}$  can go inside the shifts in the first and the last terms in the chain. So, we now get:

$$\Sigma^k(\mathcal{X} \cap \mathfrak{S}_G(\mathcal{S} \cap \mathcal{X}_1^{\leq 0})) \subseteq \mathcal{X} \cap \mathfrak{S}_G(\mathcal{S} \cap \mathcal{X}_2^{\leq 0}) \subseteq \Sigma^{-k}(\mathcal{X} \cap \mathfrak{S}_G(\mathcal{S} \cap \mathcal{X}_1^{\leq 0})).$$

Finally, we directly use Theorem 1.4.7.(i) to deduce from the above chain that  $\mathcal{X}_1^{\leq -k} \subseteq \mathcal{X}_2^{\leq 0} \subseteq \mathcal{X}_1^{\leq k}$ , and we are done.  $\blacksquare$

### 1.4.2 Sending $t$ -structures from the compacts to the preferred equivalence class upstairs

In Neeman's proof of the Antieau-Gepner-Heller conjecture that we have mentioned before, he takes a bounded  $t$ -structure on  $\text{Perf}(X)$  ( $X$  is a finite dimensional noetherian scheme), and generates with its aisle (in the sense of Theorem 1.2.11) a  $t$ -structure in the preferred equivalence class (this is exactly where the fact that  $X$  is finite dimensional becomes useful) of  $t$ -structures on  $\text{D}_{\text{qc}}(X)$ , and after this, the rest of the proof still requires some more work. Although as remarked before during the discussion of Example 1.3.6.(iv), one can apply our Theorem 1.4.4 to any essentially small triangulated category, it is still natural to wonder if one can get a statement like the starting bounded  $t$ -structure on the compacts of a triangulated category  $\mathcal{T}$  with a single compact generator  $G$  generating a  $t$ -structure in the preferred equivalence class of  $t$ -structures in  $\mathcal{T}$  in the presence of the required finiteness condition which, as per Theorem 1.4.4, should be the finiteness of  $\text{findim}((\mathcal{T}^c)^{\text{op}}, G)$ . The answer is yes.

**Theorem 1.4.10.** [6, Corollary 3.14] *Let  $\mathcal{T}$  be a triangulated category with a single compact generator  $G$ . Assume that  $\mathcal{T}^c$  has a bounded  $t$ -structure  $((\mathcal{T}^c)^{\leq 0}, (\mathcal{T}^c)^{\geq 0})$ , and also that  $\text{findim}((\mathcal{T}^c)^{\text{op}}, G)$  is finite.*

*Then, the  $t$ -structure generated on  $\mathcal{T}$  generated by the aisle of this  $t$ -structure, i.e. the  $t$ -structure given by  $(\text{Coproduct}((\mathcal{T}^c)^{\leq 0}), (\text{Coproduct}((\mathcal{T}^c)^{\leq 0})[1])^\perp)$ , is in the preferred equivalence class of  $t$ -structures on  $\mathcal{T}$ .*

For simplicity, let us assume that we are in the setup where we complete  $\mathcal{T}^c$ , the compacts in a triangulated category  $\mathcal{T}$  with a single compact generator  $G$ , with a  $G$ -good metric, to get  $\mathcal{T}_c^b$  (Example 1.3.6.(i) told us that for this exact end-result, we need a Hom vanishing condition, viz.  $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$  for  $i \gg 0$ , so we are assuming this condition as well only so that we can just keep writing  $\mathcal{T}_c^b$  instead of  $\mathfrak{S}_G(\mathcal{T}^c)$ ). As we have demonstrated already in the discussion preceding Theorem 1.4.8, in practical cases,  $\mathcal{T}_c^b$  often has an in-built (bounded)  $t$ -structure (see also the short discussion on coherent triangulated categories at the end of Section 1.7). So, one can ask if the achievement of Theorem 1.4.10 can be repeated with the starting bounded  $t$ -structure placed on  $\mathcal{T}_c^b$  instead of  $\mathcal{T}^c$ . The answer is again yes under a mild condition.

**Theorem 1.4.11.** [6, Theorem 3.13] *Assume that  $\mathcal{T}^c \subseteq \mathcal{T}_c^b$ . Take  $((\mathcal{T}_c^b)^{\leq 0}, (\mathcal{T}_c^b)^{\geq 0})$  to be a bounded t-structure on  $\mathcal{T}_c^b$ . Then,*

- i. *There exists a shift-closed full subcategory of  $\mathcal{T}^c$ ,  $\mathcal{A}$ , such that  $\text{Coproduct}((\mathcal{T}_c^b)^{\leq 0}) = \text{Coproduct}(\mathcal{A})$ . This means, by Theorem 1.2.11,  $(\mathcal{T}_c^b)^{\leq 0}$  generates a (compactly generated) t-structure on  $\mathcal{T}$ .*
- ii. *The compactly generated t-structure mentioned above is in the preferred equivalence class of t-structures on  $\mathcal{T}$  if either  $\text{findim}((\mathcal{T}^c)^{\text{op}}, G)$  is finite or  $\mathcal{T}$  admits a “big strong generator” from  $\mathcal{T}_c^b$ .*

**Warning 1.4.12.** *The statement  $\mathcal{T}^c \subseteq \mathcal{T}_c^b$  is actually equivalent to the Hom vanishing condition that  $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$  for  $i \ll 0$ , and although this is easily satisfied in many examples like when  $\mathcal{T}$  is the derived unbounded category of a ring or the derived unbounded category of complexes over a quasicompact quasiseparated scheme with quasicohherent cohomology, there are many examples outside these worlds where it is not satisfied. For example, if  $\mathcal{T}$  is taken to be the homotopy category of (left) module spectra over a connective  $\mathbb{E}_1$ -ring  $R$  with infinitely many nontrivial positive homotopy groups, then  $\text{Hom}_{\mathcal{T}}(R, R[i]) = \pi_{-i}(R)$  will not be trivial for  $i \ll 0$ .*

