Algebraic Geometry, spring term 2021

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Contents

Chapter 1

Affine and projective algebraic sets; rational normal curves, finite point sets

Below, k is an algebraically closed field of arbitrary characteristic, e.g. $\mathbb{C}$, $\bar{\mathbb{Q}}$ or also $\bar{\mathbb{F}}_p$, and $\mathbb{A}^n$ is the n-dimensional affine space over k, i.e. k^n as a set (though we will want to make use of its vector space structure occasionally as well).

Definition 1.1. For a k-vector space, $\mathbb{P}(V)$ denotes the set of 1-dimensional subspaces of V, i.e. the projective space associated to V. For the vector space k^{n+1} with componentwise addition and scalar multiplication, we also write $\mathbb{P}^n = \mathbb{P}(k^{n+1})$.

Equivalently, $\mathbb{P}^n$ is the quotient of $k^{n+1} - \{0\}$ by the equivalence relation:

$$(X_0, \dots, X_n) \sim (X_0', \dots, X_n'), \quad (X_0, \dots, X_n), (X_0', \dots, X_n') \in k^{n+1} - \{0\}$$

if there is a $\lambda \in k^*$ with $\lambda(X_0, \dots, X_n) = (X_0', \dots, X_n')$. We denote the equivalence class $[(X_0, \dots, X_n)]$ by $(X_0 : \dots : X_n)$ in that case and call the X_i's homogeneous coordinates of the point in $\mathbb{P}^n$.

Define a subset $U_i \subset \mathbb{P}^n$ by

$$U_i := \{(X_0 : \dots : X_n) \in \mathbb{P}^n \mid X_i \neq 0\}.$$

Below we will often identify U_i with $\mathbb{A}^n$ via the bijection $(X_0 : \dots : X_n) \mapsto (x_j^{(i)})_{0 \le j \le n} = (X_j / X_i)$ (so $x_i^{(i)} = 1$).

Definition 1.2. 1. An affine algebraic set $X \subset \mathbb{A}^n$ is the set of zeroes of a family $(f_\alpha)_{\alpha\in A}$, of polynomials $f_\alpha \in k[x_1, \dots, x_n]$. Since $k[x_1, \dots, x_n]$ is Noetherian, we can assume $|A| < \infty$ without loss of generality.

2. A projective algebraic set $Y \subset \mathbb{P}^n$ is the set of zeroes of a family of polynomials $(F_\alpha)_{\alpha\in A}$ with $F_\alpha \in k[X_0, \dots, X_n]$.

We have to say a few words what it means to be a zero in (2) above, i.e. what is meant by $F(p) = 0$ for a $p = (P_0 : \cdots : P_n) \in \mathbb{P}^n$. This is so because the homogeneous coordinates P_i of p are not unique, and p being a zero must be independent of the representative coordinate tuple. We do this by defining

$$F(p) = 0 :\iff F(P_0, \dots, P_n) = 0$$
$$\text{for } \textbf{all } (P_0, \dots, P_n) \in k^{n+1} - \{0\} \text{ with } [(P_0, \dots, P_n)] = p.$$

This leads to the conclusion that we can assume, without loss of generality, that the F_α in (2) of Definition 1.2 are *homogeneous*, which means the following: $S = k[X_0, \dots, X_n]$ is a graded ring, which means there is a decomposition into k-vector subspaces

$$S = \bigoplus_{m\geq 0} S_m$$

where $S_m := \langle X_0^{\alpha_0} \cdot \ldots \cdot X_n^{\alpha_n} \rangle_k$, $\alpha_0 + \ldots \alpha_m = m$, such that $S_{m_1} \cdot S_{m_2} \subset S_{m_1+m_2}$. Polynomials in S_m are called homogeneous of degree m. Now, since k is infinite, if $F \in k[X_0, \dots, X_n]$ vanishes in p as above, then all homogeneous components F_m of F with respect to the preceding direct sum decomposition vanish in p.

Remark 1.3. If $Y \subset \mathbb{P}^n$ is a projective algebraic set, then $Y_i = Y \cap U_i \subset \mathbb{A}^n$ is an affine algebraic set. To see this, consider for simplicity Y_0; the argument in the other cases being the same. If Y is defined by homogeneous polynomials F_α of degree d_α, then Y_0 is the set of zeroes of polynomials $f_\alpha(x_1, \dots, x_n)$, $x_i = X_i/X_0$, where

$$f_\alpha(x_1, \dots, x_n) := F_\alpha(X_0, \dots, X_n)/X_0^{d_\alpha} = F_\alpha(1, x_1, \dots, x_n).$$

Remark 1.4. Every affine algebraic subset $X_i \subset \mathbb{A}^n \simeq U_i \subset \mathbb{P}^n$ is the intersection of U_i with a projective algebraic subset $X \subset \mathbb{P}^n$. Again we show that for

U_0 only since the other cases are only notationally different. If X_0 is defined by $f_\alpha(x_1, \ldots, x_n)$ of degree d_α (of course not necessarily homogeneous now), we can define a suitable X by

$$F_\alpha(X_0, \ldots, X_n) := X_0^{d_\alpha} f_\alpha\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right).$$

We can summarize the preceding two remarks by saying that $X \subset \mathbb{P}^n$ is a projective algebraic subset if and only if each $X \cap U_i \subset U_i \simeq \mathbb{A}^n$ is an affine algebraic set.

Example 1.5.

1. If $W \subset k^{n+1}$ is an $(m+1)$-dimensional sub-vector space, then $\mathbb{P}(W) \subset \mathbb{P}^n$ is a projective algebraic set, called an m-dimensional projective linear subspace (for $m = 1$: line, $m = 2$: plane, $m = n - 1$: hyperplane).

2. Of course zeroes of a single homogeneous polynomial $F \in k[X_0, \ldots, X_n]$ (of degree d, say) are a projective algebraic set; this is called a hypersurface. We can assume F without multiple factors (note that $k[X_0, \ldots, X_n]$ is factorial). Then we call d the degree of the hypersurface.

3. For a more interesting example, let C be the image of the map

$$\nu\colon \mathbb{P}^1 \to \mathbb{P}^3$$
$$(X_0 : X_1) \mapsto (X_0^3 : X_0^2 X_1 : X_0 X_1^2 : X_1^3) := (Z_0 : Z_1 : Z_2 : Z_3).$$

Then C is contained in the three quadrics Q_0, Q_1, Q_2 defined as the zero sets of

$$\begin{aligned} F_0(Z) &= Z_0 Z_2 - Z_1^2 \\ F_1(Z) &= Z_0 Z_3 - Z_1 Z_2 \\ F_2(Z) &= Z_1 Z_3 - Z_2^2 \end{aligned}$$

and those define C, i.e. the zero set of them is exactly C (to see this, note that if $p \in \mathbb{P}^3$, $p = (P_0 : P_1 : P_2 : P_3)$ lies on $Q_0 \cap Q_1 \cap Q_2$, then $P_0 \neq 0$ or $P_3 \neq 0$; in the former case, $P = \nu((P_0 : P_1))$, in the latter case, $P = \nu((P_2 : P_3))$.

We continue the study of (3) in the above examples a little: although we haven't introduced any notion of "dimension" yet into our geometric study of algebraic sets, it is intuitively plausible that C should be a one-dimensional thing, a curve. It is called a *twisted cubic curve*. It is remarkable that two of the above quadratic equations do not define C, more generally:

Proposition 1.6. *For $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in k^3 - \{0\}$, write*

$$F_\lambda = \lambda_0 F_0 + \lambda_1 F_1 + \lambda_2 F_2,$$

the F_i being as in Example 1.5, (3). Denote the projective algebraic set that F_λ defines by Q_λ. Then for $[\mu] \neq [\nu] \in \mathbb{P}^2$, we have that $Q_\mu \cap Q_\nu$ is equal to the union of C and a line $L_{\mu\nu}$ intersecting C in two points.

Proof. C is defined by the 2×2 minors of

$$\begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_1 & Z_2 & Z_3 \end{pmatrix}$$

and Q_μ is the determinant of

$$\begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_1 & Z_2 & Z_3 \\ \mu'_0 & \mu'_1 & \mu'_2 \end{pmatrix}$$

where the tuple (μ'_0, μ'_1, μ'_2) agrees with μ after a signed permutation. So the locus outside of C where F_μ and F_ν vanish, is the rank ≤ 2 locus of

$$\begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_1 & Z_2 & Z_3 \\ \mu'_0 & \mu'_1 & \mu'_2 \\ \nu'_0 & \nu'_1 & \nu'_2 \end{pmatrix}$$

(where in addition the first two rows are independent). For $[\mu] \neq [\nu]$, this locus is the same as the one defined by

$$\det \begin{pmatrix} Z_0 & Z_1 & Z_2 \\ \mu'_0 & \mu'_1 & \mu'_2 \\ \nu'_0 & \nu'_1 & \nu'_2 \end{pmatrix} = \det \begin{pmatrix} Z_1 & Z_2 & Z_3 \\ \mu'_0 & \mu'_1 & \mu'_2 \\ \nu'_0 & \nu'_1 & \nu'_2 \end{pmatrix} = 0,$$

i.e. $Q_\mu \cap Q_\nu = C \cup L_{\mu\nu}$ where $L_{\mu\nu}$ is the line defined by the last two determinants. The intersection of $L_{\mu\nu}$ with C is then given by Q_λ and the two linear equations above where $\langle Q_\lambda, Q_\mu, Q_\nu \rangle = \langle F_0, F_1, F_2 \rangle$. $\square$

Proposition 1.7. *There exists a homogeneous quadratic polynomial $Q(Z_0, \ldots, Z_3)$ and a homogeneous cubic polynomial $P(Z_0, \ldots, Z_3)$ whose common zeroes are precisely C.*

Proof. One can take

$$Q(Z) = \det \begin{pmatrix} Z_0 & Z_1 \\ Z_1 & Z_2 \end{pmatrix}, \; P(z) = \det \begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_1 & Z_2 & Z_3 \\ Z_2 & Z_3 & Z_0 \end{pmatrix}.$$

Namely, if the vector (Z_0, Z_1, Z_2) and (Z_1, Z_2, Z_3) are linearly dependent, then these determinants vanish, and the converse holds as well: if the first two rows of the matrix whose determinant defines $P(z)$ are independent, and the determinant vanishes, then the last row of the matrix must be a linear combination of the first two rows. But then (Z_2, Z_3) is dependent on (Z_0, Z_1) and (Z_1, Z_2), whence the rank of the submatrix consisting of the first two rows would be 1 (taking into account $Q(z) = 0$), contradiction. □

Thus we arrive at the curious fact that C, as a set, can be defined by two polynomials, but if we look at the *ideal* $I(C) \subset k[X_0, \ldots, X_3]$ of **all** polynomials vanishing on C, this cannot be generated by 2 elements (since $\dim I(C)_2 \geq 3$, but $\dim I(C)_1 = 0$: clearly, C does not lie in a hyperplane since $X_0^3, X_0^2 X_1, X_0 X_1^2, X_1^3$ are independent). One says that C is a *set-theoretic complete intersection*, but not a *complete intersection* in $\mathbb{P}^3$. There are many open problems connected with these notions; e.g., one knows that the union of two planes intersecting only in 0 is not a set-theoretic complete intersection in $\mathbb{A}^4$, but one does not know if every curve in $\mathbb{P}^3$ is a set-theoretic complete intersection.

Let us consider the lines $L_{\mu\nu}$ more closely:

Proposition 1.8. *Every line $L \subset \mathbb{P}^3$ connecting two points two points $P, Q \in C$ occurs among the $L_{\mu\nu}$.*

Proof. Choose $R \in L$ distinct from P, Q. The three-dimensional vector space of F_λ's contains a two-dimensional subspace consisting of those vanishing in R; suppose F_{μ_0}, F_{ν_0} is a basis. But then the latter two polynomials vanish on C, hence on P, Q, R. Since they are quadratic, they are then identically zero on L. Whence $L_{\mu_0\nu_0} = L$. □

We call every algebraic set in $\mathbb{P}^3$ projectively equivalent to C a twisted cubic curve. I.e., twisted cubic curves are precisely the images of maps $\mathbb{P}^1 \to \mathbb{P}^3$ given by $[X] \mapsto (A_0(X) : \cdots : A_3(X)]$ where the $A_i(X)$'s are a basis of $k[X_0, \ldots, X_3]_3$. Generalizing this, we make

Definition 1.9. Every curve in $\mathbb{P}^d$ projectively equivalent to the image of

$$\nu_d \colon \mathbb{P}^1 \to \mathbb{P}^d,$$
$$(X_0 : X_1) \mapsto (X_0^d : X_0^{d-1}X_1 : \cdots : X_1^d)$$

is called a rational normal curve in $\mathbb{P}^d$.

Remark 1.10. 1. To make sense of the above definition, remark that the image of ν_d is really a projective algebraic set. One can take $F_{ij}(Z) = Z_iZ_j - Z_{i-1}Z_{j+1}$, $1 \leq i \leq j \leq d-1$, as a set od polynomials defining the image. The word "curve" in the above definition so far has no real mathematical meaning, but once we have introduced dimension, a rational curve will indeed be 1-dimensional.

2. Every set of $d+1$ points on a rational normal curve are linearly independent (Van der Monde determinant).

Theorem 1.11. *Through every set of $d+3$ points in general position in $\mathbb{P}^d$, there passes a unique rational normal curve.*

Here "general position" means that the assertions holds for tuples of points in a nonempty subset of $(\mathbb{P}^d)^{d+3}$ defined by some polynomial inequalities.

Proof. We prove existence first. We can assume, after applying a projectivity, that the first $d+1$ of the points, $p_1, \ldots, p_{d+1}$, are $(1 : 0 : \cdots : 0), \ldots, (0 : \cdots : 1)$ (coordinate points). Putting $G(X_0, X_1) = \prod_{k=1}^{d+1}(\nu_k X_0 - \mu_k X_1)$ and

$$H_i := \frac{G(X_0, X_1)}{(\nu_i X_0 - \mu_i X_1)},$$

we get that the image of ν_d given by

$$(X_0 : X_1) \mapsto (H_1(X_0, X_1) : \cdots : H_{d+1}(X_0, X_1))$$

passes through the coordinate points, namely maps $(\mu_i : \nu_i) \in \mathbb{P}^1$ to the i-th coordinate point. We can also assume $(\mu_i : \nu_i)$ are different from $(1 : 0)$ and

$(0 : 1)$; then, given two general additional points p_{d+2}, p_{d+3} in $\mathbb{P}^d$, we can always adjust the μ_i, ν_i so that $(1 : 0)$ maps to p_{d+2} and $(0 : 1)$ maps to p_{d+3}. This proves existence.

For uniqueness, note that every rational normal curve passing through the coordinate points $p_1, \dots, p_{d+1}$ is the image of a map given by polynomials H_i as above for certain (μ_i, ν_i). Applying a projectivity in the source, we can also assume that $(1 : 0)$ and $(0 : 1)$ map to p_{d+2}, p_{d+3}. Then such a rational normal curve is given by polynomials H_i as above, and moreover, the μ_i are fixed up to simultaneous rescaling by a constant nonzero factor α, and so are the ν_j up to a factor β. Applying the projectivity $(X_0 : X_1) \mapsto (\alpha^{-1}X_0 : \beta^{-1}X_1)$ in the source, we see that the maps corresponding to different α, β all have the same image. Hence this is the unique rational normal curve meeting all the requirements. $\square$

Another example of projective algebraic sets are finite point sets $\Gamma = \{p_1, \dots, p_N\} \subset \mathbb{P}^n$: indeed, if $q \notin \Gamma$, there is a polynomial vanishing in Γ, but not in q (take a product of N linear forms). Hence Γ is defined by polynomials of degree $\leq N$.

It is known from courses in linear algebra that

1. Two ordered point sets $(p_1, \dots, p_{n+2})$, $(q_1, \dots, q_{n+2})$ in general position can be transformed into each other by a unique projectivity $g \in \mathrm{PGL}_{n+1}(k)$.

2. In $\mathbb{P}^1$, one can transform $4 = 1 + 3$ ordered points in general position into another ordered four points in general position if and only if their cross ratios
$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$
are the same.

So when can $(n+3)$ general ordered points $(p_1, \dots, p_{n+3})$ in $\mathbb{P}^n$ be transformed into another tuple $(q_1, \dots, q_{n+3})$ of $(n+3)$-points?By Theorem 1.11, we can find rational normal curves through both sets of points. The maps defining the rational normal curves allow us to interpret these point sets as point sets
$$(p'_1, \dots, p'_{n+3}), \quad (q'_1, \dots, q'_{n+3})$$
on $\mathbb{P}^1$. By the uniqueness property of rational normal curves, and since every algebraic automorphism of $\mathbb{P}^1$ is a projectivity (cf. all biholomorphic maps

of the Riemann sphere are fractional linear), one gets that $(p_1, \ldots, p_{n+3})$ and $(q_1, \ldots, q_{n+3})$ are projectively equivalent on $\mathbb{P}^n$ if and only if $(p'_1, \ldots, p'_{n+3})$ and $(q'_1, \ldots, q'_{n+3})$ are projectively equivalent on $\mathbb{P}^1$. The latter means that

$$\text{crossratio}(p'_1, p'_2, p'_3, p'_i) = \text{crossratio}(q'_1, q'_2, q'_3, q'_i), \ \forall\, i = 4, \ldots, n+3.$$

Chapter 2

The sheaf of regular functions; algebraic varieties and regular maps

We can equip our algebraic sets with a topology.

Definition 2.1. The Zariski topology of $\mathbb{P}^n$ (resp. of $\mathbb{A}^n$) is the topology whose closed sets are the projective (resp. affine) algebraic sets; we equip every projective (resp. affine) algebraic set X in $\mathbb{P}^n$ (resp. $\mathbb{A}^n$) with the induced topology, and call this the Zariski topology of X.

Remark 2.2.

