Cohomological finiteness conditions and stable categories on rational discrete modules over TDLC groups

Rudradip Biswas

1. Objective: We want to associate to the abelian category of rational discrete modules over a TDLC group a "stable category" with good behaviour (i.e. satisfying the equivalences shown in **Theorem C**).

2. A useful homological result: For any TDLC group G, it is known (Proposition 3.2 of [3]) that the rational discrete modules form an abelian category with enough projectives and enough injectives - we will be denoting this category by $\mathfrak{A}_G$. Denote the supremum over all $\mathfrak{A}_G$-objects with finite projective (resp. injective) dimension by $\operatorname{Fin}\operatorname{ProjDim}(\mathfrak{A}_G)$ (resp. $\operatorname{Fin}\operatorname{InjDim}(\mathfrak{A}_G)$).

Theorem A: (partly new, partly inspired by [5], and partly covered by Thm. VII .2.2 of [2]) The following are equivalent for any TDLC group G.

a) Every object in $\mathfrak{A}_G$ admits a complete projective resolution, i.e. every object has a projective resolution that eventually agrees with a totally acylic complex of projectives.

b) $\operatorname{silp}(\mathfrak{A}_G)$ (defined as the supremum over the injective dimension of $\mathfrak{A}_G$-projectives) and $\operatorname{spli}(\mathfrak{A}_G)$ (defined as the supremum over the projective dimension of $\mathfrak{A}_G$-injectives) are finite.

c) Every object in $\mathfrak{A}_G$ admits a complete injective resolution, i.e. every object has an injective resolution that eventually agrees with a totally acyclic complex of injectives, and $\operatorname{Fin}\operatorname{InjDim}(\mathfrak{A}_G) < \infty$.

d) Complete cohomology computed with projective resolutions agrees with complete cohomology computed with injective resolutions (in the style of Nucinkis [5]).

Proposition B: When any of the equivalent statements of **Theorem A** are satisfied (an easy example is when $G = \mathrm{SL}_n(\mathbb{Q}_p)$ as it has finite virtual cohomological dimension), we have

$\operatorname{silp}(\mathfrak{A}_G) = \operatorname{spli}(\mathfrak{A}_G) = \operatorname{Fin}\operatorname{ProjDim}(\mathfrak{A}_G) = \operatorname{Fin}\operatorname{InjDim}(\mathfrak{A}_G) < \infty$

We can add some more invariants here like the finitistic and the global Gorenstein dimensions. The main use of **Theorem A** is that it gives very neat conditions on when every object has complete resolutions which is useful in constructing a well-behaved stable category as we describe below.

3. Candiates for stable categories and equivalences.

Theorem C: (new, in the spirit of [1, 4]) Let G be a TDLC group such that $\operatorname{silp}(\mathfrak{A}_G)$ and $\operatorname{spli}(\mathfrak{A}_G)$ are finite. Then, the following triangulated categories are equivalent (and are therefore equivalently adequate candidates for our stable category):

(i) $(\mathfrak{A}_G, \widehat{\operatorname{Ext}}^0_{\mathfrak{A}_G}(\_,\_))$ (here, the objects are all modules in $\mathfrak{A}_G$ and the Hom-sets are given by the zero-th complete cohomology groups computed with projective resolutions)

(ii) $(\mathfrak{A}_G, \widetilde{\operatorname{Ext}}^0_{\mathfrak{A}_G}(\_,\_))$ (here, the objects are all modules in $\mathfrak{A}_G$ and the Hom-sets are given by the zero-th complete cohomology groups computed with injective resolutions)

(iii) $\mathcal{D}^b(\mathfrak{A}_G)/\,\mathrm{K}^b(\mathrm{Proj}\text{-}\mathfrak{A}_G)$ (the Verdier quotient of the derived bounded category on $\mathfrak{A}_G$ and the homotopy category of bounded complexes of $\mathfrak{A}_G$-projectives)

(iv) $\mathcal{D}^b(\mathfrak{A}_G)/\,\mathrm{K}^b(\mathrm{Inj}\text{-}\mathfrak{A}_G)$ (the Verdier quotient of the derived bounded category on $\mathfrak{A}_G$ and the homotopy category of bounded complexes of $\mathfrak{A}_G$-injectives)

(v) The homotopy category of totally acylic complexes of $\mathfrak{A}_G$-projectives.