We owe the reader an explanation of what it means for  $\mathcal{T}$  to admit a “big strong generator” from  $\mathcal{T}_c^b$  as this appeared as one of the sufficient conditions in Theorem 1.4.11.(ii). The reader might be familiar with the notion of (classical) strong generation in small categories (see also Definition 1.5.8 in the next section), what we have here is just reproducing that notion in triangulated categories with arbitrary coproducts - for any class of objects  $C \subseteq \mathcal{T}$ , denote the smallest full subcategory of  $\mathcal{T}$  consisting of all coproducts of objects in  $C$  by  $\text{Coproduct}_1(C)$ , and then for  $n > 1$ , let  $\text{Coproduct}_n(C) := \text{Coproduct}_1(C) * \text{Coproduct}_{n-1}(C)$ . Under these notations, we say  $\mathcal{T}$  has a “big strong generator” (this term is not used in the literature usually; we are just using it in this survey for the ease of explanation) from  $\mathcal{T}_c^b$  if there exists an object  $H \in \mathcal{T}_c^b$  such that  $\mathcal{T} = \text{Coproduct}_n(\{H[i] : i \in \mathbb{Z}\})$  for some positive integer  $n$ . “Big strong generators” played a crucial role in the work central to the papers [5, 18] - in the next section, we will briefly discuss how this work and subsequent related theorems are relevant to our discussion.

## 1.5 Applications of the main results to algebraic geometry

We start by briefly recalling the discussion that we ended Section 1.3 with. Let  $X$  be a quasicompact quasiseparated scheme (not necessarily noetherian) and let  $Z$  be a closed subset of  $X$  with quasicompact complement. If one looks at  $\mathcal{T} = \text{D}_{\text{qc}, Z}(X)$  (the full triangulated subcategory of  $\text{D}_{\text{qc}}(X)$  only consisting of those complexes whose cohomology is supported on  $Z$ ), then  $\mathcal{T}$  has a single compact generator, which we can

denote again by  $G$ ,  $\mathcal{T}^c = \text{Perf}_Z(X)$ , and  $\mathfrak{S}_G(\mathcal{T}^c) = \mathcal{T}_c^b =$  derived category of pseudo-coherent complexes with bounded cohomology supported on  $Z$ , denoted  $D_{\text{qc},Z}^{p,b}(X)$ . If we make our lives simpler by assuming  $X$  to be noetherian (in this case, the complement of  $Z$  in  $X$  being quasicompact is automatically satisfied), then  $\mathcal{T}_c^b = D_{\text{coh},Z}^b(X)$ .

Now, as we get ready to apply our Theorem 1.4.4 in this setup, we first need the following result.

**Proposition 1.5.1.** [6, Proposition 4.13] *Let  $X$  be a quasicompact quasiseparated scheme that admits a finite open cover by  $\text{Spec}(R_i)$ 's where each commutative ring  $R_i$  has finite findim. Then, for any closed subset  $Z \subseteq X$  with quasicompact complement, both  $\text{Perf}_Z(X)$  and  $(\text{Perf}_Z(X))^{\text{op}}$  have finite findim at their classical generator in the sense of Definition 1.2.3.*

Using Proposition 1.5.1, we now get the following theorem as a special case of Theorem 1.4.4 which, as we have recalled quite a few times already, was the central result at the heart of Neeman's considerable work [19].

**Theorem 1.5.2.** *Let  $X$  be a noetherian scheme of finite dimension, and let  $Z \subseteq X$  be a closed subset. Then,  $\text{Perf}_Z(X)$  has a bounded  $t$ -structure if and only if  $\text{Perf}_Z(X) = D_{\text{coh},Z}^b(X)$ , i.e. if and only if  $Z$  is contained in the regularity locus of  $X$ .*

The careful reader will have noticed that Theorem 1.4.4 actually implies a stronger statement:

**Theorem 1.5.3.** *Let  $X$  be a quasicompact quasiseparated scheme (not necessarily noetherian), and let  $Z \subseteq X$  be a closed subset with quasicompact complement. Assume that  $X$  admits a finite open cover by open affines with each affine corresponding to a commutative ring of finite (small) finitistic dimension. Then,  $\text{Perf}_Z(X)$  has a bounded  $t$ -structure if and only if  $\text{Perf}_Z(X) = D_{\text{qc},Z}^{p,b}(X)$ .*

There are 2 obvious questions that the mind can now wander to:

- Question 1.5.4.** i. *Do we really need a finiteness condition on the scheme  $X$ ? More particularly, are there noetherian schemes  $X$  that are not regular where  $\text{Perf}(X)$  has a bounded  $t$ -structure?*
- ii. *We have seen that the finiteness condition that one places on the scheme gets reflected in the finiteness of the findim of the compacts. Should  $D_{\text{coh}}^b(X)$  also have finite findim?*

The answer to Question 1.5.4(i) is still not known, but we do have an affine regular example where the finite findim condition is not satisfied, and although the starting category is still equal to its completion computed with the good metric generated by its classical generator (due to regularity), there are at least 2 equivalence classes of bounded  $t$ -structures - take  $R = k[x_0, x_1, \dots]$  to be the polynomial ring over a field  $k$

of countably many (but infinite) variables; this is a coherent regular ring of infinite finitistic dimension, so the perfect complexes and the bounded derived category of finitely presented modules (which has a standard bounded  $t$ -structure) coincide but the standard  $t$ -structure and its derived dual, which is also a bounded  $t$ -structure on the same category, are not equivalent - we discussed this in detail in [6, Proposition 4.18]. This shows that at least in Theorem 1.4.10, the finiteness condition on the compacts cannot be dropped.

The answer to Question 1.5.4(ii) is “no”, and again one can find affine examples. It is known that for any commutative noetherian ring  $R$ ,  $D^b(R\text{-mod})$  having a classical generator implies that the regularity locus of  $R$  is open in  $\text{Spec}(R)$ . As there are noetherian local rings (such rings always have finite Krull dimension, so their small finitistic dimension is always finite) with non-open regularity locus, Warning 1.4.5 tells us that in such cases, despite the perfects (and their opposite category) having finite finitistic dimension in the sense of Definition 1.2.3 at their classical generators, the bounded derived category cannot have finite findim at any of its objects.