1. If $k = \mathbb{C}$ or another field with an interesting topology, e.g. the p-adic numbers $\mathbb{Q}_p$, then one can equip k^{n+1} with the corresponding product topology, and $\mathbb{P}^n$ with the quotient topology via the map $k^{n+1} - \{0\} \to \mathbb{P}^n$. Then all algebraic sets carry another "strong" topology, which is topologically more relevant. The Zariski topology is just convenient to talk about sets where polynomials vanish or do not vanish, but does not carry too much information otherwise.

2. In $\mathbb{P}^n$, the sets $U_F := \{F \neq 0\}$, where F ranges over all homogeneous polynomials, form a basis of the topology. In $\mathbb{A}^n$, we have an analogous basis $U_f = \{f \neq 0\}$ with f ranging over all polynomials in $k[x_1, \dots, x_n]$.

Definition 2.3. An open subset $U \subset X$, X a projective algebraic set in $\mathbb{P}^n$, will be called a quasi-projective algebraic set.

In what follows, we will define a notion of local functions on affine, projective, quasi-projective algebraic sets, equipped with the Zariski topology. Our objects of study in the sequel, called **algebraic varieties**, will then (formally) be triples

$$(X, \mathfrak{T}, \mathcal{O})$$

where X is an algebraic set of some sort, $\mathfrak{T}$ the Zariski topology on it, and $\mathcal{O}$ the local functions.

One may justifiably ask: why such complicated contortions to define the notion of a variety, which is meant to be a geometric object associated to a bunch of polynomial equations? Why is $\mathcal{O}$ necessary? Why is even $\mathcal{T}$ necessary?

The answer is that experience has shown that once one studies objects of a certain type in mathematics, which are often sets with an additional structure, one should at the same time study "maps" between those objects that preserve the given structure. Thus one studies vector spaces along with linear maps, groups along with group homomorphisms, rings with ring homomorphisms, or topological spaces with continuous maps. To get a meaningful and interesting notion of maps in our theory of zero sets of polynomials, we will need $\mathcal{O}$, and to define it, it is convenient to have $\mathcal{T}$ first.

For example, suppose you would take algebraic sets together with their Zariski topology as the fundamental structured objects of your theory, and forget (or never learn about) $\mathcal{O}$. The structure-preserving maps are then continuous maps between those topological spaces. Now look more specifically at *irreducible curves* in $\mathbb{A}^2$, i.e. zero sets of a single irreducible polynomial $f(x, y)$ where $f \in \bar{\mathbb{Q}}[x, y]$ (we take $\bar{\mathbb{Q}}$ for simplicity). Then all of these turn out to be homeomorphic! Namely, as sets, all such curves have the same cardinality, they are countable since $\bar{\mathbb{Q}}$ is. Any bijection is even a homeomorphism since the nonempty proper Zariski open subsets are complements of finite sets of points in this case (this uses the irreducibility of the $f(x, y)$). But there are facts of an algebraic nature that strongly suggest that regarding all irreducible curves in $\mathbb{A}^2$ as essentially the same (isomorphic) is much too coarse and crude: for example, consider the question whether there exist nonconstant rational functions

$$\varphi(t), \psi(t) \in \mathbb{Q}(t)$$

with

$$f(\varphi(t), \psi(t)) = 0$$

(i.e., a rational parametrization of the irreducible curve associated to f). Then it turns out that the general curve of degree $\deg(f) > 2$ does not admit such a parametrization, but all degree 1 and 2 curves do. Moreover, there are special irreducible curves of any degree that do admit such a parametrization. We certainly don't want to build our theory in such a way as to put all those algebraically totally different curves in one bag! That is why we need $\mathcal{O}$.

Definition 2.4. Let X be a topological space, k a field. For $U \subset X$ open, we denote by

$$\mathrm{Maps}(U, k)$$

the set of all functions (continuous or not) of U to k. A *sheaf of k-valued functions* on X is the datum, for any open $U \subset X$, of a subset

$$\mathcal{O}_X(U) \subset \mathrm{Maps}(U, k)$$

such that the following is true: if $U \subset X$ is open and $U = \bigcup_{i \in I} V_i$ a cover of U by open subsets $V_i \subset U \subset X$, $i \in I$, I some index set, then a function $f \in \mathrm{Maps}(U, k)$ belongs to $\mathcal{O}_X(U)$ if and only if all restrictions $f|_{V_i}$ belong to $\mathcal{O}_X(V_i)$ (for all $i \in I$).

This may seem intimidating at first sight, but is totally simple really: it just means that on any open set U in X we mark certain functions as distinguished, by painting them red, say; we call these $\mathcal{O}_X(U)$. The condition then means that being "red" is a local property: if we restrict one of our functions to the open sets of a cover, and if it is red on every one of them, then it is red globally.

Example 2.5.

1. Let $X = \mathbb{R}^1$ with the Euclidean topology, and put for $U \subset \mathbb{R}$ open, $\mathcal{O}_X(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}$. Then $\mathcal{O}_X$ is a sheaf of $\mathbb{R}$-valued functions. The property of being continuous is local.

2. For X as in (1), let $\mathcal{O}_X(U) = \{f : U \to \mathbb{R} \mid f \text{ differentiable}\}$. Then $\mathcal{O}_X$ is a sheaf of $\mathbb{R}$-valued functions. The property of being differentiable is local.

3. Let $X = \Delta_1 \sqcup \Delta_2$ be the disjoint union of two open discs in $\mathbb{R}^2$, with the induced topology from $\mathbb{R}^2$. For $U \subset X$ open, let $\mathcal{O}_X(U) = \{f : U \to \mathbb{R} \mid f \text{ constant}\}$. Then $\mathcal{O}_X$ is not a sheaf: the property of being constant is not local. For example, the function which is 1 on Δ_1 and 0 on Δ_2 is constant locally, but not constant globally.

4. Let $X = \mathbb{R}^1$ again, $\mathcal{O}_X(U) = \{f\colon U \to \mathbb{R} \mid f \text{ bounded}\}$. Then $\mathcal{O}_X$ is not a sheaf: the property of being bounded is not local. For example, $f(x) = \exp(x)$ is bounded locally, but not globally.

Definition 2.6. Let X be a topological space with a sheaf of functions $\mathcal{O}_X$, and $Y \subset X$ a subspace. Then put for $V \subset Y$ open:

$$(\mathcal{O}_X)\mid_Y (V) := \Big\{f\colon V \to k \mid \forall\, v \in V\ \exists\, U(v) \subset X \text{ open},\ U(v) \ni v, \\ \exists\, \tilde{f}_v \in \mathcal{O}_X(U(v)):\ \tilde{f}_v \mid_{U(v)\cap V} = f \mid_{U(v)\cap V}\Big\}.$$

Then $(\mathcal{O}_X)\mid_Y$ is a sheaf of functions on Y, the sheaf induced by $\mathcal{O}_X$ on Y by restriction.

Definition 2.7. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be spaces with sheaves of functions. A morphism

$$\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a continuous map $\varphi\colon X \to Y$ with the property that for all $U \subset Y$ open, and for all $f \in \mathcal{O}_Y(U)$, one has $\varphi^*(f) := f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(U))$.

We now define sheaves of functions on $\mathbb{A}^n$ and $\mathbb{P}^n$; by the construction of Definition 2.6, we can turn all quasi-projective algebraic sets (with their Zariski topology) into spaces with sheaves of functions.

Definition 2.8. For $U \subset \mathbb{A}^n$ (Zariski-)open, put

$$\mathcal{O}_{\mathbb{A}^n}(U) := \Big\{f\colon U \to k \mid \forall\, x \in U\ \exists\, V(x) \subset U \text{ open}, \\ x \in V(x),\ \exists \text{ polynomials } p,\ q \in k[x_1, \ldots, x_n],\ q(y) \neq 0\ \forall\, y \in V(x) \\ \text{such that }:\ f \mid_{V(x)} = \frac{p}{q} \mid_{V(x)}\Big\}.$$

For $U \subset \mathbb{P}^n$ open, put

$$\mathcal{O}_{\mathbb{P}^n}(U) := \Big\{f\colon U \to k \mid \forall\, x \in U\ \exists\, V(x) \subset U \text{ open}, \\ x \in V(x),\ \exists \text{ homogeneous polynomials of the same degree } p,\ q \in k[X_0, \ldots, X_n], \\ q(y) \neq 0\ \forall\, y \in V(x) \text{ such that }:\ f \mid_{V(x)} = \frac{p}{q} \mid_{V(x)}\Big\}.$$

These define sheaves of functions on the respective spaces.

This also looks complicated, but means only that the local functions are those that can be written, locally, as quotients of two polynomials (homogeneous of the same degree in the case of $\mathbb{P}^n$), with the denominator non-vanishing on the open under consideration.

Definition 2.9. Suppose $X \subset \mathbb{A}^n$ is an affine algebraic subset, or $X \subset \mathbb{P}^n$ is a projective or quasi-projective algebraic subset together with its Zariski topology $\mathfrak{T}$. We define $\mathcal{O}_X$ as $(\mathcal{O}_{\mathbb{A}^n}) \mid_X$ resp. $(\mathcal{O}_{\mathbb{P}^n}) \mid_X$. Then the triple $(X, \mathfrak{T}, \mathcal{O}_X)$ is called an affine/projective/quasi-projective **algebraic variety**. Moreover, slightly more generally, any topological space with a sheaf of k-valued functions that is isomorphic, as a space with a sheaf of functions, to an affine/projective/quasi-projective algebraic variety, will be called an affine/projective/quasi-projective algebraic variety itself. If we use algebraic variety without further qualification, we mean the most general class introduced, a quasi-projective variety.
The elements in $\mathcal{O}_X(U)$ for $U \subset X$ open, are called regular functions on U. For a variety X, an open subset $U \subset X$ as well as a closed subset $Y \subset X$ are varieties in their own right, the inclusions are morphisms. We call U an open subvariety in this case, and Y a closed subvariety of X. Subvariety without qualification will mean closed subvariety.

Remark 2.10. With these definitions, the natural projection $\mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ becomes a morphism, and the natural bijections of the subsets $U_i = \{X_i \neq 0\} \subset \mathbb{P}^n$ with $\mathbb{A}^n$ become isomorphisms, as it should be.

Definition 2.11. If $(X, \mathcal{O}_X)$ is a space with a sheaf of functions, $x \in X$, put

$$\begin{aligned} \mathcal{O}_{X,x} := \{ &\text{equivalence classes of pairs } (f, U) \\ &\text{where } U \ni x \text{ is open, } f \in \mathcal{O}_X(U), \text{ and} \\ (f,U) \sim (g,V) :\iff \ &\exists W \subset U \cap V \text{ open, } x \in W \text{ such that } f \mid_W = g \mid_W \}. \end{aligned}$$

We call $\mathcal{O}_{X,x}$ the stalk of $\mathcal{O}_X$ in x; elements of $\mathcal{O}_{X,x}$ are called germs (of functions around x).

These definitions are very slick and smooth, and leave us with a beautiful category of algebraic varieties, but the drawback is that it takes quite a while to develop an intuition for what they mean and to work with them. For example, what is $\mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n)$ or $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n)$? Here is a useful example of a morphism.

Example 2.12. Let X, Y, Z be homogeneous coordinates in $\mathbb{P}^2$, and C the plane cubic curve given by

$$C\colon ZY^2 = X(X^2 - Z^2).$$

Let $P_0 = (0:0:1) \in C$. For a (variable) point $P \in C$, let l be the line through P_0 and P and $\alpha(P)$ the third intersection point of l with C. Then the assignment $P \mapsto \alpha(P)$ defines an automorphism of C of order 2, $\alpha^2 = \mathrm{id}$.

To see this, first note that α is well-defined as a map of point sets: the polynomial defining C restricted to $l \simeq \mathbb{P}^1$ factors, and will have one zero ζ_{P_0} corresponding to P_0, one zero ζ_P corresponding to P, and a third zero $\zeta_{\alpha(P)}$ determining $\alpha(P)$ uniquely, when we take into account multiplicities.

Consider $C_Z := C \cap \{Z \neq 0\}$ and take coordinates $x = X/Z$, $y = Y/Z$ on $\mathbb{A}^2 \simeq U_Z := \{(X:Y:Z) \in \mathbb{P}^2 \mid Z \neq 0\}$. Then C_Z has an equation

$$y^2 = x(x^2 - 1).$$

If $P = (a, b) \in C_Z$, then the line l through P_0 and P is given by $x = at$, $y = bt$. Substituting yields

$$\begin{aligned} b^2t^2 &= at(a^2t^2 - 1) \\ 0 &= at(t-1)(a^2t + 1) \end{aligned}$$

So we get the third intersection point for $t = -1/a^2$, whence

$$\alpha\colon (a, b) \mapsto \left(-\frac{1}{a}, -\frac{b}{a^2}\right).$$

However, this is not a well-defined map of $U_Z \simeq \mathbb{A}^2$ into itself because the formula makes no sense for $(a, b) = (0, 0)$. However, (draw a picture!) the point $(0, 0)$ should map to $P_1 = (0:1:0)$, the unique point at ∞ on C with respect to the coordinates x, y! To prove that α is a morphism, we must rather cover C with various open sets as follows: put

$$\begin{aligned} P_1 &= \quad (0:1:0), \\ P_2 &= (0:0:1)(= P_0), \\ Q_1 &= \quad (1:0:1), \\ Q_2 &= \quad (-1:0:1) \end{aligned}$$

From the geometric definition, α exchanges P_1, P_2 and Q_1, Q_2. Put

$$\begin{aligned} U_1 &:= & C - \{P_1, P_2\} \\ U_2 &:= & C - \{P_1, Q_1, Q_2\} \\ U_3 &= & C - \{P_2, Q_1, Q_2\} \\ V_1 &:= & C - \{P_1\} = C \cap U_Z = C_Z \\ V_2 &= & C - \{P_2, Q_1, Q_2\} = C \cap \{Y \neq 0\} = C \cap U_Y = C_Y. \end{aligned}$$

Then

1. U_1, U_2, U_3 is an open cover of C.
2. V_1, V_2 is an affine open cover of C.
3. We have
$$\alpha(U_1) \subset V_1,\ \alpha(U_2) \subset V_2,\ \alpha(U_3) \subset V_1.$$

Let x, y be coordinates in U_Z (resp. V_1) as above, and $s = X/Y$, $t = Z/Y$ coordinates in U_Y (resp. V_2). Then

(A) The map $\alpha\,|_{U_1} \colon U_1 \to V_1$ is given by
$$(a, b) \mapsto \left(-\frac{1}{a}, -\frac{b}{a^2}\right).$$
Note that the right hand side are polynomials in $a, b, 1/a$ and that $x, y, 1/x \in \mathcal{O}_C(U_1)$, so $\alpha\,|_{U_1}$ is a morphism.

(B) To describe the map $\alpha\,|_{U_2} \colon U_2 \to V_2$, let us compute the image under α of a point $(x, y) = (a, b) \in U_2$ (x, y are also coordinates on $U_2 \subset V_1$) in terms of the coordinates s, t on V_2; since
$$s = \frac{x}{y},\ t = y^{-1}$$
we get
$$s(\alpha(a, b)) = \frac{a}{b} \quad t(\alpha(a, b)) = -\frac{a^2}{b}$$
and since $b^2 = a(a^2 - 1)$, this can be rewritten as
$$\alpha\,|_{U_2} \colon (a, b) \mapsto (s, t) = \left(\frac{b}{a^2 - 1}, -\frac{ab}{a^2 - 1}\right).$$

Note that $x, y, 1/(x^2-1) \in \mathcal{O}_C(U_2)$ and the right hand side of the last displayed equation is a polynomial in $a, b, 1/(a^2-1)$, so $\alpha\,|_{U_2}\colon U_2 \to V_2$ is a morphism.

(C) Finally, consider $\alpha\,|_{U_3}\colon U_3 \to V_1$. We have to compute the image of a point with coordinates $(s,t) = (c,d)$ in U_3 in terms of the coordinates x, y on V_1: since

$$x = \frac{s}{t}, \quad y = t^{-1}$$

we get

$$x(\alpha(s,t)) = -\frac{d}{c} \quad y(\alpha(s,t)) = -\frac{d}{c^2}$$

and since $d = c(c^2 - d^2)$

$$\alpha\,|_{U_3}\colon (s,t) = (c,d) \mapsto (x,y) = \left(d^2 - c^2, d(c^2-d^2) - c\right).$$

The right hand side are polynomials in c, d and $s, t \in \mathcal{O}_C(U_3)$, so $\alpha\,|_{U_3}\colon U_3 \to V_1$ is a morphism.

Hence α is a morphism itself.

This example shows that, because the requirement of being a morphism is local, checking this in practice may require passing to suitable covers by open sets and finding formulas for the map locally in terms of rational functions of the local coordinates that are suitably regular in the open sets under consideration.

We will see later that there are instances where it is easier to check that something is a morphism differently.

Chapter 3

Hilbert's Nullstellensatz, primary decomposition and geometric applications

To understand varieties, their morphisms and regular functions better, we need a little more background in commutative algebra.

An affine subvariety $X \subset \mathbb{A}^n$ in $\mathbb{A}^n$ is given by the zero set of a family of polynomials $(f_\alpha)_{\alpha \in A}$, $f \in A := k[x_1, \ldots, x_n]$. We write I for the ideal generated by the f_α in A and $X = V(I)$, V for "Verschwindungsmenge", the German for zero set, or vanishing set. For any ideal $J \subset A$, we thus use the notation

$$V(J) := \{(x_1, \ldots, x_n) \in \mathbb{A}^n \mid f(x_1, \ldots, x_n) = 0 \;\forall\, f \in I\}.$$

Definition 3.1. We write

$$I(X) = \{g \in A \mid g(x_1, \ldots, x_n) = 0 \;\forall (x_1, \ldots, x_n) \in X\}$$

for the ideal of all polynomials vanishing on X. We call it the ideal of X for short.

Theorem 3.2 (Hilbert's Nullstellensatz)**.** *We have*

$$I(V(I)) = \sqrt{I} := \{f \in A \mid \exists\, n \colon f^n \in I\}.$$

Here $\sqrt{I}$ is called the radical of the ideal I.