(vi) The homotopy category of totally acylic complexes of $\mathfrak{A}_G$-injectives.

(vii) $\underline{\mathrm{GProj}}(\mathfrak{A}_G)$ ($\mathrm{GProj}(\mathfrak{A}_G)$ is a Frobenius category with the class of projective-injectives given by the $\mathfrak{A}_G$-projectives; the notation here denotes its stable category where we keep all the objects and kill all morphisms that factor through an $\mathfrak{A}_G$-projective). To get to (iii) from (i), consider a module as a complex concentrated in degree 0; for (iii) to (v), take complete projective resolutions (possible due to Thm A); for (v) to (vii), take the zero-th syzygy functor (see Def. 3.7 of [4]); and for (vii) to (i), use the inclusion functor. Composing these, get $(i) \cong (iii) \cong (v) \cong (vii)$. Repeat the analogous treatment with "module as a deg 0 concentrated complex", "taking complete injective resolutions" (again, possible due to Thm A), and the zero-th cosyzygy functor, to get $(ii) \cong (iv) \cong (vi)$. $(i) \cong (ii)$ by Thm A, and $(iii) \cong (iv)$ as $\mathrm{silp}(\mathfrak{A}_G), \mathrm{spli}(\mathfrak{A}_G) < \infty$.

Note that all the categories in **Theorem C** except two, namely $(\mathfrak{A}_G, \widehat{\mathrm{Ext}}^0_{\mathfrak{A}_G}(\_,\_))$ and $(\mathfrak{A}_G, \widetilde{\mathrm{Ext}}^0_{\mathfrak{A}_G}(\_,\_))$, are clearly triangulated categories. Without the assumption that $\mathrm{silp}(\mathfrak{A}_G)$ and $\mathrm{spli}(\mathfrak{A}_G)$ are finite, these two categories need not even be triangulated. Since we know very little about $\mathfrak{A}_G$-injectives, the presence of (vi) above is noteworthy.

3. Finishing remarks: We are insisting on these equivalences because they are useful for making progress on stratification questions in the spirit of, for example, Benson-Iyengar-Krause or Barthel-Heard-Sanders. This is why even if we can replace $\mathfrak{A}_G$ with a more refined abelian category associated to G using the full force of *Condensed Maths*, our use of and dependence on complete resolutions (both projective and injective) to establish "good behaviour" of our stable category will remain.

Ending open question: Can we achieve **Theorem C** with a finiteness assumption on just one of $\mathrm{silp}(\mathfrak{A}_G)$ and $\mathrm{spli}(\mathfrak{A}_G)$?

References

[1] A. Beligiannis. *The homological theory of contravariantly finite subcategories: Auslander-Buchweitz contexts, Gorenstein categories and (co-)stabilization*, Comm. Algebra 28 (2000) 4547—4596.

[2] A. Beligiannis and I. Reiten. *Homological and homotopical aspects of torsion theories*, Mem. Amer. Math. Soc. 188 (2007), no. 883, viii+207 pp.

[3] I. Castellano and T. Weigel. *Rational discrete cohomology for totally disconnected locally compact groups*, J. Algebra 453 (2016), 101—159.

[4] N. Mazza and P. Symonds. *The stable category and invertible modules for infinite groups*, Adv. Math. 358 (2019), 106853, 26 pp.

[5] B. Nucinkis. *Complete cohomology for arbitrary rings using injectives*, J. Pure Appl. Algebra 131 (1998), no. 3, 297—318.