So, if  $\mathcal{S} = D^b(R\text{-mod})$  for  $R$  a noetherian local ring with non-open regularity locus, then although  $\mathcal{S}$  has a bounded  $t$ -structure, the hypothesis, in Theorem 1.4.4, of  $\text{findim}(\mathcal{S}^{\text{op}}, G)$  being finite for some  $G \in \mathcal{S}$  cannot be satisfied. In this case, we can still construct  $G$ -good metrics  $\{\langle G \rangle^{(-\infty, -n]}\}_{n \in \mathbb{N}}$  for arbitrary complexes  $G \in D^b(R\text{-mod})$ , and the completions with respect to them won't necessarily be exactly  $\mathcal{S}$  again. In a way, one can consider this to be an illustration of Theorem 1.4.4 failing without the finiteness assumption, but one can also say that we cheated a little bit here because our category did not have a classical generator here (recall Warning 1.4.5).

**Open question 1.5.5.** *Can there be an essentially small triangulated category  $\mathcal{S}$  with a single classical generator  $G$  such that  $\mathcal{S}$  has a bounded  $t$ -structure but  $\mathfrak{S}_G(\mathcal{S}) \neq \mathcal{S}$ ? If not, is there a generalised version of Theorem 1.4.4 without the findim finite hypothesis?*

**Applications of Theorem 1.4.8.** Based on the discussion that started this section, we can see that Theorem 1.4.8 directly implies the following:

**Theorem 1.5.6.** *Let  $X$  be a quasicompact quasiseparated scheme that admits a finite open affine cover by  $\text{Spec}(R_i)$ 's where each commutative ring  $R_i$  has finite findim. Then, for any closed subset  $Z \subseteq X$  with quasicompact complement, any triangulated category sandwiched between  $\text{Perf}_Z(X)$  and  $D_{\text{qc}, Z}^{p, b}(X)$  has only one equivalence class of bounded  $t$ -structures.*

**Corollary 1.5.7.** *(an immediate consequence for affines) Let  $R$  be a commutative noetherian ring with finite Krull dimension or small finitistic dimension. Then, all bounded  $t$ -structures on  $D^b(R\text{-mod})$  (and on any triangulated category lying between the perfects and this category) are equivalent.*

For notational simplicity, if we again assume  $X$  to be finite dimensional and noetherian, we see that just for  $D_{\text{coh},Z}^b(X)$ , one could use only Theorem 1.4.11.(ii) to deduce that all bounded  $t$ -structures are equivalent. Neeman had visited this question himself in his paper [19], and he showed this result in precisely three cases - (i) when  $Z$  is contained in the regularity locus of  $X$  (as discussed before, this corresponds to the “ $\mathcal{T}^c = \mathcal{T}_c^b$ ” scenario), (ii)  $D_{\text{coh},Z}^b(X)$  has a dualizing complex, and (iii) when  $Z = X$  and  $X$  is separated and quasiexcellent (in this case, based on the work of [5] which builds on the work of [18],  $D_{\text{qc}}(X)$  admits a “big strong generator” from  $D_{\text{coh}}^b(X)$  - this is the only place where “big strong generators” become relevant in this survey). So, Theorem 1.4.11, and especially Theorem 1.4.8, help us obtain, in the avatar of Theorem 1.5.6, substantial improvements on the best results on these questions that previously existed. In fact, the way Neeman had shown in the three cases mentioned above that all bounded  $t$ -structures on  $D_{\text{coh},Z}^b(X)$  were equivalent was by showing that they all generated a  $t$ -structure in the preferred equivalence class upstairs (i.e. in  $D_{\text{qc},Z}(X)$ ); so his method was in line with what Theorem 1.4.11.(ii) achieves and completely different from Theorem 1.4.8. To appreciate the extra room that Theorem 1.4.8 provides, one needs to investigate, assuming  $\text{Perf}_Z(X) \neq D_{\text{coh},Z}^b(X)$ , whether there can be or are triangulated categories sandwiched strictly between  $\text{Perf}_Z(X)$  and  $D_{\text{coh},Z}^b(X)$  that can have a bounded  $t$ -structure. We have recorded this as an open question in the next section (see Open Question 1.6.10).

Although, as mentioned above, when  $D_{\text{qc}}(X)$  has a “big strong generator” from  $D_{\text{coh}}^b(X)$ , Neeman has already shown that all bounded  $t$ -structures on  $D_{\text{coh}}^b(X)$  are equivalent, his proof does not reach the same conclusion when the only available information regarding strong generation of  $D_{\text{coh}}^b(X)$  is that  $D_{\text{coh}}^b(X)$  has a strong generator in the classical sense. Let us recall what classical strong generation means.

**Definition 1.5.8.** (standard) *Let  $\mathcal{S}$  be an essentially small triangulated category. For any full subcategory  $\mathcal{A} \subseteq \mathcal{S}$ , let  $\text{coprod}_1(\mathcal{A}) :=$  the smallest full subcategory of  $\mathcal{S}$  containing finite coproducts of objects of  $\mathcal{A}$ , and for  $n > 1$ ,  $\text{coprod}_n(\mathcal{A}) := \text{coprod}_1(\mathcal{A}) * \text{coprod}_{n-1}(\mathcal{A})$ . We say  $\mathcal{S}$  has a strong generator if there exists a positive integer  $n$  and an object  $G$  such that the smallest full subcategory of  $\mathcal{S}$  containing  $\text{coprod}_n(\{G[i] : i \in \mathbb{Z}\})$  and closed under all direct summands is all of  $\mathcal{S}$ . When this happens,  $G$  is called a strong generator of  $\mathcal{S}$ .*

There is an interesting connection between an essentially small  $\mathcal{S}$  admitting a strong generator and its findim in the sense of Definition 1.2.3.