Proof. It is clear that $\sqrt{I} \subset I(V(I))$, so it suffices to prove the opposite inclusion.

Step 1. We prove that if $V(I) = \emptyset$, then $I = A$. Suppose by contradiction that $I \subsetneq A$. Then $I \subset \mathfrak{m}$ for some maximal ideal in A. Now Step 1 will be complete once we can show that every maximal ideal of A is of the form

$$\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$$

for some $a_i \in k$. Indeed, then $V(I)$ will contain the point $(a_1, \ldots, a_n)$, hence not be empty. Now the assertion that every maximal ideal is of the form above is equivalent to the claim that the quotient field

$$L = k[x_1, \ldots, x_n]/\mathfrak{m}$$

is isomorphic to k via the inclusion $k \subset L$, and since k is algebraically closed, this is in turn equivalent to showing that L is algebraic over k. We can assume that $x_1, \ldots, x_l \in L$ are algebraically independent over k and that $x_{l+1}, \ldots, x_n$ are algebraic over $k(x_1, \ldots, x_l) \subset L$.

Lemma 3.3. *Let R be a Noetherian ring, $S \supset R$ a finitely generated R-algebra. If $T \subset S$ is an R-algebra such that S is a finitely T-module, then T is a finitely generated R-algebra.*

Let us first indicate how the Lemma allows us to finish the proof of Step 1. Apply it with $R = k$, $S = L$, $T = k(x_1, \ldots, x_l)$. The hypotheses are then satisfied, hence we would get that $k(x_1, \ldots, x_l)$ is a finitely generated k-algebra, which is absurd unless $l = 0$: if rational functions $z_1, \ldots, z_N$

$$z_i = \frac{P_i(x_1, \ldots, x_l)}{Q_i(x_1, \ldots, x_l)}$$

were generators, and if $f \in k[x_1, \ldots, x_l]$ is irreducible, then $1/f$ must be a polynomial in the z_i with coefficients in k. But then f would have to divide one of the Q_i since $A = k[x_1, \ldots, x_l]$ is a UFD, leading to the absurd conclusion that there are only finitely many (monic) irreducible polynomials in A.

Proof. (of Lemma 3.3) Let $\xi_1, \ldots, \xi_p$ be R-algebra generators of S and let

$s_1, \ldots, s_q$ be T-module generators of S. Thus $\exists\, t_{\iota\kappa},\ t'_{\iota\kappa\lambda} \in T$ such that

$$\xi_\iota = \sum_\kappa t_{\iota\kappa} s_\kappa \tag{3.1}$$

$$s_\iota \cdot s_\kappa = \sum_\lambda t'_{\iota\kappa\lambda} s_\lambda. \tag{3.2}$$

Let T_0 be the R-sub-algebra of T which is generated by the $t_{\iota\kappa},\ t'_{\iota\kappa\lambda}$ over R. Then S is finitely generated as a T_0-module (clear because of formulas (3.1) and (3.2)), and then T is finitely generated as a T_0-module: indeed, T_0 is Noetherian since R is, and a sub-module of a finitely generated module over a Noetherian ring is finitely generated. Hence the claim. □

Step 2. Having completed Step 1, let us show how it quickly implies the entire assertion of the Nullstellensatz. We have to show that if $f \in I(V(I))$, then for some integer $m > 0$: $f^m \in I$. Put

$$J := \langle I, x_{n+1} \cdot f(x_1, \ldots, x_n) - 1 \rangle \subset k[x_1, \ldots, x_n, x_{n+1}]$$

(the brackets indicate to take the ideal generated). Then $V(J) = \emptyset$, thus by Step 1, $J = k[x_1, \ldots, x_{n+1}]$. Thus if

$$B := k[x_1, \ldots, x_{n+1}]/(x_{n+1} \cdot f - 1),$$

then $I \cdot B = (1)$, so there is an equation $1 = \sum g_i a_i$ with $g_i \in I$, $a_i \in B$ in B. Hence in the ring B we get an equation

$$1 = h_0 + h_1 x_{n+1} + \cdots + h_m x_{n+1}^m,\ h_i \in I.$$

Thus in the ring $A = k[x_1, \ldots, x_n]$ we have

$$f^m = f^m h_0 + \cdots + h_m$$

and this is in I. □

We can use the above to describe rings of regular functions in several cases more simply:

Corollary 3.4. *We have*

$$\mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n) \simeq k[x_1, \ldots, x_n]$$

and more generally: if $X \subset \mathbb{A}^n$ is a closed subvariety and $I(X)$ is prime (we will shortly interpret this simplifying hypothesis geometrically), then

$$\mathcal{O}_X(X) \simeq k[x_1, \ldots, x_n]/I(X).$$

One customarily calls $k[x_1, \ldots, x_n]/I(X)$ the affine coordinate ring *of X.*

Proof. Suppose $f \in \mathcal{O}_X(X)$. By definition, this means that there is an open cover $\{U_\alpha\}$ of X such that

$$f \mid_{U_\alpha} = \frac{h_\alpha}{k_\alpha} \mid_{U_\alpha}, \quad k_\alpha \neq 0 \text{ on } U_\alpha,\ h_\alpha, k_\alpha \in k[x_1, \ldots, x_n].$$

Since every Zariski open set on X is a union of the sets $U_g = \{x \in X \mid g(x) \neq 0\}$, $g \in k[x_1, \ldots, x_n]$, and since the fact that a set of U_{g_α} covers is equivalent, by the Nullstellensatz, to the ideal generated by the g_α being the unit ideal in $k[x_1, \ldots, x_n]/I(X)$, we can assume that $U_\alpha = U_{g_\alpha}$, some *finitely* many $g_\alpha \in k[x_1, \ldots, x_n]$. So $\alpha \in A$, A an index set with $|A| < \infty$.

The ideal (k_α) has no zero on X, hence, by the Nullstellensatz again, $(k_\alpha, I(X)) = (1)$. Thus there is an equation (which can be viewed as an algebraic partition of unity, if you are familiar with partitions of unity from Differential Topology)

$$1 = \sum_{\alpha \in A} l_\alpha k_\alpha \text{ in } k[x_1, \ldots, x_n]/I(X).$$

Put $P_f := \sum_\alpha l_\alpha h_\alpha$. Then

$$k_\beta P_f = \sum_{\alpha \in A} l_\alpha h_\alpha k_\beta = \sum_{\alpha \in A} l_\alpha h_\beta k_\alpha = h_\beta$$

on U_β, since

$$\frac{h_\alpha}{k_\alpha} = \frac{h_\beta}{k_\beta} \text{ in } \mathrm{Quot}(k[x_1, \ldots, x_n]/I(X)),$$

where $\mathrm{Quot}(k[x_1, \ldots, x_n]/I(X))$ is the quotient field of $k[x_1, \ldots, x_n]/I(X)$. Namely, $h_\alpha k_\beta - h_\beta k_\alpha$ vanishes on the open $U_\alpha \cap U_\beta$ in X, hence is in $I(X)$, since $I(X)$ is prime. $\square$

Remark 3.5. The statement of Corollary 3.4 also holds without the assumption that $I(X)$ is prime, but the proof is a bit messier then and we omit it.

Corollary 3.6. *We have* $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$.

Proof. Because of Corollary 3.4, we have in $\mathbb{A}^n \simeq U_0 = \{X_0 \neq 0\} \subset \mathbb{P}^n$ that an $f \in \mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n)$ can be written as

$$f = p\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right),\ p \in k[x_1, \ldots, x_n].$$

As a rational function in $k(X_0, \ldots, X_n)$, f is the quotient r/s of two homogeneous polynomials r, s of the same degree with the denominator not vanishing identically on $X_0 = 0$; hence

$$rX_0^{\deg(p)} = X_0^{\deg(p)} p\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right) s,$$

but X_0 does not divide s and $X_0^{\deg(p)}$ divides the polynomial

$$X_0^{\deg(p)} p\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right)$$

only when p is constant. □

The same method of proof even shows that $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n - H_1 \cap H_2) = k$ where H_1, H_2 are two different hyperplanes in $\mathbb{P}^n$.

Corollary 3.7. *Let* $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ *be closed affine subvarieties. Every morphism* $f \colon X \to Y$ *is of the form*

$$f = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

with $f_i \in k[x_1, \ldots, x_n]$.

Proof. This follows from Corollary 3.4 (and Remark 3.5) since $T_j \circ f$, T_j the j-th coordinate function on $\mathbb{A}^m$, is regular on X. □

Theorem 3.8. *Every radical ideal* $I \subset A = k[x_1, \ldots, x_n]$ *is a finite intersection of prime ideals* $\mathfrak{p}_i$ *with* $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ *for* $i \neq j$*, unique up to reordering.*

Proof. Let I_0 be a maximal element in the family of all radical ideals $I \subset A$ which are not a finite intersection of prime ideals.Then I_0 is clearly not prime itself. Pick $a, b \in A$ with $a, b \notin I_0$ such that $ab \in I_0$. Put

$$I_1 = \sqrt{(I_0, a)}, \quad I_2 = \sqrt{(I_0, b)}.$$

By construction, I_1, I_2 are finite intersections of prime ideals. But $I_0 = I_1 \cap I_2$, contradiction. (To see that really $I_0 = I_1 \cap I_2$, take $f \in I_1 \cap I_2$; then there are integers $m, n > 0$ with $f^m \in (I_0, a)$ and $f^n \in (I_0, b)$ whence $f^{m+n} \in I_0$ because $ab \in I_0$. Consequently, $f \in I_0$).

Thus we have proved existence of a representation of any radical ideal as an intersection of prime ideals; uniqueness is easy under the above irredundancy hypothesis: if

$$I = \bigcap_i \mathfrak{p}_i = \bigcap_j \mathfrak{q}_j,$$

then clearly for all i we have $\mathfrak{p}_i \supset \bigcap_j \mathfrak{q}_j$. Then $\mathfrak{q}_k \subset \mathfrak{p}_i$ for some k. Vice versa, we also have $\mathfrak{p}_l \subset \mathfrak{q}_k$ for some l. By irredundancy, $\mathfrak{p}_l = \mathfrak{p}_i = \mathfrak{q}_k$. So there is a one-one correspondence between the $\mathfrak{p}$'s and $\mathfrak{q}$'s. $\square$

Definition 3.9. A variety X is called **irreducible** if for all closed subvarieties $Y, Z \subset X$ with $X = Y \cup Z$, we have $Y = X$ or $Z = X$. Otherwise, X is called **reducible**.

Remark 3.10. An affine subvariety $X \subset \mathbb{A}^n$ is irreducible if and only if $I(X)$ is prime. Indeed, if $Y \subsetneq X$ and $Z \subsetneq X$ are proper subvarieties, there is an $f \in I(Y)$, $f \notin I(X)$ and there is a $g \in I(Z)$, $g \notin I(X)$. If $X = Y \cup Z$, then $fg \in I(X)$, so $I(X)$ is not prime. And conversely, if f, g are some elements in $k[x_1, \dots, x_n]$ with $fg \in I(X)$, then $X = V(I(X), f) \cup V(I(X), g)$. So if $I(X)$ is not prime, then X is reducible.

The preceding remark holds also for projective subvarieties $X \subset \mathbb{P}^n$ with the following adjustments: first, putting $S = k[x_0, \dots, x_n]$, a graded ring, we can consider

$$I(X) := \{F \in S \mid F \equiv 0 \text{ on } X\}.$$

This is a *homogeneous* ideal in $S = \bigoplus_{m \geq 0} S_m$, which means that if a is in I, all its homogeneous components with respect to the direct sum decomposition of S are. We call $I(X)$ the homogeneous ideal of X. Then

$$S/I(X)$$

is another graded ring, the so-called *homogeneous coordinate ring* of X. It is easy to see that X is irreducible if and only if $I(X)$ is prime (with the same proof).

Because of Theorem 3.8 we get immediately

Theorem 3.11. *Every (possibly reducible) variety X is a finite union of irreducible subvarieties X_i with $X_i \subsetneq X_j$ for $i \neq j$, and in a unique way (up to reordering). The X_i are called the* irreducible components *of X.*

This is a simply the geometric translation of Theorem 3.8, but to be sure that everything works in the projective case as well one has to remark: if $I \subset S$ is homogeneous and radical, and

$$I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$$

a representation as an intersection of primes, then all the $\mathfrak{p}_i$ are homogeneous ideals as well. For, I being homogeneous is equivalent to

$$I = I^\lambda := \{f(\lambda x_0, \dots, \lambda x_n) \mid f \in I\}$$

for infinitely many $\lambda \in k^*$. This implies that for every i there is a j with $\mathfrak{p}_i = \mathfrak{p}_j^\lambda$ for infinitely many $\lambda = \lambda_1, \lambda_2, \dots$ since applying $(-)^\lambda$ gives another representation of I as an irredundant intersection of primes, and it is unique up to reordering. But then $\mathfrak{p}_i^{\lambda_j \lambda_1^{-1}} = \mathfrak{p}_i$ for all j, hence $\mathfrak{p}_i$ is homogeneous.

Remark 3.12. As an exercise, one should formulate and prove an extension of the Nullstellensatz for projective varieties now. Moreover, one should verify that

$$\mathcal{O}_{\mathbb{A}^n}(U_f) = k[x_1, \dots, x_n]_f \text{ (localization)},$$
$$\mathcal{O}_{\mathbb{P}^n}(U_F) = k[x_0, \dots, x_n]_{(F)} \text{ (homogeneouslocalization)}$$

where $f \in k[x_1, \dots, x_n]$ is a polynomial, $F \in k[x_0, \dots, x_n]$, and $U_f \subset \mathbb{A}^n$ resp. $U_F \subset \mathbb{P}^n$ are the *principal open subsets* where f resp. F do not vanish. The same holds for any affine resp. projective subvarieties.

Corollary 3.7 gives a nice characterization of morphisms between affine subvarieties. We can do something similar for projective subvarieties.

Theorem 3.13. *Let $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ be projective subvarieties, and $f\colon X \to Y$ a morphism. Then there is a $N \times (m+1)$ matrix $A = (F_{ij})$ of homogeneous polynomials $F_{ij} \in k[X_0, \dots, X_n]$ such that $\deg F_{ij}$ is constant in each row,*

$$\mathrm{rk}(A(x)) = 1 \quad \forall\, [x] \in X,$$

$\mathbb{P}(\mathrm{Im}(A(x)^t)) \in Y$ for all $[x] \in X$, and $f([x]) = \mathbb{P}(\mathrm{Im}(A(x)^t))$ for all $[x] \in X$. Conversely, every such matrix determines a morphism $X \to Y$ in the preceding manner.

Proof. Denote homogeneous coordinates in the target $\mathbb{P}^m$ by $(T_0 : \cdots : T_m)$ and $U_i = \{T_i \neq 0\} \simeq \mathbb{A}^m \subset \mathbb{P}^m$. The sets $V_i := f^{-1}(U_i)$ are open and cover X. We can write

$$V_i = \bigcup_{j=1}^{r} X_{G_{ij}}$$

$G_{ij} \in k[X_0, \dots, X_n]$ homogeneous, $X_{G_{ij}}$ principal open. Then $f(X_{G_{ij}}) \subset U_i \cap Y \subset U_i \simeq \mathbb{A}^m$, whence

$$f\mid_{X_{G_{ij}}} = \left(\frac{P_0(X_0, \dots, X_n)}{G_{ij}^{\mu_0}} : \cdots : 1 : \cdots : \frac{P_m(X_0, \dots, X_n)}{G_{ij}^{\mu_m}} \right) \tag{3.3}$$

with $\deg P_k = \deg(G_{ij}) \cdot \mu_k$ and 1 is in the i-th position. This is so because

$$\mathcal{O}_X(X_{G_{ij}}) = (k[X_0, \dots, X_n]/I(X))_{(G_{ij})}.$$

Clearing denominators in (3.3), we get one matrix row for A. Doing this for all i, j, we get A. $\square$

Example 3.14. Let us illustrate the preceding Theorem 3.13 in one geometric situation. Suppose $C \subset \mathbb{P}^2$ is given by

$$X^2 + Y^2 - Z^2 = 0.$$

Then: the line $Y - Z = 0$ intersects C in one point, namely $P = (0 : 1 : 1)$, the line $Y + Z = 0$ intersects C in one point $(0 : -1 : 1)$, and we want to consider the stereographic projection of that conic from P onto the line $Y = 0$:

$$f \colon C - \{(0 : 1 : 1)\} \to \mathbb{P}^1$$
$$(X : Y : Z) \mapsto (X : Z - Y).$$

Thus, geometrically, we construct the line through P and a point $R \in C - \{P\}$ and let $f(R)$ be the intersection of that line with the line $Y = 0$. Now define $\tilde{f}$ by f on $C - \{P\}$ and $\tilde{f}(P) = (1 : 0)$ (the point at infinity on $\mathbb{P}^1 \simeq \{Y = 0\}$. Then $\tilde{f}$ is a morphism, which can be seen as follows: let $(S : T)$ be homogeneous coordinates on $\mathbb{P}^1$, with $U_S = \{S \neq 0\} \simeq \mathbb{A}^1$, $U_T = \{T \neq 0\} \simeq \mathbb{A}^1$. Then

$$\tilde{f}^{-1}(U_S) = C - \{(0 : 1 : -1)\},$$
$$\tilde{f}^{-1}(U_T) = C - \{(0 : 1 : 1)\}.$$

Then $\tilde{f}$ is regular on $\tilde{f}^{-1}(U_T)$, and on $\tilde{f}^{-1}(U_S)$ $\tilde{f}$ can be given by

$$(X : Y : Z) \mapsto (Y + Z : X).$$

The matrix

$$A = \begin{pmatrix} X & Z - Y \\ Y + Z & X \end{pmatrix}$$

has rank 1 everywhere on C, and is a matrix defining $\tilde{f}$ in the sense of Theorem 3.13.