**Proposition 1.5.9.** [6, Proposition 4.5] *Let  $\mathcal{S}$  be an essentially small triangulated category with a strong generator  $G$ . Assume additionally that  $\text{Hom}_{\mathcal{S}}(G, G[i]) = 0$  for  $i \ll 0$ . Then,  $\text{findim}(\mathcal{S}, G)$  is finite.*

The above result gives us the following immediate corollary that answers the point raised in the discussion preceding Definition 1.5.8, and again, we are sticking to noetherian schemes for notational simplicity.

**Corollary 1.5.10.** *Let  $X$  be a noetherian scheme of finite dimension such that  $D_{\text{coh}}^b(X)$  has a strong generator. Then, all bounded  $t$ -structures on it are equivalent.*

*Proof.* Note that a category admitting a strong generator implies that its opposite category admits one as well. Take  $\mathcal{S} = D_{\text{coh}}^b(X)$  in Proposition 1.5.9, and we are done by Theorem 1.4.8. ■

As the reader will have noticed, usually we take the starting category to be something like  $\text{Perf}(X)$  and realise  $D_{\text{coh}}^b(X)$  as the completion, but in the above proof, we are directly applying Proposition 1.5.9 to  $D_{\text{coh}}^b(X)$ .

## 1.6 Applications of the main results to representation theory

### 1.6.1 Statements equivalent to an algebra having finite global dimension

Let us start with the following fact which is by now quite classical and standard.

**Fact 1.6.1.** [2, Example 2.5.(a)] *For any artin algebra  $A$ ,  $D^b(A\text{-mod})$  admits a silting object iff  $A$  is of finite global dimension.*

First, we apply Theorem 1.4.4 to the case where  $\mathcal{S} = K^b(A\text{-proj})$ , the category of perfect complexes over an artin algebra  $A$ , with  $G =$  the regular module  $A$  as a left module over itself. Recall from the discussion following Definition 1.2.3 that  $\text{findim}(K^b(A\text{-proj}), A) = \text{findim}(A)$ .

**Theorem 1.6.2.** *Let  $A$  be an artin algebra such that  $\text{findim}(A^{\text{op}})$  is finite. Then,  $A$  is of finite global dimension iff  $K^b(A\text{-proj})$  has a bounded  $t$ -structure.*

*Proof.* A direct application of Theorem 1.4.4 says assuming  $\text{findim}(A^{\text{op}}) < \infty$ , if  $K^b(A\text{-proj})$  has a bounded  $t$ -structure, then  $K^b(A\text{-proj}) = D^b(A\text{-mod})$  which in turn is equivalent to  $A$  having finite global dimension. Of course, when  $K^b(A\text{-proj}) = D^b(A\text{-mod})$ ,  $K^b(A\text{-proj})$  has the standard  $t$ -structure which is bounded, so we are done. ■

The above result can be seen as a characterisation of when the perfect complexes can admit a bounded  $t$ -structure, although it does not quite connect the perfect complexes to the satisfiability of the findim conjecture. However, because of this conjecture, we should want to remove the  $\text{findim}(A^{\text{op}}) < \infty$  hypothesis. There was a result due to Adachi-Mizuno-Yang that had almost solved this problem years ago, but it turns

out that instead of their result achieving more than Theorem 1.6.2, our result actually improves/strengthens their result. Before stating their result, we need to state the definition of algebraic (bounded)  $t$ -structures.

**Definition 1.6.3.** *An algebraic  $t$ -structure on an essentially small triangulated category  $\mathcal{S}$  is a bounded  $t$ -structure with a heart that is a length category with only finitely many isomorphism classes of simple objects in this heart.*

For some classes of algebras  $A$ , all bounded  $t$ -structures on  $D^b(A\text{-mod})$  are algebraic (see [1, Example 3.21]), but that is still for the bounded derived category. In general, clearly not all bounded  $t$ -structures need to be algebraic. However, a category admitting an algebraic  $t$ -structure implies a very neat (and somewhat expected) conclusion about its findim.

**Proposition 1.6.4.** [6, Lemma 4.1.(3)] *Any triangulated category that admits an algebraic  $t$ -structure has finite finitistic dimension. In particular, if a triangulated category  $\mathcal{S}$  has an algebraic  $t$ -structure  $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$  and if the sum of (pairwise non-isomorphic) simples in the heart of this  $t$ -structure is denoted  $G$ , then  $\text{findim}(\mathcal{S}, G) = \text{findim}(\mathcal{S}^{\text{op}}, G) = 0$ .*

Adachi-Mizuno-Yang proved the following:

**Theorem 1.6.5.** [1, Proposition 4.12] *For any finite-dimensional algebra  $A$  over a field,  $K^b(A\text{-proj})$  has an algebraic  $t$ -structure  $\iff A$  has finite global dimension.*

Clearly, Proposition 1.6.4 tells us that {algebras where the perfects have an algebraic  $t$ -structure}  $\subseteq$  {algebras whose opposite algebra have finite small finitistic dimension}. So, Theorem 1.6.2 is a generalisation of or an improvement on Proposition 1.6.5, but for a moment, let us not look at it this way. Let us instead appreciate how we now have two completely different methods that are not related even a little bit for proving Theorem 1.6.5.

**Proof of Theorem 1.6.5 using the new method.** As mentioned, this is just the combination of Theorem 1.6.2 and Proposition 1.6.4. So, the main non-trivial ingredient is Theorem 1.4.4. ■

**Adachi-Mizuno-Yang's proof of Theorem 1.6.5.** Only the " $\implies$ " direction needs proving. Take an algebraic  $t$ -structure on  $K^b(A\text{-proj})$ . Write  $G$  for the direct sum of pairwise non-isomorphic simple objects in its heart. Then, this  $G$  is a single compact generator of  $D(A\text{-Mod})$ , and a well-known result of Keller [12, Lemma 6.1] now tells us that  $\mathbf{R}\text{Hom}_A(G, -) : D(A\text{-Mod}) \longrightarrow D(\mathbf{R}\text{Hom}_A(G, G))$  is a triangle equivalence which restricts to a triangle equivalence between  $D^b(A\text{-mod})$  and the finite-dimensional derived category of dg  $\mathbf{R}\text{Hom}_A(G, G)$ -modules, i.e. the full subcat-

egory of  $D(\mathbf{R}\mathrm{Hom}_A(G, G))$  consisting of those dg-modules whose total cohomology is finite dimensional over the base; this latter category has a silting object namely  $\mathbf{R}\mathrm{Hom}_A(G, G)$ , which means that  $D^b(A\text{-mod})$  has a silting object, and therefore  $A$  has finite global dimension by Fact 1.6.1.

So here, the main nontrivial ingredient comes from Keller's work and the observation about the existence of the silting object in the right category. ■

Now, let us see how many statements we can make equivalent to an algebra having finite global dimension. The result below is essentially a collection of existing results plus some new observations based out of our new results, but we are still recording it as a proposition for the ease of exposition.

**Proposition 1.6.6.** *For any artin algebra  $A$ , the following statements are equivalent.*

- i.  $A$  has finite global dimension.
- ii.  $\mathbf{K}^b(A\text{-proj})$  has a  $t$ -structure whose aisle generates, on  $D(A\text{-Mod})$ , a  $t$ -structure that is equivalent to the standard  $t$ -structure of  $D(A\text{-Mod})$ .
- iii. The standard  $t$ -structure of  $D(A\text{-Mod})$  restricts to a  $t$ -structure on  $\mathbf{K}^b(A\text{-proj})$ .
- iv.  $\mathbf{K}^b(A\text{-proj})$  has an algebraic  $t$ -structure.
- v.  $D^b(A\text{-mod})$  has a silting object.

*Proof.* We have already established that  $(i) \iff (iv) \iff (v)$ .

The  $(i) \implies (iii)$  direction is trivial. The  $(iii) \implies (i)$  requires some work, but it has been done before - Keller and Nicolás showed (see [13, Non-example 7.8]) that the standard  $t$ -structure restricts to the perfects only if  $A$  has finite dimension.

To see that  $(i) \implies (ii)$ , note that when  $A$  has finite global dimension, we have the standard bounded  $t$ -structure on the bounded derived category (same as the perfects in this case), and the aisle of this  $t$ -structure literally generates the standard  $t$ -structure on  $D(A\text{-Mod})$ .

So now, we are left with showing  $(ii) \implies (i)$ . This part is new but can be deduced from a careful reading of [19, Proof that Theorem 0.1 follows from Lemma 3.4]. Before we start the proof, note that  $(ii) \implies (i)$  is actually a generalisation or a strengthening of  $(iii) \implies (i)$ .

Now, for the proof, let  $(D(A\text{-Mod})^{\leq 0}, D(A\text{-Mod})^{\geq 0})$  be the standard  $t$ -structure on  $D(A\text{-Mod})$ . Let  $(\mathcal{U}, \mathcal{V})$  be a  $t$ -structure on  $\mathbf{K}^b(A\text{-proj})$  as guaranteed by the hypothesis of  $(ii)$ . Recall from Theorem 1.2.11 that this means that as  $t$ -structures on  $D(A\text{-Mod})$ ,  $(\mathrm{Coprod}(\mathcal{U}), (\mathrm{Coprod}(\mathcal{U})[1])^\perp) \simeq (D(A\text{-Mod})^{\leq 0}, D(A\text{-Mod})^{\geq 0})$ . There must be an integer  $n$  such that

$$D(A\text{-Mod})^{\leq -n} \subseteq \mathrm{Coprod}(\mathcal{U}) \subseteq D(A\text{-Mod})^{\leq n}.$$

Now, take an object  $F \in \mathbf{D}^b(A\text{-mod})$  and assume without loss of generality that  $F \in \mathbf{D}(A\text{-Mod})^{\geq 0}$ . The standard  $t$ -structure is in the preferred equivalence class here as recalled in the discussion following Definition 1.2.13, and we know that with  $\mathcal{T} = \mathbf{D}(A\text{-Mod})$ ,  $\mathcal{T}_c^- = \mathbf{K}^-(A\text{-proj}) = (\text{in this case}) \mathbf{D}^-(A\text{-mod})$  (see Example 1.3.6.(ii)).  $F \in \mathcal{T}_c^-$  and by the definition of  $\mathcal{T}_c^-$ , we get that for any integer  $M$ , there is a distinguished triangle  $D \rightarrow E \rightarrow F$  with  $E$  perfect and  $D \in \mathbf{D}(A\text{-Mod})^{\leq -M}$ .

Put  $M = 1 + 2n$ . Now,

- a.  $\mathbf{D}(A\text{-Mod})^{\leq -n} \subseteq \text{Coprod}(\mathcal{U}) \implies \mathbf{D}(A\text{-Mod})^{\leq -1-2n} \subseteq (\text{Coprod}(\mathcal{U}))[1+n]$ ,
- b.  $\text{Coprod}(\mathcal{U}) \subseteq \mathbf{D}(A\text{-Mod})^{\leq n} \implies \mathbf{D}(A\text{-Mod})^{\geq 0} \subseteq (\text{Coprod}(\mathcal{U}))[1+n]^\perp$ .

We can truncate  $E$  with respect to the  $t$ -structure  $(\mathcal{U}, \mathcal{V})$ , so we have a distinguished triangle  $E^{\leq -1-n} \rightarrow E \rightarrow E^{\geq -n}$  where  $E^{\leq -1-n} \in \mathcal{U}[1+n]$  and  $E^{\geq -n} \in \mathcal{V}[n]$ . Clearly,  $E^{\leq -1-n} \in (\text{Coprod}(\mathcal{U}))[1+n]$ , and  $E^{\geq -n} \in ((\text{Coprod}(\mathcal{U}))[1+n])^\perp$ . So, both  $D \rightarrow E \rightarrow F$  and  $E^{\leq -1-n} \rightarrow E \rightarrow E^{\geq -n}$  are truncation triangles for  $E$  with respect to the same  $t$ -structure on  $\mathbf{D}(A\text{-Mod})$ , which implies that  $F \simeq E^{\geq -n}$  and hence perfect, and we are done.  $\blacksquare$

There are many more statements equivalent to  $A$  having finite global dimension including some very interesting ones like  $\mathbf{D}^b(A\text{-mod})$  having a Serre functor, and several others - we would like to invite the reader to consult [15, Theorem 6.4.13] (and also [1, Proposition 4.12]). We end this subsection with the following open question that anyone who believes in the finitistic dimension conjecture should expect a positive answer to based on Theorem 1.6.2 and Proposition 1.6.6.