Now one can show that in this case

$$\tilde{f} \colon C \to \mathbb{P}^1$$

cannot be defined by homogeneous polynomials $F_0, F_1 \in k[X_0, X_1, X_2]$ of the same degree *without common zeroes* on C. I.e., we cannot find a 1×2 matrix as in Theorem 3.13 defining $\tilde{f}$. To see this, assume by contradiction that such F_0, F_1 would exist. Then note that

$$\tau \colon \mathbb{A}^1 \simeq C - \{(0 : -1 : 1)\}$$

via the inverse of the projection onto $\{Y = 0\}$ from $(0 : -1 : 1)$. We have $\tau(t) = (2t : -t^2 + 1 : t^2 + 1)$. Now

$$\frac{X}{Z - Y} \text{ and } \frac{F_0}{F_1}$$

define the same rational function in $k(t)$ on $\mathbb{A}^1$ via τ since $XF_1 - F_0(Z - Y) = 0$ on C. Now F_1 vanishes in $(0 : 1 : 1)$, but F_0 does not, since $(0 : 1 : 1)$ maps to $(1 : 0)$. Now

$$\tau^* \left(\frac{X}{Z - Y} \right) = \frac{1}{t}$$

has a simple pole in $0 \in \mathbb{A}^1$. Hence the same must hold for

$$\tau^* \left(\frac{F_0}{F_1} \right) = \frac{F_0(2t, -t^2 + 1, t^2 + 1)}{F_1(2t, -t^2 + 1, t^2 + 1)}.$$

Consequently, $F_1(2t, -t^2 + 1, t^2 + 1)$ is a polynomial in $k[t]$ with a simple zero at 0 and no further zero, since zeroes of $F_1(2t, -t^2 + 1, t^2 + 1)$ correspond bijectively to zeroes of $F_1(X,Y,Z)$ on $C - \{(0:-1:1)\}$ and $(1:0)$ has only one preimage under $\tilde{f}$ there. Thus

$$F_1(2t, -t^2+1, t^2+1) = ct, \quad c \in k^*$$

and then $F_1(X,Y,Z) - (1/2)X$ is a polynomial vanishing identically on C since $\tau^*(F_1(X,Y,Z) - (1/2)X) = 0$ in $k[t]$. So $F_1(X,Y,Z)$ has the further zero $(0:-1:1)$ on C, but this must also a zero of $F_0(X,Y,Z)$ since $(0:-1:1) \mapsto (0:1)$ under $\tilde{f}$. Hence F_0, F_1 do have a common zero on C, contradicting our assumption.

Besides morphism and isomorphisms, there is another type of "map" and equivalence in algebraic geometry that is extremely important:

Definition 3.15. Let X and Y be irreducible varieties. A **rational map** $f\colon X \dashrightarrow Y$ is given by an equivalence class of pairs (f_U, U) where $U \subset X$ is Zariski open and nonempty (hence dense) and $f_U\colon U \to Y$ is a morphism. Two pairs (f_U, U) and (f_V, V) are considered equivalent if $(f_U)\mid_{U\cap V} = (f_V)\mid_{U\cap V}$. A **rational function** on X is a rational map to $\mathbb{A}^1$. A rational map f is called **dominant** if the image of f_U is dense in Y for any representative f_U of f. Varieties X, Y are called **birational** if there exist dominant rational maps

$$f\colon X \dashrightarrow Y, \quad g\colon Y \dashrightarrow X$$

such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ as rational maps.

Thus a rational map f as above need not be everywhere defined on X, which we indicate by the dotted arrow above. Hence it is not a "map", but this is consistent with the "red herring principle" in mathematical terminology, which means that a red herring need neither be red nor a herring in mathematics. Anyway, varieties with dominant rational maps do form a category, so everything is perfectly fine.

For example, $\mathbb{A}^2$ and $\mathbb{P}^2$ are birational, but not isomorphic (for example because $\mathbb{P}^2$ has no nonconstant global regular functions whereas $\mathbb{A}^2$ has lots of them).

Chapter 4

Segre embeddings, Veronese maps and products

If $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are affine subvarieties, clearly $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m \simeq \mathbb{A}^{n+m}$ is an affine subvariety. (But note that the product topology on $X \times Y$ is in general strictly **weaker than the Zariski topology**!) What about the same question for projective subvarieties $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$? The first question is how to turn the *set* $\mathbb{P}^n \times \mathbb{P}^m$ into a projective variety.

Definition 4.1. We define a topology on the set $\mathbb{P}^n \times \mathbb{P}^m$, which we call the Zariski topology, in the following way: its closed sets are zero sets $V(f_1, \dots, f_N)$ in $\mathbb{P}^n \times \mathbb{P}^m$ of polynomials $f_i \in k[X_0, \dots, X_n, Y_0, \dots, Y_m]$ which are homogeneous in the X's and Y's separately, i.e.

$$f_i = \sum_{a_0+\dots+a_n=d, b_0+\dots+b_m=e} c_{a_0,\dots,a_n;b_0,\dots,b_m} X_0^{a_0} \dots X_n^{a_n} Y_0^{b_0} \dots Y_n^{b_n}.$$

Thus f_i is "bi-homogeneous" of bi-degree (d, e).

Theorem 4.2. *Let $(Z_{ij})_{0 \le i \le n, 0 \le j \le m}$ be homogeneous coordinates in $\mathbb{P}^{nm+n+m}$. Let $\mathfrak{a}$ be the homogeneous ideal in $k[Z_{ij}]$ generated by*

$$Z_{ij} Z_{kl} - Z_{il} Z_{kj} \quad \forall\, i, j, kl$$

(so all 2×2-minors of the matrix (Z_{ij})). Then the map

$$\begin{aligned} s_{n,m} \colon \mathbb{P}^n \times \mathbb{P}^m &\to V(\mathfrak{a}) \subset \mathbb{P}^{nm+n+m} \\ ((X_0 : \dots : X_n), (Y_0 : \dots : Y_m)) &\mapsto (Z_{ij}) := (X_i Y_j) \end{aligned}$$

is a homeomorphism and $V(\mathfrak{a})$ *is an irreducible projective subvariety. One calls* $s_{n,m}$ *the* Segre embedding *and* $V(\mathfrak{a}) =: \Sigma_{n,m}$ *the* Segre variety.

Proof. **Step 1.** We show that $s_{n,m}$ is *injective*. Thus suppose that $s_{n,m}(X,Y) = s_{n,m}(X',Y')$. It means that there is a nonzero constant λ such that for all i,j we have $X_iY_j = \lambda X_i'Y_j'$. Moreover, X, Y representing points in projective spaces, there is a tuple of indices i_0, j_0 with $X_{i_0} \neq 0$, $Y_{j_0} \neq 0$. Thus both X_{i_0}' and Y_{i_0}' have to be nonzero as well. Putting

$$\mu := \frac{X_{i_0}}{X_{i_0}'}, \quad \nu := \frac{Y_{j_0}}{Y_{j_0}'}$$

(whence $\lambda = \mu\nu$), we calculate

$$X_iY_{j_0} = \lambda X_i'Y_{j_0}' = \mu\nu X_i'Y_{j_0}' = \mu X_i'Y_{j_0}$$

and thus $X_i = \mu X_i'$ for all i. Thus $X = X'$ in projective space, and analogously we get $Y = Y'$.

Step 2. The map $s_{n,m}$ is *surjective*. Suppose Z_{ij} satisfy $Z_{ij}Z_{kl} = Z_{il}Z_{kj}$ and not all Z's are zero, $Z_{i_0j_0} \neq 0$, say. Put

$$X_i := \frac{Z_{ij_0}}{Z_{i_0j_0}}, \quad Y_j = \frac{Z_{i_0j}}{Z_{i_0j_0}}.$$

Then

$$(Z_{i_0j_0})^2 X_iY_j = Z_{ij_0}Z_{i_0j} = Z_{ij}Z_{i_0j_0}$$

which means $s(X,Y) = [(Z_{ij})]$.

Step 3. The map $s_{n,m}$ is a *homeomorphism*. Indeed, the topology we put on $\mathbb{P}^n \times \mathbb{P}^m$ has a basis consisting of the sets

$$\{(X,Y) \mid f(X,Y) \neq 0,\ f \text{ bihomogeneous of bidegree } (d,d)\}.$$

For, by definition, the topology has a basis consisting of

$$\{(X,Y) \mid f(X,Y) \neq 0,\ f \text{ bihomogeneous of bidegree } (d,e)\}.$$

Suppose $d \geq e$ (the opposite case being similar). Then

$$\{(X,Y) \mid f(X,Y) \neq 0\} = \bigcup_{j=0}^{m} \left\{(X,Y) \mid (Y_j^{d-e}f)(X,Y) \neq 0\right\}$$

and $Y_j^{d-e} f$ has bidegree (d, d).

Now every polynomial $f(X, Y)$ which is bihomogeneous of bidegree (d, d) can be written as $F(\dots, X_i Y_j, \dots)$ where F is a homogeneous polynomial in the Z_{ij} of degree d. That means that the open set $f \neq 0$ in $\mathbb{P}^n \times \mathbb{P}^m$ gets mapped to the open set $V(\mathfrak{a}) \cap (F \neq 0)$. Thus $s_{n,m}$ is a homeomorphism.

Step 4. The image $V(\mathfrak{a})$ is irreducible. This follows from a general topological Lemma.

Lemma 4.3. *Let X, Y be irreducible topological spaces. If on $X \times Y$ a topology is given which induces the given topology on $\{x\} \times Y \simeq Y$ and $X \times \{y\} \simeq X$ for all $x \in X$ and all $y \in Y$, then $X \times Y$ is irreducible.*

Proof. Let $X \times Y = S \cup T$ be a decomposition into closed subsets. For all $x \in X$ we have

$$\{x\} \times Y = [S \cap (\{x\} \times Y)] \cup [T \cap (\{x\} \times Y)] \,.$$

Since Y is irreducible, it follows that for all x we have $\{x\} \times Y \subset S$ or $\{x\} \times Y \subset T$. Let s_y be the map $s_y \colon X \to X \times Y$ given by $s_y(x) = (x, y)$. Then

$$S' := \bigcap_{y \in Y} s_y^{-1}(S) = \{x \mid (x, y) \in S \; \forall\, y\} = \{x \mid \{x\} \times Y \subset S\}.$$

Clearly, S' is also closed. Similarly,

$$T' := \bigcap_{y \in Y} s_y^{-1}(T) = \{x \mid \{x\} \times Y \subset T\}$$

is closed. Since we saw that $X = S' \cup T'$, the irreducibility of X implies $X = S'$ or $X = T'$. Thus $X \times Y = S$ or $X \times Y = T$. $\square$

Thus $\Sigma_{n,m} = V(\mathfrak{a})$ is irreducible and this concludes the proof of Theorem 4.2. $\square$

We now give $\mathbb{P}^n \times \mathbb{P}^m$ the structure of a projective variety, and every product $X \times Y$ of (quasi-)projective varieties the structure of (quasi-)projective variety that comes from identifying it with the image under $s_{n,m}$.

Example 4.4. The variety $\Sigma_{1,1} = s_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^3$ is the image of the map

$$((X_0 : X_1), (Y_0 : Y_1)) \mapsto (X_0Y_0 : X_0Y_1 : X_1Y_0 : X_1Y_1) =: (Z_0 : Z_1 : Z_2 : Z_3)$$

i.e., the quadric

$$\det \begin{pmatrix} Z_0 & Z_1 \\ Z_2 & Z_3 \end{pmatrix} = 0$$

in $\mathbb{P}^3$. In general, $s_{n,m}$ maps each $\mathbb{P}^n \times \{y\}$ and $\{x\} \times \mathbb{P}^m$ onto projective linear subspaces of $\mathbb{P}^{(n+1)(m+1)-1}$; in this case this gives two families of lines on $\Sigma_{1,1}$, namely

$$V(Z_1 = \lambda Z_0,\ Z_3 = \lambda Z_2)_\lambda \in \mathbb{P}^1,$$
$$V(Z_2 = \lambda' Z_0,\ Z_3 = \lambda' Z_1)_{\lambda' \in \mathbb{P}^1}.$$

These are the loci where either the rows or columns of the above matrix satisfy a linear dependency relation.

In fact, any line $L \subset \mathbb{P}^3$ on $\Sigma_{1,1}$ belongs to one of these two families: suppose we write L as the image of a map $\mathbb{P}^1 \to \mathbb{P}^3$

$$(\lambda : \mu) \mapsto (l_1(\lambda, \mu) : \cdots : l_4(\lambda, \mu))$$

where the $l_i(\lambda, \mu)$ are linear forms. The condition that L lie on $\Sigma_{1,1}$ means that

$$\det \begin{pmatrix} l_1(\lambda, \mu) & l_2(\lambda, \mu) \\ l_3(\lambda, \mu) & l_4(\lambda, \mu) \end{pmatrix} \equiv 0.$$

Without loss of generality, after possibly interchanging rows or columns and transposing, we may assume $l_1 \neq 0$ and l_1, l_2 linearly independent. Then $l_1 l_4 = l_2 l_3$ implies that $l_1 \mid l_3$ and $l_2 \mid l_4$, hence $l_3 = cl_1$ and $l_4 = cl_2$, some $c \in k$.

Remark 4.5. In coordinate free form, the Segre map can be given as

$$s\colon \mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes W)$$
$$[v] \times [w] \mapsto [v \otimes w].$$

Remark 4.6. The product $X \times Y$ of varieties X, Y as we defined it is the categorical product of varieties. This means that the projections $\mathrm{pr}_X\colon X \times Y \to X$ and $\mathrm{pr}_Y\colon X \times Y \to Y$ are morphisms, and for all varieties Z and morphisms $\alpha\colon Z \to X$, $\beta\colon Z \to Y$ there is a unique morphism $\alpha \times \beta\colon Z \to X \times Y$ such that

$$\begin{tikzcd}
 & Z \arrow[d, "\alpha\times\beta"] \arrow[ddl, bend right, "\alpha"'] \arrow[ddr, bend left, "\beta"] & \\
 & X \times Y \arrow[dl, "\mathrm{pr}_X"] \arrow[dr, "\mathrm{pr}_Y"] & \\
X & & Y
\end{tikzcd}$$

commutes. To see this note that purely set-theoretically it is clear how we have to define $\alpha \times \beta$ if it is to exist at all. Then we just have to check that $\alpha \times \beta$ is a morphism. To see this suppose $r_0 \in Z$ is a point with $\alpha(r_0) = p$, $\beta(r_0) = q$. Suppose p is in $X_0 \neq 0$ in $\mathbb{P}^n$. Then, in a neighborhood of r_0 the map α can be given by

$$r \mapsto (1 : f_1(r) : \dots : f_n(r))$$

with the f_i regular in a neighborhood of r_0. After possibly renumbering coordinates Y_i in the target $\mathbb{P}^m$ we can also assume that β is locally around r_0 given by

$$\beta(r) = (1 : g_1(r) : \dots : g_m(r))$$

with the g's regular in a neighborhood of r_0. Then $\alpha \times \beta$ is, locally around r_0, given by

$$r \mapsto (1 : \dots : f_i(r) : \dots : g_j(r) : \dots : f_i(r)g_j(r) : \dots) \in \mathbb{P}^{nm+n+m},$$

which is regular.

Remark 4.7. If $f\colon X \to Y$ is a morphism between projective varieties $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$, then

$$\Gamma_f := \{(x, f(x)) \mid x \in X\} \subset X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m$$

is a (closed) subvariety. Indeed, by the preceding Remark 4.6, the map $f \times \mathrm{id}_Y\colon X \times Y \to Y \times Y$ is a morphism, and

$$\Gamma_f = (f \times \mathrm{id}_Y)^{-1}(\Delta_Y)$$

where $\Delta_Y \subset Y \times Y$ is the diagonal. But $\Delta_Y = \Delta_{\mathbb{P}^m} \cap (Y \times Y)$ is closed because $\Delta_{\mathbb{P}^m}$ can be defined by $X_iY_j - X_jY_i = 0$.

We turn to the subject of *Veronese embeddings*. Note that the homogeneous polynomials F of degree m in the variables $X_0, \dots, X_n$ form a vector space of dimension

$$\binom{m+n}{m}.$$

This can be seen by the following count: decompositions

$$\alpha_0 + \cdots + \alpha_n = m$$

of m into $n+1$ nonnegative integers α_i correspond bijectively to subsets of n elements in $\{1, \dots, m+n\}$ via

$$1 \leq \alpha_0 + 1 < \alpha_0 + \alpha_1 + 2 < \cdots < \alpha_0 + \cdots + \alpha_{n-1} + n \leq m + n.$$

Now put

$$N := \binom{m+n}{m} - 1$$

and for each decomposition into nonnegative integers $i_0 + \cdots + i_n = m$, introduce a symbol $v_{i_0 \dots i_n}$, and consider these as homogeneous coordinates in $\mathbb{P}^N$. Then the m-th **Veronese map** is the map

$$v_m \colon \mathbb{P}^n \to \mathbb{P}^N$$
$$v_{i_0 \dots i_n} := u_0^{i_0} \dots u_n^{i_n}$$

where $u_0, \dots, u_n$ are homogeneous coordinates in $\mathbb{P}^n$. This is visibly a morphism. The image $v_m(\mathbb{P}^n) \subset \mathbb{P}^N$ is called **Veronese variety**. Indeed, this image is a projective subvariety of $\mathbb{P}^N$, which we check as follows:

For all points in $v_m(\mathbb{P}^n)$ we have the relations

$$v_{i_0 \dots i_n} v_{j_0 \dots j_n} = v_{k_0 \dots k_n} v_{l_0 \dots l_n} \; \forall \, i_0 + j_0 = k_0 + l_0, \dots, i_n + j_n = k_n + l_n. \quad (4.1)$$

If (4.1) holds, then we have $v_{0,\dots,0,m,0\dots,0} \neq 0$ for m sitting in some position in the multi-index. If this were not so, we could pick a $v_{l_0 \dots l_n}$ such that the maximum of the indices $l_0, \dots, l_n$, call it $l_{\max}$, is maximal among all indices that occur in the coordinates $v_{j_0 \dots j_n}$ with $v_{j_0 \dots j_n} \neq 0$, and such that $1 \leq l_{\max} \leq m - 1$. But this would lead to a contradiction using the equations (4.1), for then there will be another index $\tilde{l}$ among the $l_0, \dots, l_n$ with $1 \le \tilde{l} \le m-1$ and we can write

$$(v_{l_0 \dots l_n})^2 = v_{\dots, l_{\max}+1, \dots, \tilde{l}-1, \dots} v_{\dots, l_{\max}-1, \dots, \tilde{l}+1, \dots}.$$

But then $v_{\dots, l_{\max}+1, \dots, \tilde{l}-1, \dots}$ is nonzero and has greater maximum index!

So we have get a $v_{0,\dots,0,m,0\dots,0} \neq 0$. Suppose for simplicity of notation that the m is in the first place: $v_{m,0,0,\dots} \neq 0$. Then we can set

$$u_0 := v_{m,0,0,\dots}, \quad u_i := v_{m-1,0,\dots,0,1,0,\dots}, \; i \ge 2$$

where the 1 in the formula for u_i is in the i-th position. This defines a regular inverse map to v_m on the open set $v_{m,0,\dots} \neq 0$. Thus we conclude that (1) $v_m(\mathbb{P}^n)$ is defined by the equations (4.1) and the map $v_m \colon \mathbb{P}^n \to v_m(\mathbb{P}^n) \subset \mathbb{P}^N$ is an isomorphism onto the image.

For $n = 1$, we get rational normal curves $v_m(\mathbb{P}^1) \subset \mathbb{P}^N$. The image of $v_2 \colon \mathbb{P}^2 \to \mathbb{P}^5$ is classically called the *Veronese surface*.

Of course we can also aplly v_m to any subvariety $X \subset \mathbb{P}^n$ and get a subvariety $v_m(X) \subset \mathbb{P}^N$. X and $v_m(X)$ are then isomorphic as varieties. This leads to the following somewhat surprising result.

Theorem 4.8. *Every projective variety X is isomorphic to an intersection of a Veronese variety with a linear space. In particular, every projective variety is isomorphic to an intersection of quadrics.*

Proof. The zero set of a homogeneous polynomial $F(u_0, \dots, u_n)$ of degree d is the same as the zero set of all polynomials $\{u_i F\}_{i=0,\dots,n}$. Hence we can assume that $X \subset \mathbb{P}^n$ is defined by homogeneous polynomials $F_j(u_0, \dots, u_n) = 0$, all of which have the *same* degree D. This means that F_j is a linear polynomial in the monomials $u_0^{i_0} \dots u_n^{i_n}$, $i_0 + \dots + i_n = D$, hence $v_D(X) \subset \mathbb{P}^N$ is an intersection of $v_D(\mathbb{P}^n)$ with a linear subspace. $\square$

Of course not every X is isomorphic to a *complete intersection* of quadrics, the homogeneous ideal of X in any embedding need not be generated by quadrics. So the algebraic consequences of Theorem 4.8 are not so strong.

Remark 4.9. We can also describe the Veronese map in a coordinate free manner (for $\mathrm{char}(k) = 0$) as

$$\begin{aligned} v_d \colon \mathbb{P}(V) &\to \mathbb{P}(\mathrm{Sym}^d V) \\ [v] &\mapsto [v^d]. \end{aligned}$$

If $\mathrm{char}(k) = p$, the p-powers of linear forms in $\mathbb{P}(\mathrm{Sym}^p V)$ are not a rational normal curve, but lie on a line.

Chapter 5

Grassmannians, flag manifolds, Schubert varieties

We want to make the set $\mathrm{Grass}(p, V)$ of all p-dimensional sub-vector spaces of an n-dimensional k-vector space V into a projective variety (equivalently, the set of all $(p-1)$-dimensional projective linear subspaces of $\mathbb{P}(V)$).

Definition 5.1. We call elements $\omega \in \Lambda^p V$ (the p-th graded piece of the exterior algebra) *p-forms*. We call ω *completely reducible* if there are $v_1, \ldots, v_p \in V$ such that $\omega = v_1 \wedge \cdots \wedge v_p$.

The k-sub-vector space of V

$$\mathrm{Ann}(\omega) = \{v \in V \mid v \wedge \omega = 0\}$$

is called the *annihilator* of ω.

Theorem 5.2. *Let ω_1 resp. ω_2 be completely reducible $p-$ resp. $q-$forms. Then:*

1.

$$\mathrm{Ann}(\omega_1) \supset \mathrm{Ann}(\omega_2) \iff \exists\, \omega \in \Lambda^{p-q} V : \omega_1 = \omega \wedge \omega_2$$

unless $\omega_2 = 0$, $\omega_1 = e_1 \wedge \cdots \wedge e_n$ for some basis $e_1, \ldots, e_n$ of V.

2.

$$\mathrm{Ann}(\omega_1) \cap \mathrm{Ann}(\omega_2) = \{0\} \iff \omega_1 \wedge \omega_2 \neq 0.$$

3.

$$\mathrm{Ann}(\omega_1) \cap \mathrm{Ann}(\omega_2) = \{0\} \implies \mathrm{Ann}(\omega_1) + \mathrm{Ann}(\omega_2) = \mathrm{Ann}(\omega_1 \wedge \omega_2).$$

Proof. For (1) we note that if $v \wedge \omega_2 = 0$, then $v \wedge \omega \wedge \omega_2 = \pm\omega \wedge v \wedge \omega_2 = 0$, thus $\Longleftarrow$ holds. To prove the converse, consider some completely reducible

$$\tilde{\omega} = v_1 \wedge \cdots \wedge v_p.$$

If $v_1, \ldots, v_p$ are linearly dependent then $\tilde{\omega} = 0$. Suppose that they are linearly independent, $v_1 \wedge \cdots \wedge v_p \neq 0$. Then we will show that

$$\mathrm{Ann}(v_1 \wedge \cdots \wedge v_p) = \langle v_1, \ldots, v_p \rangle. \tag{5.1}$$

What is clear is that $\supset$ holds in the preceding formula. Let $v_1, \ldots, v_p, v_{p+1}, \ldots, v_n$ be a basis of V. Then to show (5.1), it suffices to show

$$\sum_{i=1}^{n} a_i v_i \in \mathrm{Ann}(v_1 \wedge \cdots \wedge v_p) \implies a_i = 0 \; \forall\, i > p.$$

This is clear since

$$0 = \left(\sum_{i=1}^{n} a_i v_i\right) \wedge (v_1 \wedge \cdots \wedge v_p) = \sum_{i=p+1}^{n} a_i v_i \wedge v_2 \wedge \cdots \wedge v_p$$

and $v_i \wedge v_1 \wedge \cdots \wedge v_p$, $p+1 \leq i \leq n$, are linearly independent. Suppose now $\mathrm{Ann}(\omega_1) \supset \mathrm{Ann}(\omega_2)$, $\omega_1 = v_1 \wedge \cdots \wedge v_p$, $\omega_2 = v_1' \wedge \cdots \wedge v_q'$. If $\omega_1 = 0$, the conclusion of (1) is trivial, and if ω_2 is zero, the annihilator of ω_1 is the entire space, hence ω_1 is zero or $\omega_1 = e_1 \wedge \cdots \wedge e_n$ for a basis $e_1, \ldots, e_n$ of V; in the first case, the conclusion of (1) holds trivially, in the second case it holds, too. If $\omega_1 \neq 0$, $\omega_2 \neq 0$, we get by the above considerations

$$\langle v_1, \ldots, v_p \rangle \supset \langle v_1', \ldots, v_q' \rangle.$$

Thus we can choose a basis of $\mathrm{Ann}(\omega_1)$ of the form $(v_1', \ldots, v_q', v_{q+1}', \ldots, v_p')$ whence

$$\omega_1 = \det(A) v_1' \wedge \cdots \wedge v_q' \wedge v_{q+1}' \wedge \ldots v_p'$$

where A is the base change matrix from $(v_1', \ldots, v_p')$ to $(v_1, \ldots v_p)$. Thus there exists $\omega = \pm \det(A) v_{q+1}' \wedge \ldots v_p'$ with $\omega_1 = \omega \wedge \omega_2$.

For (2) and (3), first note that

$$(v_1 \wedge \cdots \wedge v_p) \wedge (v_1' \wedge \cdots \wedge v_q') \neq 0$$

if and only if $v_1, \ldots, v_p, v'_1, \ldots, v'_q$ are linearly independent if and only if

$$\langle v_1, \ldots, v_p \rangle \cap \langle v'_1, \ldots, v'_q \rangle = \{0\}.$$

This shows (2), and (3) follows from what has been proven so far (in particular (5.1)). $\square$

Corollary 5.3. *The map*

$$\mathrm{Ann}\colon \{[\omega] \in \mathbb{P}(\Lambda^p V) \mid \omega \text{ completely reducible}\} \to \mathrm{Grass}(p, V)$$

is bijective.

Proof. Every p-dimensional sub-vector space $W \subset V$ is in the image since if $W = \langle v_1, \ldots, v_p \rangle$, then $W = \mathrm{Ann}(v_1 \wedge \cdots \wedge v_p)$. Theorem 5.2, (1) gives: $\mathrm{Ann}(\omega_1) = \mathrm{Ann}(\omega_1) \implies \omega_1 = \omega \wedge \omega_2$ with $\omega \in \Lambda^0 V = k$. $\square$

Hence the map

$$i_{\mathrm{Pl}} := \mathrm{Ann}^{-1}\colon \mathrm{Grass}(p, V) \to \mathbb{P}(\Lambda^p V)$$

is a bijection onto its image. This map i_{Pl} is called the *Plücker embedding.* If $W = \langle v_1, \ldots, v_p \rangle$ and A the $p \times n$-matrix with rows the coordinates of $v_1, \ldots, v_p$ (with respect to some basis $e_1, \ldots, e_n$ of V), then W gets mapped under i_{Pl} to

$$\sum_{1 \le i_1 < \cdots < i_p \le n} M^{i_1, \ldots, i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}$$

where$M^{i_1, \ldots, i_p}$ is the minor of A belonging to the columns $i_1, \ldots, i_p$. One calls the $(M^{i_0, \ldots, i_p})$ the *Plücker coordinates* of W.

To check that $\mathrm{Grass}(p, V) \subset \mathbb{P}(\Lambda^p V)$ is a projective subvariety, we have to exhibit polynomial equations which characterize the completely reducible p-forms.

Theorem 5.4. *A p-form $\omega \in \Lambda^p V$, $\omega \neq 0$, is completely reducible if and only if* $\dim \mathrm{Ann}(\omega) = p$. *Otherwise we always have* $\dim \mathrm{Ann}(\omega) < p$. *Moreover,* $\mathrm{Grass}(p, V) \subset \mathbb{P}(\Lambda^p V)$ *is a projective variety.*

Proof. From the proof of Theorem 5.2 we know that if $\omega \neq 0$ and ω is completely reducible, then $\dim \mathrm{Ann}(\omega) = p$. Suppose conversely that $\dim \mathrm{Ann}(\omega) =$

r (we will need the case $r = p$). Suppose that $\mathrm{Ann}(\omega) = \langle v_1, \ldots, v_r \rangle$ and that $(v_1, \ldots, v_r, v_{r+1}, \ldots, v_n)$ is a basis of V. Write

$$\omega = \sum_{1 \le i_1 < \cdots < i_p \le n} \omega_{i_1 \ldots i_p} v_{i_1} \wedge \cdots \wedge v_{i_p}.$$

The condition $v_i \wedge \omega = 0$ for all $i = 1, \ldots, r$ translates into $\omega_{i_1 \ldots i_p} = 0$ for $\{1, \ldots, r\} \not\subset \{i_1, \ldots, i_p\}$. This means that for $\omega \neq 0$, we get $r \le p$ and $\omega = (v_1 \wedge \cdots \wedge v_r) \wedge (\text{something})$. For $r = p$, we obtain the claim.

Now this implies that $[\omega] \in \mathrm{Grass}(p, V)$ is equivalent to the rank of the map

$$\varphi(\omega) \colon V \to \Lambda^{p+1} V, \quad v \mapsto \omega \wedge v$$

being $n - p$ (namely, $\ker(\varphi(\omega)) = \mathrm{Ann}(\omega)$). Now $\mathrm{rk}(\varphi(\omega)) < n - p$ cannot happen since always $\dim \mathrm{Ann}(\omega) \le p$. Hence

$$[\omega] \in \mathrm{Grass}(p, V) \iff \mathrm{rk}(\varphi(\omega)) \le n - p. \tag{5.2}$$

The map

$$\Phi \colon \Lambda^p V \to \mathrm{Hom}(V, \Lambda^{p+1} V), \quad \omega \to \varphi(\omega)$$

is k-linear, i.e. the entries of a matrix for $\varphi(\omega)$ are linear in the homogeneous coordinates of $\mathbb{P}(\Lambda^p V)$. The $(n - p + 1) \times (n - p + 1)$-minors of this matrix define $\mathrm{Grass}(p, V)$. □

Remark 5.5. Actually the above equations are easy to produce but they do not generate the homogeneous ideal of $\mathrm{Grass}(p, V)$.

Now consider the special case $p = 2$. We will study this in a little more detail.

Theorem 5.6. *Assume* $\mathrm{char}(k) \neq 2$. *Then an element* $0 \neq \omega \in \Lambda^2 V$ *is completely reducible if and only if* $\omega \wedge \omega = 0$.

Proof. The direction $\implies$ is clear, and for $\impliedby$ we do induction over $n = \dim V$, $n = 2$ being trivial. For the induction step, let $(v_1, \ldots, v_{n+1})$ be a basis of V. Then we can write

$$\omega = v_{n+1} \wedge \omega_1 + \omega_2$$

where ω_1 is a linear combination of v_i, $1 \le i \le n$, and ω_2 is a linear combination of $v_i \wedge v_j$, $1 \le i < j \le n$. Now $\omega \wedge \omega = 0$ translates into

$$\omega_2 \wedge \omega_2 + 2 v_{n+1} \wedge \omega_1 \wedge \omega_2 = 0$$

because $(v_{n+1} \wedge \omega_1) \wedge (v_{n+1} \wedge \omega_1) = 0$ and ω_2 is in the center of $\Lambda^\bullet V$. Since $\omega_2 \wedge \omega_2$ does not contain v_{n+1} in its expansion with respect to basis elements in V, we have separately

$$\omega_2 \wedge \omega_2 = 0 \quad v_{n+1} \wedge \omega_1 \wedge \omega_2 = 0.$$

By the induction hypothesis, this implies that ω_2 is completely reducible. Since $\omega_1 \wedge \omega_2$ does not contain v_{n+1} in its basis expansion, we conclude that even $\omega_1 \wedge \omega_2 = 0$, i.e., $\omega_1 \in \operatorname{Ann}(\omega_2)$. Thus $\dim \operatorname{Ann}(\omega_2) = 2$ (by Theorem 5.4 and because we just proved that it is completely reducible), hence

$$\omega_2 = \omega_1' \wedge \omega_1$$

for some ω_1', unless $\omega_1 = 0$ in which the case the proof is already complete anyway. Then

$$\omega = v_{n+1} \wedge \omega_1 + \omega_1' \wedge \omega_1 = (v_{n+1} + \omega_1') \wedge \omega_1$$

is completely reducible. □

Corollary 5.7. *The variety* $\operatorname{Grass}(2, V) \subset \mathbb{P}(\Lambda^2 V)$*,* $n = \dim V \geq 3$*, is an intersection of quadrics.*

Proof. Indeed, in a basis $e_1, \ldots, e_n$ of V, the condition that

$$\omega = \sum_{1 \leq i_1 < i_2 \leq n} \omega_{i_1 i_2} e_{i_1} \wedge e_{i_2}$$

be completely reducible is, by the Theorem 5.6,

$$\left(\sum_{1 \leq i_1 < i_2 \leq n} \omega_{i_1 i_2} e_{i_1} \wedge e_{i_2} \right) \wedge \left(\sum_{1 \leq j_1 < j_2 \leq n} \omega_{j_1 j_2} e_{j_1} \wedge e_{j_2} \right) = 0.$$

This is equivalent to

$$\sum \omega_{i_1 i_2} \omega_{j_1 j_2} \operatorname{sgn} \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ k_1 & k_2 & k_3 & k_4 \end{pmatrix} = 0.$$

Thus we get one equation for each quadruple of indices $1 \leq k_1 < k_2 < k_3 < k_4 \leq n$ and the sum above then runs over those indices $1 \leq i_1 < i_2 \leq n$ and $1 \leq j_1 < j_2 \leq n$ with $\{i_1, i_2, i_3, i_4\} = \{k_1, k_2, k_3, k_4\}$. With sgn we mean the sign of the permutation

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ k_1 & k_2 & k_3 & k_4 \end{pmatrix}.$$

□

In particular, if $n = 4$, we can consider $\mathrm{Grass}(2, V) \subset \mathbb{P}(\Lambda^2 V) \simeq \mathbb{P}^5$ given by

$$\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} = 0.$$

This quadric is called the *Plücker quadric.* It parametrizes projective lines in $\mathbb{P}^3$.

We can in general consider the subset

$$\Sigma_l(M) \subset \mathrm{Grass}(p, V)$$

where $M \subset V$ is an m-dimensional subspace, and $\Sigma_l(M)$ consists of those p-dimensional $W \subset V$ with $\dim(W \cap M) \geq l$.

Proposition 5.8. *The subset* $\Sigma_l(M)$ *is a subvariety of* $\mathrm{Grass}(p, V) \subset \mathbb{P}(\Lambda^p V)$ *which is a section of* $\mathrm{Grass}(p, V)$ *by a linear subspace of* $\mathbb{P}(\Lambda^p V)$.

Proof. Indeed,

$$\Sigma_l(M) = \{[\omega] \in \mathrm{Grass}(p, V) \mid \omega \wedge v_1 \wedge \cdots \wedge v_{m-l+1} = 0 \; \forall \, v_1, \ldots, v_{m-l+1} \in M\}\,.$$

This means that the span of W and M is at most $p+m-l$-dimensional. $\square$

Customarily, the $\Sigma_l(M)$ are called (special) **Schubert varieties** in $\mathrm{Grass}(p, V)$.