**Open question 1.6.7.** *For a fixed artin algebra  $A$ , are the following two statements equivalent?*

- i.  $\mathbf{K}^b(A\text{-proj})$  has a bounded  $t$ -structure.
- ii.  $\mathbf{K}^b(A\text{-proj})$  has an algebraic  $t$ -structure.

## 1.6.2 Equivalence of bounded $t$ -structures when they exist

Although the concept of equivalence of two  $t$ -structures had not been introduced when the Adachi-Mizuno-Yang paper [1] was written, they proved a result that can be neatly rephrased in this language.

**Proposition 1.6.8.** *(follows from [1, Lemma 3.22]) On any essentially small triangulated category  $\mathcal{S}$ , all bounded  $t$ -structures are equivalent if  $\mathcal{S}$  admits an algebraic  $t$ -structure.*

As the standard  $t$ -structure on  $\mathbf{D}^b(A\text{-mod})$ , for any artin algebra  $A$ , is an algebraic  $t$ -structure, we get the following immediate corollary.

**Corollary 1.6.9.** *For any artin algebra  $A$ , all bounded  $t$ -structures on  $D^b(A\text{-mod})$  are equivalent.*

Corollary 1.6.9 can also be proved without using the language of algebraic  $t$ -structures - it is implied by Proposition 1.5.9 and Theorem 1.4.8 as  $D^b(A\text{-mod})$  has a strong generator.

As mentioned in the discussion following Theorem 1.5.6, it will be worthwhile to find triangulated categories between  $K^b(A\text{-proj})$  and  $D^b(A\text{-mod})$  that can have a bounded  $t$ -structure because only then will we be using Theorem 1.4.8 to its full extent. Derived bounded categories of many resolving subcategories in the sense of Auslander-Reiten do lie between  $K^b(A\text{-proj})$  and  $D^b(A\text{-mod})$ , but recall from Warning 1.2.8 that derived bounded categories of exact categories do not come with an in-built bounded  $t$ -structure. If one takes the category of finitely generated  $A$ -Gorenstein projectives,  $A\text{-gproj}$ , then there is actually some existing work on when  $A\text{-gproj}$  can be an abelian category ([14, Section 2]) but it turns out, according to this work, that when  $A\text{-gproj}$  is an abelian category,  $A$  is Gorenstein, so  $D^b(A\text{-gproj})$  is not strictly sandwiched between  $K^b(A\text{-proj})$  and  $D^b(A\text{-mod})$  as  $D^b(A\text{-gproj}) = D^b(A\text{-mod})$ .

**Open question 1.6.10.** *Find examples of triangulated categories that lie strictly between  $K^b(A\text{-proj})$  and  $D^b(A\text{-mod})$  for artin algebras  $A$  (or, analogously, between  $\text{Perf}(X)$  and  $D_{\text{coh}}^b(X)$ , for noetherian schemes  $X$ ) that have bounded  $t$ -structures or can have them under some conditions on  $A$  (or  $X$ ).*

### 1.6.3 A possible use of lifting $t$ -structures

Assume  $K^b(A\text{-proj})$  has a bounded  $t$ -structure, for some artin algebra  $A$ . As noted in the discussion after Theorem 1.6.2, anyone who believes in the finitistic dimension conjecture should expect that  $A$  must have finite global dimension. But Proposition 1.6.6.(i)-(ii) tells us that we will achieve precisely this if we can show, without any extra assumption, that the aisle of this  $t$ -structure generates a  $t$ -structure equivalent to the standard one on  $D(A\text{-Mod})$ . At the same time, we do know, courtesy Corollary 1.6.9, that all bounded  $t$ -structures on  $D^b(A\text{-mod})$  are equivalent without any extra assumptions. Let us interject now with a technical observation that will help further this discussion.

**Observation 1.6.11.** *Let  $\mathcal{T}$  be a triangulated category with a single compact generator  $G$  and assume that  $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$  for  $|i| \gg 0$ . Fix a  $G$ -good metric, and let  $(\mathcal{U}, \mathcal{V})$  be a  $t$ -structure on  $\mathcal{T}^c$  that is extendable with respect to this metric (in this case, it turns out that such a  $t$ -structure has to be bounded above - see [6, Lemma 3.3.(1) and Remark 3.4]). Then,  $\text{Coprod}(\mathcal{U}) = \text{Coprod}(\mathfrak{S}_G(\mathcal{U}))$ .*

**Proof sketch.** We are only sketching this proof because one has to make free use of the language of good extensions in the sense of [21, Notation 0.14]. We will only be using

some good extension facts as a blackbox, the embedding  $\mathcal{T}^c \hookrightarrow \mathcal{T}$  is a good extension, and because of this, instead of dealing with colimits in  $\mathcal{T}^c\text{-Mod}$ , we are allowed to deal with honest homotopy colimits in  $\mathcal{T}$ . From the definition of homotopy colimits in triangulated categories, it now follows that  $\mathfrak{S}_G(\mathcal{U}) \subseteq \text{Coproduct}(\mathcal{U})$ , and therefore  $\text{Coproduct}(\mathfrak{S}_G(\mathcal{U})) \subseteq \text{Coproduct}(\mathcal{U})$ . As  $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$  for  $i \gg 0$ , we have that  $\mathfrak{S}_G(\mathcal{T}^c) = \mathcal{T}_c^b$  by Example 1.3.6. So, Theorem 1.4.2 tells us that  $(\mathfrak{S}_G(\mathcal{U}), \mathfrak{S}_G(\mathcal{V}))$  is a  $t$ -structure on  $\mathcal{T}_c^b$  with an additive equivalence  $\mathfrak{S}_G(\mathcal{V}) \simeq \mathcal{V}$ . So,  $\mathfrak{S}_G(\mathcal{U}) = ({}^{\perp_{\mathcal{T}_c^b}} \mathcal{V})[-1]$ , and  $\mathcal{U} = ({}^{\perp_{\mathcal{T}^c}} \mathcal{V})[-1]$ . As  $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$  for  $i \ll 0$ ,  $\mathcal{T}^c \subseteq \mathcal{T}_c^b$  (see the discussion in Warning 1.4.12), and that implies  $\mathcal{U} \subseteq \mathfrak{S}_G(\mathcal{U})$ . Therefore,  $\text{Coproduct}(\mathcal{U}) \subseteq \text{Coproduct}(\mathfrak{S}_G(\mathcal{U}))$ . ■

Going back to our discussion, Observation 1.6.11 tells us that if we can show that the bounded  $t$ -structure on  $\text{K}^b(A\text{-proj})$  that we started with, when lifted to  $\text{D}^b(A\text{-mod})$  using Theorem 1.4.2, will stay bounded (not just bounded above as promised by Theorem 1.4.2.(iii)), then we will be done. Why is that so? That is because as all bounded  $t$ -structures on  $\text{D}^b(A\text{-mod})$  are equivalent, the  $t$ -structures on  $\text{D}(A\text{-Mod})$  that get generated by their aisles (recall Theorem 1.4.11.(i)) are also all equivalent, but this latter equivalence class should contain the standard  $t$ -structure of  $\text{D}(A\text{-Mod})$  because the  $t$ -structure on  $\text{D}(A\text{-Mod})$  generated by the aisle of the standard bounded  $t$ -structure on  $\text{D}^b(A\text{-mod})$  is nothing but the standard  $t$ -structure.

We leave the following as an open question.

- Open question 1.6.12.** i. *For  $A$  an artin algebra, if  $\text{K}^b(A\text{-proj})$  has a bounded  $t$ -structure, show that the  $t$ -structure lifted to  $\text{D}^b(A\text{-mod})$  under  $\mathfrak{S}_A(-)$  is also bounded.*
- ii. *In the general setting of Observation 1.6.11, characterise the  $t$ -structures on  $\mathcal{T}_c^b$  that can be written as  $(\mathfrak{S}_G(\mathcal{U}), \mathfrak{S}_G(\mathcal{V}))$  for some bounded above  $t$ -structure  $(\mathcal{U}, \mathcal{V})$  on  $\mathcal{T}^c$ .*

#### 1.6.4 The mysterious case of the singularity category

Let us ask the following question.

**Question 1.6.13.** *Does  $\text{D}_{\text{sg}}(A)$ , for any artin algebra  $A$ , always have finite findim at its classical generator?*

For now, we have the following answer.

**Proposition 1.6.14.** [6, Corollary 4.3.(1)] *The answer to Question 1.6.13 is yes if  $A$  is Gorenstein.*

As  $D_{\text{sg}}(A)$  has a strong generator, one might have assumed, based on Proposition 1.5.9, that the answer to Question 1.6.13 would be “yes”. But, as the careful reader will have noticed, for  $G = A/\text{rad}(A)$  which is a classical generator of this category (note that when strong generators exist, all classical generators become strong generators), for Proposition 1.5.9 to be applicable, we need a Hom vanishing condition on  $G$  which we know is not satisfied in this case as long as the singularity category is nontrivial (recall the discussion following Observation 1.2.10). More precisely, denoting  $\mathcal{S} := D_{\text{sg}}(A)$  for some  $A$  of infinite global dimension, we need  $\text{Hom}_{\mathcal{S}}(G, G[i]) = 0$  for  $i \ll 0$  to use Proposition 1.5.9, but in reality,  $\text{Hom}_{\mathcal{S}}(G, G[i]) \neq 0$  for all  $i \in \mathbb{Z}$ .

This is where a very interesting result proved by Jan Stovicek becomes extremely useful.

**Theorem 1.6.15.** *For any artin algebra  $A$ ,  $\text{Findim}(A^{\text{op}}) < \infty \iff$  the closure of  $A/\text{rad}(A)$ , in  $D_{\text{sg}}(A)$ , under only finite extensions and summands is the whole of  $D_{\text{sg}}(A)$ .*

I must add that this result has still not appeared in print. It was proved in 2022 by Stovicek, and around the same time, upon request, he sent me a copy of a part of this paper that was still in progress, containing a fully detailed proof that I duly read carefully and verified. Based on my most recent communication with him about this issue, Theorem 1.6.15 (or rather, a generalisation of it) will appear in a future paper by Jan Stovicek and Souvik Dey. The following corollary is now straightforward.

**Corollary 1.6.16.**  *$\text{Findim}(A^{\text{op}}) < \infty \implies \text{findim}(D_{\text{sg}}(A), A/\text{rad}(A)) = 0$ , for any artin algebra  $A$ .*

So, if one finds an example of  $A$  with a “no” answer for Question 1.6.13, they will have found a counterexample to the big finitistic dimension conjecture.

We have learned from the work of Xiao-Wu Chen and Zhengfang Wang [9, Theorem A] that there is a very interesting triangle equivalence between the singularity category of a finite-dimensional algebra  $kQ/I$  over a field  $k$  and the category of perfect complexes over the dg Leavitt path algebra associated to the quiver without sinks that one obtains from the radical quiver of  $kQ/I$  (see [9, Definition 10.4]) by removing sinks repeatedly. If  $D_{\text{sg}}(kQ/I)$  has finite findim, then the category of these perfect complexes will also have finite findim. For discrete rings, we have seen that the finiteness of the findim of the category of perfect complexes reflects or respects the finiteness of the honest small finitistic dimension of the ring itself. But this intuition, on the face of it, is not of much use here because we do not have a ready notion of findim that can be applied to these dg Leavitt path algebras. Put differently, suppose one works out an adequate notion of small finitistic dimension for dg Leavitt path algebras whose finiteness reflects the finiteness of the findim of the category of perfect complexes over them. Then, if they can find a dg Leavitt path algebra with infinite findim in this sense such that the category of perfect complexes over it is triangle equivalent to  $D_{\text{sg}}(A)$  for

some finite-dimensional algebra  $A$ , then  $A$  will provide a negative answer to Question 1.6.13 and  $A^{\text{op}}$  will have infinite Findim.