For $1 \leq a_1 < a_2 < \cdots < a_t \leq n$ consider the subset

$$\begin{aligned} \mathrm{Flag}(a_1, \ldots, a_t; V) &= \{(W_1, \ldots, W_t) \mid W_1 \subset \cdots \subset W_t\} \\ &\subset \mathrm{Grass}(a_1, V) \times \cdots \times \mathrm{Grass}(a_t, V) \end{aligned}$$

where the latter product is a projective variety via the Segre embedding. Elements in $\mathrm{Flag}(a_1, \ldots, a_t; V)$ are called *flags of type* $(a_1, \ldots, a_t)$ in V.

Proposition 5.9. *The subset* $\mathrm{Flag}(a_1, \ldots, a_t; V)$ *is a subvariety of* $\mathrm{Grass}(a_1, V) \times \cdots \times \mathrm{Grass}(a_t, V)$.

We call it the **flag variety** of flags of type $(a_1, \ldots, a_t)$ in V.

Proof. If

$$\mathrm{pr}_{ij} \colon \mathrm{Grass}(a_1, V) \times \cdots \times \mathrm{Grass}(a_t, V) \to \mathrm{Grass}(a_i, V) \times \mathrm{Grass}(a_j, V)$$

is the projection, then

$$\mathrm{Flag}(a_1, \dots, a_t; V) = \bigcap_{1 \le i < j \le t} \mathrm{pr}_{ij}^{-1} \left(\mathrm{Flag}(a_i, a_j, V) \right),$$

so it suffices to show that some $\mathrm{Flag}(a_1, a_2, V) \subset \mathrm{Grass}(a_1, V) \times \mathrm{Grass}(a_2, V)$ is closed.

Step 1. We first describe local charts on Grassmannians. For $\mathrm{Grass}(p, V)$, $V \simeq k^n$, consider the sub-vector space $\Gamma = \langle e_{p+1}, \dots, e_n \rangle$ where $e_1, \dots, e_n$ is the standard basis of $V = k^n$. Consider

$$U := \{ W \in \mathrm{Grass}(p, V) \mid W \cap \Gamma = \{0\} \}.$$

If W is spanned by the rows of

$$\begin{pmatrix} w_{11} & \dots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{p,1} & \dots & w_{pn} \end{pmatrix},$$

then $W \in U$ if the minor (=Plücker coordinate of W)

$$\det \begin{pmatrix} w_{11} & \dots & w_{1p} \\ \vdots & \ddots & \vdots \\ w_{p,1} & \dots & w_{pp} \end{pmatrix}$$

is nonzero. Thus $U \subset \mathrm{Grass}(p, V)$ is open in $\mathrm{Grass}(p, V)$ and isomorphic to $\mathbb{A}^{p(n-p)}$; to see the last statement it suffices to consider the map

$$\mathrm{Mat}_k(p \times (n-p)) \simeq \mathbb{A}^{p(n-p)} \to U$$

which associates to a matrix

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,n-p} \\ \vdots & \ddots & \vdots \\ a_{p,1} & \dots & a_{p,n-p} \end{pmatrix}$$

the Plücker coordinates of the subspace W of k^n which is spanned by the rows of the $p \times n$ matrix

$$\begin{pmatrix} 1 & \dots & 0 & a_{1,1} & \dots & a_{1,n-p} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & a_{p,1} & \dots & a_{p,n-p} \end{pmatrix}.$$

If instead of Γ above we take

$$\Gamma_{i_1\dots i_{n-p}} = \langle e_{i_1}, \dots, e_{i_{n-p}}\rangle, \quad 1 \le i_1 < \dots < i_{n-p} \le n,$$

then the construction runs analogously: the subspaces W with $W \cap \Gamma_{i_1\dots i_{n-p}} = \{0\}$ form an open subset $U_{j_1,\dots,j_p} \simeq \mathbb{A}^{p(n-p)}$ of $\mathrm{Grass}(p, V)$, where $(j_1, \dots, j_p)$ are the indices complementary to $(i_1, \dots, i_{n-p})$ in $\{1, \dots, n\}$, and $U_{j_1,\dots,j_p}$ is then given by

$$\det \begin{pmatrix} w_{1j_1} & \dots & w_{1j_p} \\ \vdots & \ddots & \vdots \\ w_{p,j_1} & \dots & w_{pj_p} \end{pmatrix} \neq 0.$$

The $U_{j_1,\dots,j_p}$ form a cover of $\mathrm{Grass}(p, V)$ by affine spaces.

Step 2. Consider now

$$\mathrm{Flag}(a_1, a_2, V) \subset \mathrm{Grass}(a_1, V) \times \mathrm{Grass}(a_2, V) \subset \mathbb{P}(\Lambda^{a_1} V) \times \mathbb{P}(\Lambda^{a_2} V).$$

The open sets

$$U_{j_1\dots j_{a_1}} \times U_{l_1\dots l_{a_2}} \simeq \mathbb{A}^{a_1(n-a_1)} \times \mathbb{A}^{a_2(n-a_2)}$$

$1 \le j_1 < \dots < j_{a_1} \le n$, $1 \le l_1 < \dots < l_{a_2} \le n$, cover $\mathrm{Grass}(a_1, V) \times \mathrm{Grass}(a_2, V)$, and it suffices to show that

$$\mathrm{Flag}(a_1, a_2, V) \cap (U_{j_1\dots j_{a_1}} \times U_{l_1\dots l_{a_2}})$$

is closed in $\mathbb{A}^{a_1(n-a_1)} \times \mathbb{A}^{a_2(n-a_2)}$. Without loss of generality we can assume $\{l_1, \dots, l_{a_2}\} = \{1, \dots, a_2\}$. An $M \in U_{l_1\dots l_{a_2}}$ is given by a matrix

$$\begin{pmatrix} 1 & \dots & 0 & m_{1,1} & \dots & m_{1,n-a_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & m_{a_2,1} & \dots & m_{a_2,n-a_2} \end{pmatrix}.$$

Moreover, $S \in U_{j_1,\dots,j_{a_1}}$ is given by a matrix

$$\begin{pmatrix} s_{1,1} & \dots & s_{1,n} \\ \vdots & \ddots & \vdots \\ s_{a_1,1} & \dots & s_{a_1,n} \end{pmatrix}$$

and the columns corresponding to $j_1, \ldots, j_{a_1}$ form the unit matrix. Then $S \subset M$ means that the matrix

$$\begin{pmatrix} s_{1,1} & \ldots & \ldots & \ldots & \ldots & s_{1,n} \\ \vdots & \ddots & \ddots & \ddots & \ldots & \vdots \\ s_{a_1,1} & \ldots & \ldots & \ldots & \ldots & s_{a_1,n} \\ 1 & \ldots & 0 & m_{1,1} & \ldots & m_{1,n-a_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & m_{a_2,1} & \ldots & m_{a_2,n-a_2} \end{pmatrix}$$

has rank $\leq a_2$, i.e., all $(a_2+1) \times (a_2+1)$-minors vanish.This finishes the proof. $\square$

Remark 5.10. The subset $G = \mathrm{GL}_n(k) \subset \mathbb{A}^{n^2}$ is open, hence a variety. What is more, the group composition and inverse map are morphisms. The varieties

$$\mathrm{Grass}(p, V) \text{ and } \mathrm{Flag}(a_1, \ldots, a_t, V)$$

are *homogeneous spaces* for G, which means that G acts transitively on them and the obvious group actions

$$G \times \mathrm{Grass}(p, V) \to \mathrm{Grass}(p, V),$$
$$G \times \mathrm{Flag}(a_1, \ldots, a_t, V) \to \mathrm{Flag}(a_1, \ldots, a_t, V)$$

are morphisms.

Chapter 6

Images of projective varieties under morphisms

If $f\colon X \to Y$ is a morphism of varieties, is it true that the image $f(X) \subset Y$ is a quasi-projective subvariety? I.e., is it locally closed? Unfortunately (or fortunately), things aren't quite as simple:

Example 6.1. Look at the morphism

$$\begin{aligned} f\colon \mathbb{A}^2 &\to \mathbb{A}^2 \\ (x,y) &\mapsto (x,xy). \end{aligned}$$

Then $\operatorname{Im}(f) = \{(a,b) \in \mathbb{A}^2 \mid a \neq 0\} \cup \{(0,0\}$. This is not open in its closure $\mathbb{A}^2$.

However, everything is nicer when the source is projective.

Theorem 6.2. *Let X be a projective variety and Y any variety, $f\colon X \to Y$ a morphism. The $f(X) \subset Y$ is closed.*

Proof. The graph $\Gamma_f \subset X \times Y$ is closed by by Remark 4.7 and $f(X) = \operatorname{pr}_2(\Gamma_f)$ where $\operatorname{pr}_2\colon X \times Y \to Y$ is the projection onto the second factor. So it suffices to show:

If $Z \subset X \times Y$ is closed and X is projective, then $\operatorname{pr}_2(Z) \subset Y$ is closed.

Now if $X \subset \mathbb{P}^n$ then Z is also closed in $\mathbb{P}^n \times Y$ and $\operatorname{pr}_2(Z)$ is equal to the image of the projection $\mathbb{P}^n \times Y \to Y$. Thus we can also assume $X = \mathbb{P}^n$.

Now let $Y = \bigcup_{i \in I} U_i$ be a finite cover of Y by affine open subsets. Then

$$X \times Y = \bigcup X \times U_i, \quad Z = \bigcup_{i \in I} Z \cap (X \times U_i),$$

$$\mathrm{pr}_2(Z) = \bigcup_{i \in I} \mathrm{pr}_2 \left(Z \cap (X \times U_i) \right),$$

so it suffices to show that $\mathrm{pr}_2 \left(Z \cap (X \times U_i) \right)$ is closed in U_i. Hence we can furthermore assume that Y is affine, $Y \subset \mathbb{A}^m$. Moreover, it is then $X \times Y$ closed in $X \times \mathbb{A}^m$ and $\mathrm{pr}_2(Z)$ equal to the image of Z under the projection $X \times \mathbb{A}^m \to \mathbb{A}^m$, so we can even assume $Y = \mathbb{A}^m$.

Thus let $Z \subset \mathbb{P}^n \times \mathbb{A}^m$ be closed. Because of Theorem 4.2, Z can be defined by equations

$$F_i(T_0, \ldots, T_n, t_1, \ldots, t_m) = 0, \quad i = 1, \ldots, N,$$

where $F_i \in k[T_0, \ldots, T_n, t_1, \ldots, t_m]$ is a polynomial which is homogeneous, of degree d_i, say, in the coordinates $T_0, \ldots, T_n$ on $\mathbb{P}^n$ (but of course not necessarily in the coordinates $t_1, \ldots, t_m$ on $\mathbb{A}^m$). For all $a = (a_1, \ldots, a_m) \in k^m \simeq \mathbb{A}^m$ put

$$Z_a := \{ (T_0 : \cdots : T_n) \in \mathbb{P}^n \mid F_i(T_0, \ldots, T_n, a_1, \ldots, a_m) = 0 \; \forall \, i = \ldots, N \}$$

(which can be thought of as the fiber of Z over a point $a \in \mathbb{A}^m$). Then Z_a is empty if and only if $(0, 0, \ldots, 0)$ is the only solution (in $\mathbb{A}^{n+1}$) of the equations $F_i(T_0, \ldots, T_n, a_1, \ldots, a_m) = 0 \; \forall \, i = \ldots, N$. This in turn is equivalent, by the Nullstellensatz 3.2, to the fact that the radical $\sqrt{I_a}$ of the ideal I_a generated by the $F_i(T_0, \ldots, T_n, a_1, \ldots, a_m)$ in $k[T_0, \ldots, T_n]$ is equal to $(T_0, \ldots, T_n)$. We can also phrase this as

$$Z_a = \emptyset \iff (T_0, \ldots, T_n)^s \subset I_a \text{ for some } s \geq 0.$$

Thus we get the characterization

$$\begin{aligned} \mathrm{pr}_2(Z) &= \{ a \in \mathbb{A}^m \mid Z_a \neq \emptyset \} \\ &= \bigcap_{s \geq 0} \{ a \in \mathbb{A}^m \mid (T_0, \ldots, T_n)^s \not\subset I_a \}. \end{aligned}$$

Thus it is sufficient to show that each set

$$Y_s := \{ a \in \mathbb{A}^m \mid (T_0, \ldots, T_n)^s \not\subset I_a \}$$

is closed in $\mathbb{A}^m$. Now remark that $(T_0, \dots, T_n)^s \subset I_a$ if and only if every homogeneous polynomial of degree s in $k[T_0, \dots, T_n]$ can be written as

$$\sum_{i=1}^{N} F_i(T_0, \dots, T_n, a_1, \dots, a_m) Q_i(T_0, \dots, T_n)$$

for some $Q_i(T_0, \dots, T_n) \in k[T_0, \dots, T_n]_{s-d_i}$. This leads us to consider the k-linear map

$$\varphi_a \colon \bigoplus_{i=1}^{N} k[T_0, \dots, T_n]_{s-d_i} \to k[T_0, \dots, T_n]_s$$
$$(Q_1, \dots, Q_N) \mapsto \sum_{i=1}^{N} F_i(T_0, \dots, T_n, a_1, \dots, a_m) Q_i(T_0, \dots, T_n),$$

and to reformulate once more: $a \in \mathbb{A}^m \backslash Y_s$ if and only if φ_a is surjective, thus $a \in Y_s$ if and only if $\operatorname{rk}(\varphi_a) < \dim k[T_0, \dots, T_n]_s =: d$. But then the $d \times d$-minors of any matrix representing φ_a are polynomials, with coefficients in k, in the $a_1, \dots, a_m$, which then define Y_s. □

Corollary 6.3. *If $f \colon X \to Y$ is a morphism, and X is projective and irreducible, and Y is affine, then f is constant.*

Proof. Suppose $Y \subset \mathbb{A}^m$ and let $x_i \colon \mathbb{A}^m \to \mathbb{A}^1$ be the i-th coordinate function. Then the composition

$$X \xrightarrow{f} Y \hookrightarrow \mathbb{A}^m \xrightarrow{x_i} \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$$

is a morphism from X to $\mathbb{P}^1$ whose image is closed in $\mathbb{P}^1$ by Theorem 6.2. A closed subset of $\mathbb{P}^1$ is either the whole of $\mathbb{P}^1$, which is not the case in the present situation since f maps into $\mathbb{A}^m$, or a finite point set. Since X is irreducible, this has to consist of exactly one point. Since this holds for all $i = 1, \dots, m$, the assertion follows. □

Let $f \colon X \to Z$, $g \colon Y \to Z$ morphisms of (quasi-)projective varieties. Let

$$X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

Since this is the preimage of the diagonal $\Delta_Z \subset Z \times Z$ under the morphism $X \times Y \to Z \times Z$, $(x, y) \mapsto (f(x), g(y))$, this is a quasi-projective variety. We call it the **fiber product** of $f \colon X \to Z$ and $g \colon Y \to Z$.

Definition 6.4. A morphism $f\colon X \to Z$ of varieties is called *proper* if for all varieties Y and for all morphisms $g\colon Y \to Z$ and for all closed subsets $W \subset X \times_Z Y$, the image of W under the projection $X \times_Z Y \to Y$ is closed in Y.

Theorem 6.5. *If X is projective, then every morphism $f\colon X \to Z$ to another variety Z is proper.*

Proof. In the proof of Theorem 6.2 we saw that, in the situation and notation of Definition 6.4, the projection $X \times Y \to Y$ maps closed sets to closed sets. But $X \times_Z Y$ is closed in $X \times Y$, and W is closed in $X \times_Z Y$, hence also in $X \times Y$. □

How does the image of a morphism of general varieties $f\colon X \to Y$ look like?

Remark 6.6. If X is not necessarily projective, one can still show that the image of $f\colon X \to Y$ in Y is always a *constructible set* (this is a theorem due to Chevalley). Here the constructible sets in Y are the smallest family of subsets in Y which (1) contains all the open subsets, (2) is stable under finite intersections, and (3) stable under taking complements.

As applications of the fore-going theory, the reader may try to prove the following assertions for herself or himself:

1. Consider $k[X_0, \ldots, X_n]_d$ as an affine space
$$\mathbb{A}^N, \quad N = \binom{n+d}{d}.$$
Prove that the subset of reducible polynomials is closed.

2. Prove that for $n > 1$, the varieties
$$\mathbb{A}^n - \{\text{point}\}, \quad \mathbb{P}^n - \{\text{point}\}$$
are isomorphic neither to affine nor to projective varieties. (To show that $\mathbb{A}^n - \{(0, \ldots, 0)\}$ is not affine, compute the global regular functions $\mathcal{O}_{\mathbb{A}^n - \{(0,\ldots,0)\}}(\mathbb{A}^n - \{(0, \ldots, 0)\})$ and remark that for any affine variety Z, a proper ideal $I \subsetneq \mathcal{O}_Z(Z)$ defines a non-empty subset by the Nullstellensatz).

3. If $X \subset \mathbb{P}^n$ is a projective subvariety which is not a finite point set, and $Y \subset \mathbb{P}^n$ is a projective subvariety defined by a single (non-constant) homogeneous polynomial $F \subset k[X_0, \ldots, X_n]$, then $X \cap Y \neq \emptyset$.

4. Let $n \geq 2$ and $f \colon \mathbb{P}^n \to \mathbb{P}^1$ a morphism. Show that f is constant. (Note that the projective closure of the zero set of a polynomial p in $\mathbb{A}^n$ is defined by the homogenization of the polynomial).

Chapter 7

Finite morphisms, Noether normalization

In this Chapter and Chapters 8,9 (on dimension theory and its applications), all varieties are tacitly assumed to be irreducible unless explicitly mentioned otherwise. This is more a matter of convenience than necessity, but it will simplify some proofs.

Let $B \supset A$ a ring extension. We assume A and B Noetherian. Then $b \in B$ is called *integral* over A if there is an equation

$$b^k + a_1 b^{k-1} + \cdots + a_k = 0, \quad a_i \in A, \text{ some } k > 1.$$

Moreover, B is called *integral over* A if every element in B is integral over A.

Remark 7.1. If B is a finitely generated A-algebra, then B is integral over A if and only if B is a finite A-module.

To see this, it is sufficient to prove that: $b \in B$ integral over $A \iff A[b]$ is a finitely generated A-module. Now the direction $\implies$ is clear. To show $\impliedby$, assume that $A[b]$ is a finite A-module, and let $\omega_1, \ldots, \omega_r$ be A-module generators. Then if $c \in A[b]$ is arbitrary, we get equations

$$c\omega_i = \sum_{j=1}^{r} a_{ij}\omega_j, \quad i = 1, \ldots, r, \ a_{ij} \in A.$$

We can rewrite this as

$$(c\mathrm{Id} - M)\begin{pmatrix}\omega_1\\ \vdots \\ \omega_r\end{pmatrix} = 0, \quad M = (a_{ij}).$$

Multiplying through with the adjoint of $c\mathrm{Id} - M$ we obtain

$$\det(c\mathrm{Id} - M)\omega_i = 0, \quad i = 1, \dots, r.$$

Since $1_A = 1_B$ can be linearly combined from the ω_i, we see that $\det(c\mathrm{Id} - M) = 0$ is an integrality equation (monic equation) for c over A.

Definition 7.2. Let X, Y be affine varieties, and $f\colon X \to Y$ a morphism with $\overline{f(X)} = Y$. Then f is called *finite* if

$$f^*\colon \mathcal{O}_Y(Y) \hookrightarrow \mathcal{O}_X(X)$$

is an integral ring extension.

Remark 7.3. If f is finite as in Definition 7.2, then f has finite fibers: indeed, suppose $X \subset \mathbb{A}^n$ and let $t_1, \dots, t_n$ be the coordinate functions on $\mathbb{A}^n$. Then t_i assumes only finitely many values on the fiber $f^{-1}(y)$, $y \in Y$, since we have a monic equation

$$t_i^k + a_1 t_i^{k-1} + \cdots + a_k = 0 \quad a_i \in \mathcal{O}_Y(Y)$$

and if $x \in f^{-1}(y)$

$$t_i(x)^k + a_1(y)t_i(x)^{k-1} + \cdots + a_k(y) = 0$$

has only finitely many roots $t_i(x)$.

Theorem 7.4. *If $f\colon X \to Y$ is a finite morphism between affine varieties, $\overline{f(X)} = Y$, then f is surjective.*

Proof. Let $y = (y_1, \dots, y_n) \in Y \subset \mathbb{A}^n$ with $\mathfrak{m}_y \subset \mathcal{O}_Y(Y)$ the maximal ideal of y, $\mathfrak{m}_y = (t_1 - y_1, \dots, t_n - y_n)$. The variety $f^{-1}(y)$ has equations $f^*(t_i) = y_i$, $i = 1, \dots, n$. Thus by the Nullstellensatz 3.2

$$f^{-1}(y) = \emptyset \iff (f^*(t_1) - y_1, \dots, f^*(t_n) - y_n) = \mathcal{O}_X(X).$$

In other words, the fiber is empty if and only if $\mathfrak{m}_y \cdot \mathcal{O}_X(X) = \mathcal{O}_X(X)$. Hence the assertion follows from Lemma 7.5 below. $\square$

Lemma 7.5. *Suppose $B \supset A$ is finite. Then, if $\mathfrak{a} \subset A$ is an ideal, we have the implication*

$$\mathfrak{a} \subsetneq A \implies \mathfrak{a}B \subsetneq B.$$

Proof. B contains 1_A, thus for every $a \in A$ with $aB = 0$, we have $a = 0$. On the other hand, if $\mathfrak{a} \neq A$, then $0 \notin 1 + \mathfrak{a}$. Thus Lemma 7.5 in turn follows from Lemma 7.6 below. $\square$

Lemma 7.6 (Nakayama's Lemma)**.** *Suppose M is a finite A-module, $\mathfrak{a} \subset A$ an ideal. Suppose that for every $a \in 1 + \mathfrak{a}$, the equation $aM = 0$ can only hold if $M = 0$. Then $\mathfrak{a}M = M$ implies $M = 0$.*

Proof. Suppose M is generated by $\omega_1, \ldots, \omega_r$ as an A-module. Then the hypothesis $\mathfrak{a}M = M$ translates into the existence of a matrix T with entries in $\mathfrak{a}$ such that

$$(\mathrm{Id} - T)\begin{pmatrix}\omega_1 \\ \vdots \\ \omega_r\end{pmatrix} = 0.$$

Multiplying by the adjoint matrix again, we obtain that $\det(\mathrm{Id} - T) \in 1 + \mathfrak{a}$ annihilates M, which by hypothesis implies $M = 0$. $\square$

Corollary 7.7. *Every finite morphism as in Theorem 7.4 maps closed sets to closed sets.*

Proof. If $Z \subset X$ is closed (and, without loss of generality, irreducible), the map $f|_Z \colon Z \to \overline{f(Z)}$ is finite. Hence Theorem 7.4 yields $\overline{f(Z)} = f(Z)$. $\square$

Finiteness is a local property:

Theorem 7.8. *If $f \colon X \to Y$ is a morphism of affine varieties, and if every $y \in Y$ has an affine open neighborhood $U \ni y$ such that $V = f^{-1}(U)$ is affine and $f|_{f^{-1}(U)} \colon V = f^{-1}(U) \to U$ is finite, then f is finite.*

Proof. We can assume that U is principal open and cover Y with finitely many U_{g_α} with $g_\alpha \in \mathcal{O}_Y(Y)$. That those cover means

$$(g_\alpha) = \mathcal{O}_Y(Y)$$

by the Nullstellensatz. Then

$$V_\alpha := f^{-1}(U_{g_\alpha}) = U_{f^*(g_\alpha)}$$

is principal open, and

$$\mathcal{O}_X(V_\alpha) = B_{g_\alpha}, \quad B := \mathcal{O}_X(X),$$
$$\mathcal{O}_Y(U_{g_\alpha}) = A_{g_\alpha}, \quad A := \mathcal{O}_Y(Y).$$

The hypothesis says that $B[1/g_\alpha]$ is a finite $A[1/g_\alpha]$-module, generated by elements $\omega_{i,\alpha}$, say. We can even assume $\omega_{i,\alpha} \in B$ after clearing denominators. We claim that all the $\omega_{i,\alpha}$ taken together form a generating system for B/A: indeed, every $b \in B$ has a representation for all α as

$$b = \sum_i \frac{a_{i,\alpha}}{g_\alpha^{n_\alpha}} \omega_{i,\alpha}, \text{ some } n_\alpha \in \mathbb{N}.$$

But $(g_\alpha^{n_\alpha}) = 1$ since the $g_\alpha^{n_\alpha}$ have no common zeroes on Y, hence there are $h_\alpha \in A$ with $\sum_\alpha g_\alpha^{n_\alpha} h_\alpha = 1$. Thus

$$b = b \sum_\alpha g_\alpha^{n_\alpha} h_\alpha = \sum_i \sum_\alpha a_{i,\alpha} h_\alpha \omega_{i,\alpha}.$$

□

Thus we can define:

Definition 7.9. A morphism $f \colon X \to Y$ of (quasi-projective) varieties is called *finite* if for all $y \in Y$ there is an open affine neighborhood $V \ni y$ such that $f^{-1}(V) =: U \subset X$ is affine and $f|_U \colon U \to V$ is finite.

Theorem 7.8 tells us that "finite" in this new sense coincides with the notion of "finite" introduced in Definition 7.2.

Remark that we immediately obtain that for finite f in the sense of Definition 7.9, it continues to be true that all fibers $f^{-1}(y)$ are finite, and every such f is surjective.

Theorem 7.10. *If $f \colon X \to Y$ is a morphism of varieties and $f(X) \subset Y$ is dense, then $f(X)$ contains an open subset of Y.*

Proof. We can assume X, Y affine (and irreducible). Via f^*, we have an inclusion of coordinate rings $\mathcal{O}_Y(Y) \subset \mathcal{O}_X(X)$. We introduce notation for quotient fields:

$$k(X) = \mathrm{Quot}(\mathcal{O}_X(X)), \quad k(Y) = \mathrm{Quot}(\mathcal{O}_Y(Y))$$

and put $r := \operatorname{tr.deg}(k(X)/k(Y))$. Suppose that $u_1, \dots, u_r \in \mathcal{O}_X(X)$ are algebraically independent over $k(Y)$. Then we have the inclusions

$$\mathcal{O}_X(X) \supset \mathcal{O}_Y(Y)[u_1, \dots, u_r] = \mathcal{O}(Y \times \mathbb{A}^r) \supset \mathcal{O}_Y(Y)$$

which means we can factor f as $f = g \circ h$ where

$$h \colon X \to Y \times \mathbb{A}^r, \quad g = \mathrm{pr}_1 \colon Y \times \mathbb{A}^r \to Y.$$

By construction, every element $v \in \mathcal{O}_X(X)$ is algebraic over $\mathcal{O}(Y \times \mathbb{A}^r)$, thus av is integral for a suitable $a \in \mathcal{O}(Y \times \mathbb{A}^r)$. Let $v_1, \dots, v_m$ be k-algebra generators of $\mathcal{O}_X(X)$ and $a_i \in \mathcal{O}(Y \times \mathbb{A}^r)$ such that $a_i v_i$ is integral over $\mathcal{O}(Y \times \mathbb{A}^r)$. Put $F = a_1 \cdot \dots \cdot a_m$. Now, by construction, the functions v_i restricted to $\{h^*(F) \neq 0\} \subset X$ are integral over $\mathcal{O}(Y \times \mathbb{A}^r)[1/F]$, i.e.,

$$h\,|_{\{h^*(F) \neq 0\}} \colon \{h^*(F) \neq 0\} \to U_F \subset Y \times \mathbb{A}^r$$

is finite. Thus by Theorem 7.4, $h(\{h^*(F) \neq 0\}) = \{F \neq 0\}$. In other words, $U_F \subset h(X)$. It remains to show that $g(U_F)$ contains an open subset of Y. But g is just a projection! So if $F = \sum F_\alpha(y) T^\alpha$, T a tuple of coordinates on $\mathbb{A}^r$, $F_\alpha \in \mathcal{O}_Y(Y)$, then $g(U_F) \supset \bigcup U_{F_\alpha}$. $\square$

Theorem 7.10 may look unspectacular at first sight, but it really expresses a very fundamental and remarkable property of morphisms between varieties. For example, an analogue of it does not hold in the differentiable category (think of a "dense wind" $\mathbb{R}^1 \to T = \mathbb{R}^2/\mathbb{Z}^2$, $x \mapsto (x, \sqrt{2}x) \bmod \mathbb{Z}^2$).

Theorem 7.11. *Let $X \subset \mathbb{P}^n$ be a projective variety. Let $L \subset \mathbb{P}^n$ be a d-dimensional (projective) linear subspace with $L \cap X = \emptyset$. Let*

$$\pi_L \colon X \to \mathbb{P}^{n-d-1}$$

be the projection with center L; this means that if L is given by linear equations $L_0 = \dots = L_{n-d-1} = 0$, then

$$\pi_L(x) := (L_0(x) : \dots : L_{n-d-1}(x)).$$

Then $\pi := \pi_L \colon X \to \pi_L(X)$ is finite.

Before embarking on the proof we state and prove a few corollaries.

Corollary 7.12 (Noether normalization)**.** *For every irreducible projective variety X, there is a finite morphism $f\colon X \to \mathbb{P}^n$ for some $n \in \mathbb{N}$.*

Proof. Suppose $X \subset \mathbb{P}^N$. If there is a point $x \in \mathbb{P}^N \backslash X$, then we project to $\mathbb{P}^{N-1}$ from x, and by Theorem 6.2 the image is closed, and the projection is finite onto its image. If the image is not yet the entire $\mathbb{P}^{N-1}$, we continue in the same manner. □

Corollary 7.13. *If X is affine, then there is a finite morphism $f\colon X \to \mathbb{A}^n$ for some $n \in \mathbb{N}$.*

Proof. Suppose $X \subset \mathbb{A}^N \subset \mathbb{P}^N$, and let $\overline{X} \subset \mathbb{P}^N$ its closure. If $X \neq \mathbb{A}^N$, we can project from $x \in \mathbb{P}^N \backslash \mathbb{A}^N$, $x \notin \overline{X}$. Then X gets mapped into $\mathbb{A}^{N-1} \subset \mathbb{P}^{N-1}$ and we conclude by the same pattern as in the proof of Corollary 7.12. □

Corollary 7.14. *If $F_0, \dots, F_s \in k[X_0, \dots, X_n]_m$ are homogeneous polynomials of degree $m > 0$ without common zeroes on a projective variety $X \subset \mathbb{P}^n$, then*

$$f = (F_0 : \cdots : F_s)\colon X \to f(X)$$

is finite. Thus morphisms which can be defined by "one row" in the sense of Theorem 3.13 are finite.

Proof. This follows from Theorem 7.11 since f is the composition of the m-th Veronese embedding of X and a linear projection. □

Proof. (of Theorem 7.11) Let $y_0, \dots, y_{n-d-1}$ be homogeneous coordinates in $\mathbb{P}^{n-d-1}$ so that π is given by $y_j = L_j(x)$, $j = 0, \dots, n-d-1$, $x = (x_0 : \cdots : x_n) \in X$. Put

$$U_i := \pi^{-1}\left(\mathbb{A}^{n-d-1}_{y_i \neq 0}\right) \cap X$$

the affine open subset of X given by $L_i \neq 0$. We show that

$$\pi\colon U_i \to \mathbb{A}^{n-d-1}_{y_i \neq 0} \cap \pi(X)$$

is finite. Every function $g \in \mathcal{O}_X(U_i)$ is of the form

$$g = \frac{G_i(x_0, \dots, x_n)}{L_i^m}$$

where G_i is homogeneous of degree $\deg G_i = m$. Consider the map

$$\tilde{\pi}\colon X \to \mathbb{P}^{n-d}$$
$$x \mapsto (z_0 : \cdots : z_{n-d}), \quad z_j := (L_j(x))^m,\ j \leq n-d-1,\ z_{n-d} := G_i(x).$$

It is a morphism since the L_j have no common zeroes on X. So by Theorem 6.2, $\tilde{\pi}(X) \subset \mathbb{P}^{n-d}$ is closed. Suppose $F_1, \ldots, F_s$ are homogeneous equations for $\tilde{\pi}(X)$. Since the L_j have no common zeroes on X, the point $(0 : \cdots : 0 : 1) \in \mathbb{P}^{n-d}$ is not in $\tilde{\pi}(X)$, which means

$$z_0 = \cdots = z_{n-d-1} = F_1 = \cdots = F_s = 0$$

has no solution in $\mathbb{P}^{n-d}$. By the Nullstellensatz 3.2, we get

$$(z_0, \ldots, z_{n-d-1}, F_1, \ldots, F_s) \supset (z_0, \ldots, z_{n-d})^k \quad \text{some } k > 0.$$

In particular, we have an equation

$$z_{n-d}^k = \sum_{j=0}^{n-d-1} H_j z_j + \sum_{j=1}^{s} P_j F_j$$

for some polynomials H_j, P_j. Then

$$\Phi(z_0, \ldots, z_{n-d}) := z_{n-d}^k - \sum_{j=0}^{n-d-1} H_j^{(k-1)} z_j \equiv 0 \text{ on } \tilde{\pi}(X)$$

where $H_j^{(k-1)}$ is the homogeneous component of degree $k-1$ of H_j. This can be rephrased by saying that

$$\Phi(L_0^m, \ldots, L_{n-d-1}^m, G_i) \equiv 0 \text{ on } X.$$

Then dividing by L_i^{mk} we get an integrality equation for g over $\mathcal{O}(\mathbb{A}^{n-d-1}_{y_i \neq 0} \cap \pi(X))$. $\square$

Chapter 8

Dimension theory

In Definition 3.15, we defined a rational function on an irreducible variety X as a rational map to $\mathbb{A}^1$. This means that the totality of all rational functions can be described as

$$k(X) = \varinjlim_{U \subset X \text{ open}, U \neq \emptyset} \mathcal{O}_X(U)$$

and $k(X)$ is a ring. In the following Remark we summarize some properties that are immediate from the definition.

Remark 8.1.

1. Not only is $k(X)$ a ring, it is a *field*: if $[(f, U)] \in k(X)$, and $f \neq 0$, then $[(1/f, U - \{f = 0\})]$ is its inverse.
2. If X is affine, then $k(X) = \mathrm{Quot}(\mathcal{O}_X(X))$.
3. If X is projective, then
$$k(X) = \left\{ \frac{f}{g} \mid f, g \in k[X_0, \dots, X_n]/I(X),\ f, g \text{ homogeneous of the same degree},\ g \not\equiv 0 \right\}.$$
4. If $U \subset X$ is open and nonempty, then $k(X) \simeq k(U)$.

Definition 8.2. Let X be an irreducible variety. Then we define its **dimension** by putting

$$\dim X := \mathrm{tr.deg}_k k(X).$$

If X is reducible, we set $\dim X := \max_i \dim X_i$ where X_i is an irreducible component of X. If $Y \subset X$ is a (closed) subvariety

$$\mathrm{codim}_X Y := \dim X - \dim Y$$

is called the **codimension** of Y in X. Irreducible varieties of dimension 1, 2, 3 . . . are called curves, surfaces, threefolds, . . .

In the sequel all varieties will be irreducible unless otherwise stated.

Remark 8.3.

1. If X is projective resp. affine, and $f\colon X \to \mathbb{P}^n$ resp. $\mathbb{A}^n$ a finite morphism (which always exists by Corollaries 7.12 and 7.13), then $n = \dim X$. Namely, in those cases $k(X)$ is a finite algebraic extension of $k(T_1, \dots, T_n)$. This also shows that our definition of dimension is intuitively reasonable.