## 1.7 Regularity and negative $K$ -theory

**Definition 1.7.1.** *For simplicity, we deal with categories with a single classical generator. We declare an essentially small triangulated category  $\mathcal{S}$  regular if it admits a classical generator  $G$  such that  $\mathfrak{S}_G(\mathcal{S}) = \mathcal{S}$ .*

If  $\mathcal{S} = \text{Perf}(X)$  for a noetherian scheme  $X$ , then clearly,  $\text{Perf}(X)$  is regular as per Definition 1.7.1 iff  $X$  is regular in the classical sense. As recalled in the introduction, Antieau, Gepner, and Heller have shown that if a small stable  $\infty$ -category has a bounded  $t$ -structure, then it has trivial  $(-1)$ -th  $K$ -group, so it follows that when  $\text{Perf}(X)$  is regular, its  $K_{-1}$  is trivial. Given the long history of regularity in various contexts implying the vanishing of either the first negative  $K$ -group, or under extra assumptions, all negative  $K$ -groups, it is natural to ask the following question in our familiar  $\mathcal{T}^c - \mathcal{T}_c^b$  setup. Assume all our triangulated categories are homotopy categories of stable  $\infty$ -categories.

**Open question 1.7.2.** *Let  $\mathcal{T}$  be a triangulated category with a single compact generator  $G$  such that  $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$  for  $i \gg 0$ . We know that in this case,  $\mathfrak{S}_G(\mathcal{T}^c) = \mathcal{T}_c^b$ . If  $\mathcal{T}^c = \mathcal{T}_c^b$ , is  $K_{-1}(\mathcal{T}^c) = 0$ ?*

Note that if the answer is “yes”, then our “categorical obstruction” to bounded  $t$ -structures (for categories whose opposite category has finite findim), i.e. the triviality of the Verdier quotient  $\mathcal{T}_c^b/\mathcal{T}^c$  (recall that whenever there is a bounded  $t$ -structure,  $\mathcal{T}^c \subseteq \mathcal{T}_c^b$ ), will be finer than the Antieau-Gepner-Heller  $K$ -theoretic obstruction.

There is a direct way to see that the answer is also “yes” if  $\mathcal{T}$  is *coherent*. This notion of coherence comes from [21, Definition 5.1], but we present a more workable definition (see [21, Example 5.3]) that says that  $\mathcal{T}$  is coherent if (i)  $\mathcal{T}$  is weakly approximable (we do not intend to introduce this concept in detail, the interesting reader can check [20, Definition 0.25]; just note that all derived unbounded categories that we have discussed in this survey satisfy this property), and (ii) there is a  $t$ -structure in the preferred equivalence class of  $t$ -structures on  $\mathcal{T}$  that restricts to give a  $t$ -structure on  $\mathcal{T}_c^-$  (recall that when  $\mathcal{T} = \text{D}(R\text{-Mod})$ , for  $R$  any ring,  $R$  is left-coherent iff the standard  $t$ -structure of  $\text{D}(R\text{-Mod})$  restricts to  $\mathcal{T}_c^-$  - for the most updated reference on this that also covers connective  $\mathbb{E}_1$ -rings, check [17, Proposition 7.2.4.18] along with [6, Theorem A.5.(1)]). Whenever (ii) is satisfied in this definition of coherence, said  $t$ -structure restricts to  $\mathcal{T}_c^b$  to give a bounded  $t$ -structure there. So, when  $\mathcal{T}$  is coherent,  $\mathcal{T}^c = \mathcal{T}_c^b$  means  $\mathcal{T}^c$  has a bounded  $t$ -structure, and a direct application of the afore-

mentioned Antieau-Gepner-Heller result gives us a positive answer to Open Question 1.7.2.

I currently do not know of any way of approaching this question beyond the coherent case.

## 1.8 Findim as a possible smallness measure

We end on a philosophical point. Let us look at the following question which is admittedly a little vague.

**Question 1.8.1.** *Can finite findim be a detector of smallness in triangulated categories?*

“Smallness” does not mean “compactness” here. What we are asking is - if, for example, we have an essentially small triangulated category that should conjecturally be, say, trivial, should one expect it (or its opposite category) to have finite findim, preferably unconditionally? Many conjectures in representation theory, and in homological algebra in general, can be phrased in terms of the vanishing of some triangulated category. A very neat example in relation to Tachikawa’s second conjecture can be found in [11, Corollary 4.9], and we were able to show in our work [6, Example 4.4] that the relevant essentially small triangulated category (and its op) in this case does have finite findim unconditionally. There should be many more such instances from all over the subject and beyond.

**Final comment, separate from Section 1.8.** There are a few avenues of applications of our main results and related topics that I have decided to not include in this survey. Most notably, in [6, Appendix A], we computed the completions of perfect modules over a connective  $\mathbb{E}_1$ -ring, applied Theorems 1.4.4 and 1.4.8 to this setup, and got many interesting results and connections with existing notions of regularity for  $\mathbb{E}_1$ -rings. In this survey, I have deliberately tried to not bring up  $\mathbb{E}_1$ -rings as much as possible. There were also some applications to dg algebras, and some interesting properties of our “big” Findim for big triangulated categories (that was briefly touched on in Remark 1.2.6) that I have not delved into here. If the reader is interested in learning more about these areas, they will be well advised to read our paper. My decisions regarding not discussing them too much or at all here were made mainly based on the amount of space I wanted to give to the applications in algebraic geometry and representation theory (Sections 1.5 and 1.6). Also, some of those topics are quite far removed from the subjects usually addressed at ICRA, and as this survey corresponds to last year’s ICRA meeting, this too was a genuine factor.

■

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