2. If $U \subset X$ is open and nonempty, then $\dim U = \dim X$.

3. $\dim X = 0$ (and X possibly reducible) $\iff$ X is a finite point set. Namely, $\Longleftarrow$ is clear. For the direction $\Longrightarrow$ assume without loss of generality that X is irreducible, affine, $X \subset \mathbb{A}^n$. Then the coordinate functions t_i on X are algebraic over k, hence can take only finitely many values.

4. For varieties X, Y, $\dim(X \times Y) = \dim X + \dim Y$. Indeed, it suffices to show this for $X \subset \mathbb{A}^m$, $Y \subset \mathbb{A}^n$ affine. Let $\dim X = d$, $\dim Y = e$, $x_1, \dots, x_m$, $y_1, \dots, y_n$ affine coordinates on $\mathbb{A}^m$, $\mathbb{A}^n$ such that $x_1, \dots, x_d$ are algebraically independent in $\mathbb{C}(X)$, $y_1, \dots, y_e$ are algebraically independent in $\mathbb{C}(Y)$.

   Now $\mathcal{O}_{X\times Y}(X\times Y)$ is generated by $x_1, \dots, x_m, y_1, \dots, y_n$ which are algebraically dependent on $x_1, \dots, x_d, y_1, \dots, y_e$. Thus $\dim(X\times Y) \le d+e$. Let us show that $x_1, \dots, x_d, y_1, \dots, y_e$ are algebraically independent on $X \times Y$, which will prove $\dim(X \times Y) \ge d + e$. Suppose

   $$F(x_1, \dots, x_d, y_1, \dots, y_e) = 0 \text{ on } X \times Y.$$

   Since this means that for all $\tilde{x} \in X$ we have $F(\tilde{x}, y_1, \dots, y_e) = 0$ on Y, the algebraic independence of the y's implies that every coefficient $a_i(x_1, \dots, x_d)$ of F (viewed as a polynomial in $y_1, \dots, y_e$) is zero on X. Since $x_1, \dots, x_d$ are also algebraically independent, it follows that each a_i must be the zero polynomial.

5. It is $\dim \mathrm{Grass}(r, n) = r(n-r)$ since $\mathrm{Grass}(r, n)$ is irreducible (because $\mathrm{GL}_r(k)$ is irreducible and acts transitively on $\mathrm{Grass}(r, n)$) and since $\mathrm{Grass}(r, n)$ contains open subsets isomorphic to $\mathbb{A}^{r(n-r)}$ as we saw in the proof of Proposition 5.9.

A good dimension function should decrease on proper subvarieties:

Theorem 8.4. *If X, Y are not necessarily irreducible varieties, $X \subset Y$ closed, then $\dim X \le \dim Y$. If Y is irreducible, then $\dim X = \dim Y$ happens only if $X = Y$.*

Proof. It suffices to prove that for X, Y affine and irreducible, $X \subset Y \subset \mathbb{A}^N$ closed. Suppose $t_1, \dots, t_N$ are coordinates on $\mathbb{A}^N$ and $\dim Y = n$. Thus each $n+1$ out of the $t_1, \dots, t_N$ are algebraically dependent on Y, whence the same holds for X and

$$\text{tr.deg}_k k(X) \le \text{tr.deg}_k k(Y).$$

If $\dim X = \dim Y = n$, then after reordering we can assume that $t_1, \dots, t_n$ are algebraically independent on X. These are then also algebraically independent on Y. Let $u \in \mathcal{O}_Y(Y)$, $u \ne 0$. Then there is a polynomial $p(t_1, \dots, t_n, u)$ with

$$p = a_0(t_1, \dots, t_n)u^k + \cdots + a_k(t_1, \dots, t_n) \equiv 0 \text{ on } Y.$$

We can assume $a_k(t_1, \dots, t_n)$ not identically zero on Y by assuming $p(t_1, \dots, t_n, u)$ of smallest degree in u. Suppose that u vanishes on X. Then $a_k(t_1, \dots, k_n) \equiv 0$ on X, and by the algebraic independence of the $t_1, \dots, t_n$ on X, we have $a_k(t_1, \dots, t_n) = 0$ on all of $\mathbb{A}^N$, hence on Y, a contradiction. So u does not vanish on X if it does not vanish on Y. This means $X = Y$. □

Theorem 8.5. *Every irreducible component of a hypersurface (=the zero locus of a single nonzero polynomial, homogeneous in the projective case) in $\mathbb{A}^n$ (or $\mathbb{P}^n$) has codimension 1.*

Proof. Suppose without loss of generality that $X \subset \mathbb{A}^n$ is given by an irreducible polynomial $F \ne 0$. After reordering the coordinates $t_1, \dots, t_n$ we may assume that t_n occurs in F. Then $t_1, \dots, t_{n-1}$ are algebraically independent on X, for $G(t_1, \dots, t_{n-1}) \equiv 0$ on X would imply, by the Nullstellensatz that F would divide a power G^l, some $l > 0$, and this is only possible if G is the zero polynomial. Hence, $\dim X = n - 1$, using Theorem 8.4 ($\dim X = n$ is impossible because then $X = \mathbb{A}^n$. □

In fact, we have a converse to the latter result.

Theorem 8.6. *If $X \subset \mathbb{A}^n$ is a closed subvariety all of whose irreducible components have codimension 1, then X is a hypersurface and I_X is a principal ideal. The same holds in the case of a projective $X \subset \mathbb{P}^n$.*

Proof. Again it suffices to do the proof for $X \subset \mathbb{A}^n$ affine and irreducible. Since $X \neq \mathbb{A}^n$, there is a polynomial $F \neq 0$ in $k[x_1, \ldots, x_n]$ with $F \equiv 0$ on X. Since X is irreducible, one irreducible factor of F, F' say, also vanishes identically on X. Let $Y := \{F' = 0\} \subset \mathbb{A}^n$. Then Y is irreducible: indeed, $GH \equiv 0$ on Y implies, by the Nullstellensatz, $F' \mid (GH)^l$, some $l > 0$, thus $G \equiv 0$ or $H \equiv 0$ on Y. But $X \subset Y$. Hence Theorems 8.4 and 8.5 imply $X = Y$. Now I_X is generated by F' since by the Nullstellensatz $I_X = \sqrt{(F')}$ and (F') is a prime ideal, hence radical. □

The next result says that the dimension function does not drop too much.

Theorem 8.7. *If $X \subset \mathbb{P}^N$ is an irreducible projective variety, $F \in k[X_0, \ldots, X_N]_m$ with $m > 0$ and $F \not\equiv 0$ on X, then for the intersection* $\dim(X \cap \{F = 0\}) = \dim X - 1$*, i.e.* $(X \cap \{F = 0\}$ *contains at least one component of dimension* $\dim X - 1$.

In particular, a projective variety contains subvarieties of any dimension $\leq \dim X$. Also we immediately get the following "combinatorial" characterization of dimension:

Corollary 8.8. *One can define the dimension of a projective variety as the maximal integer n such that there is a chain*

$$Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_n \neq \emptyset$$

of irreducible subvarieties $Y_i \subset X$.

Proof. (of Theorem 8.7) For every projective variety $S \subset \mathbb{P}^N$, reducible or not, one can find a homogeneous polynomial $G(X_0, \ldots, X_N)$ (of every degree $m > 0$) which does not vanish identically on any component of S (for example, pick a point in every component and take a power of a linear form which does not vanish in any of those points).

Now suppose $X \subset \mathbb{P}^N$ and $F \not\equiv 0$ on every component of X. Then Theorem 8.4 implies

$$X^{(1)} := X \cap \{F = 0\}$$

has dimension strictly smaller than $\dim X$. Now pick a homogeneous polynomial F_1, of degree $\deg F$, which does not vanish on any component of $X^{(1)}$. Continuing in this fashion we get a chain of subvarieties (possibly reducible)

$$X = X^{(0)} \supset X^{(1)} \supset \cdots \supset X^{(i+1)} := X^{(i)} \cap \{F_i = 0\},$$

$F_0 := F$, $\dim X^{(i+1)} < \dim X^{(i)}$ as long as $X^{(i)}$ is nonempty. If $\dim X = n$, then $X^{(n+1)}$ is empty. This means that $F_0 = F, \dots, F_n$ have no common zeroes on X. Now, if X is irreducible, we get by Corollary 7.14 that

$$f \colon X \to \mathbb{P}^n$$
$$f(X) = (F_0(X) : \cdots : F_n(X))$$

is finite onto its image $f(X) \subset \mathbb{P}^n$. Hence $\dim X = \dim f(X) = n$, and by Theorem 8.4, $f(X) = \mathbb{P}^n$. Now, if, arguing by contradiction, we had $\dim X^{(1)} < n-1$, the already $X^{(n)}$ would be empty. I.e., already $F_0, \dots, F_{n-1}$ would have no common zeroes on X. But then $(0 : \cdots : 0 : 1)$ wouldn't be in the image of f, a contradiction. □

This leads to a third way of characterizing dimension which is more geometric:

Corollary 8.9. *The dimension n of a projective variety $X \subset \mathbb{P}^N$ is equal to $N - s - 1$ where s is the maximal dimension of a linear subspace $L \subset \mathbb{P}^N$ with $X \cap L = \emptyset$.*

Proof. If L is of dimension $s \geq N - n$, then L can be defined by $\leq n$ equations, so Theorem 8.7 gives $\dim X \cap L \geq 0$, so $X \cap L \neq \emptyset$. On the other hand, if in the proof of Theorem 8.7 we take the F_i equal to $L_0, \dots, L_n$, then $(L_0 = \cdots = L_n = 0) = L$ satisfies $\dim L = N - n - 1$ and $X \cap L = \emptyset$. □

We can also regard Theorem 8.7 as giving a strong existence result for solution sets of polynomial equations:

Corollary 8.10. *The zero set of r homogeneous polynomials $F_1, \dots, F_r$ with $\deg(F_i) > 0$, $i = 1, \dots, r$, on an n-dimensional projective variety has dimension $\geq n - r$; so in particular, if $r \leq n$, then solutions exist.*

This implies for example that in $\mathbb{P}^2$, any two curves intersect since by Theorem 8.6, these are defined by one equation; this is false on $\mathbb{P}^1 \times \mathbb{P}^1$. Lines of the same ruling do not intersect. This proves also that $\mathbb{P}^2$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ although the two are birational.

We can also use Corollary 8.10 to see that any curve of degree ≥ 3 in $\mathbb{P}^2$ has an inflection point, and for many similar geometric existence results.

We now prove a strengthening of Theorem 8.7.

Theorem 8.11. *Let X be an irreducible projective variety X in $\mathbb{P}^N$, and F a homogeneous polynomial of positive degree which does not vanish identically on X. Then every irreducible component of $X \cap \{F = 0\}$ has dimension* $\dim X - 1 =: n - 1$.

Proof. We go back to the set-up we produced in the course of the proof of Theorem 8.7: there we produced a chain of (possibly reducible) subvarieties

$$X = X^{(0)} \supsetneq X^{(1)} \supsetneq \cdots \supsetneq X^{(i+1)} = X^{(i)} \cap \{F_i = 0\}$$

with $F_0, \dots, F_n$ homogeneous polynomials of the same degree m without common zeroes on X, $F_0 = F$. Then we saw that

$$f \colon X \to \mathbb{P}^n, \ x \mapsto (F_0(x) : \cdots : F_n(x))$$

is finite onto its image $f(X)$. Let $\mathbb{A}^n_{(x_i \neq 0)} \subset \mathbb{P}^n$ be the standard affine chart, and $U_i \subset X$ its preimage under f. This is open and affine (the latter since we can realize f as the composition of a Veronese embedding with a linear projection).

Now it suffices to prove that every component of $(X \cap \{F = 0\}) \cap U_i$ has dimension $n-1$ (for all i). The latter set is the zero set, in U_i, of the function

$$\varphi := \frac{F}{F_i} \in \mathcal{O}_X(U_i).$$

Put $U := U_i$. Then f restricted to U gives a map

$$f \colon U \to \mathbb{A}^n, \ x \mapsto (f_1(x), \dots, f_n(x))$$

which is finite onto its image, and $f_1 = f$ is the dehomogenization of F. We have to show that every component of $\{\varphi = 0\}$ in U has dimension $\geq n - 1$. To do this we will prove that $f_2, \dots, f_n$ are algebraically independent on every component of $\varphi = f_1 = 0$ in U. Let $P \in k[t_2, \dots, t_n]$. To prove that if P is not the zero polynomial, then $R := P(f_2, \dots, f_n) \not\equiv 0$ on every component of $\varphi = 0$, it suffices to show

$$\forall\, Q \in \mathcal{O}_U(U) \colon RQ \equiv 0 \text{ on } (\varphi = 0) \implies Q \equiv 0 \text{ on } (\varphi = 0). \tag{8.1}$$

For, if

$$\{u \in U \mid \varphi(u) = 0\} = U^{(1)} \cup \cdots \cup U^{(t)}$$

is the decomposition into irreducible components and, for example, $R \equiv 0$ on $U^{(1)}$, then we could take a Q which vanishes on $U^{(2)} \cup \cdots \cup U^{(t)}$, but not on $U^{(1)}$. Then we would have $RQ \equiv 0$ on $(\varphi = 0)$, but $Q \not\equiv 0$ on $(\varphi = 0)$, a contradiction.

By the Nullstellensatz, condition (8.1) is equivalent to

$$\varphi \mid (RQ)^l \text{ some } l > 0 \implies \varphi \mid Q^k \text{ some } k > 0. \tag{8.2}$$

Thus Theorem 8.11 follows from the next Lemma. □

Lemma 8.12. *Let $B = k[T_1, \ldots, T_n]$ and $A \supset B$ an integral domain which is integral over B. Let $x := T_1$, and suppose $B \ni y = p(T_2, \ldots, T_n) \neq 0$. Then for all $u \in A$ we have that*

$$x \mid (yu)^l \text{ in } A \text{ for some } l > 0 \implies x \mid u^k \text{ for some } k > 0.$$

Proof. This uses that x, y are relatively prime in B. In fact, replacing y^l by z and u^l by v, it suffices to show: if x, z are relatively prime in $k[T_1, \ldots, T_n]$, then

$$x \mid zv \text{ in } A \implies x \mid v^k \text{ some } k > 0.$$

This means, intuitively, that the property that x, z are relatively prime is "inherited" in a certain sense by the overring $A \supset B$, which is integral over B.

Put $K = \mathrm{Quot}(B) = k(T_1, \ldots, T_n)$. If $t \in A$ is integral over B, then t is algebraic over K. Let $F(T)$, $T = (T_1, \ldots, T_n)$, be the minimal polynomial (with leading coefficient 1) of t over K. Then

$$t \text{ integral over } B \iff F(T) \in B[T].$$

Indeed, to see $\implies$, $\impliedby$ being trivial, remark that if $G(t) = 0$ for a $G \in B[T]$ with leading coefficient 1, then $G(T) = F(T)H(T)$ in $K[T]$. But B is factorial, whence by Gauss's Lemma, $F(T), H(T) \in B[T]$.

Now if $zv = xw$, $v, w \in A$, let

$$F(T) = T^k + b_1 T^{k-1} + \cdots + b_k$$

be the minimal polynomial of w, $b_i \in B$. Then the minimal $G(T)$ of $v = (xw)/z$ is

$$\left(\frac{x}{z}\right)^k F\left(\frac{zT}{x}\right),$$

namely,

$$G(T) = T^k + \frac{xb_1}{z} T^{k-1} + \cdots + \frac{x^k b_k}{z^k}. \tag{8.3}$$

Since v is integral over B,

$$\frac{x^i b_i}{z^i} \in B \quad \forall\, i.$$

But then, since x, z are relatively prime, $z^i \mid b_i$, and substituting v for T in (8.3) yields $x \mid v^k$. $\square$

Corollary 8.13. *Let $X \subset \mathbb{P}^N$ be an irreducible* quasi-projective *variety, and $F \neq 0$ a homogeneous polynomial of positive degree such that $F \not\equiv 0$ on X. Then every irreducible component of $X \cap \{F = 0\}$ has codimension* 1.

Proof. It suffices to apply Theorem 8.11 to the closure $\overline{X} \subset \mathbb{P}^N$, and remark that $X \cap \{F = 0\} = (\overline{X} \cap \{F = 0\}) \cap X$. $\square$

The next result also follows directly from Theorem 8.11 now.

Corollary 8.14. *If $X \subset \mathbb{P}^N$ is irreducible and quasi-projective,* $\dim X = n$*, and $Y \subset X$ is the zero set of homogeneous polynomials of positive degrees $F_1, \ldots, F_m$, then every component of Y has dimension $\geq n - m$.*

Theorem 8.15. *Let $X, Y \subset \mathbb{P}^N$ be irreducible, quasi-projective varieties,* $\dim X = n$, $\dim Y = m$*. Then every component Z of $X \cap Y$ has dimension $\geq n + m - N$. If X and Y are projective and $n + m \geq N$, then $X \cap Y \neq \emptyset$.*

Proof. To prove the first assertion, it suffices to consider the case when $X, Y \subset \mathbb{A}^N$ are affine. Then

$$X \cap Y \simeq (X \times Y) \cap \Delta_{\mathbb{A}^N}$$

and the diagonal $\Delta_{\mathbb{A}^N} \subset \mathbb{A}^{2N}$ is defined by N equations. Then the first assertion follows from Corollary 8.14. The second assertion follows by considering the affine cones over X, Y and applying the first assertion. $\square$

We can also say that

$$\operatorname{codim}_{\mathbb{P}^n} \bigcap_{I=1}^{r} X_i \leq \sum_{i=1}^{r} \operatorname{codim}_{\mathbb{P}^n}(X_i)$$

for any finite number r of irreducible quasi-projective subvarieties $X_i \subset \mathbb{P}^n$.

The next theorem describes how the dimensions of the fibers of a morphism can vary and has very many applications.

Theorem 8.16 (Theorem about the (Upper-Semicontinuity of) Fiber Dimension)**.** *Let $f\colon X \to Y$ be a morphism of irreducible varieties, and suppose that $f(X) = Y$, $\dim X = n$, $\dim Y = m$. Then we have $m \le n$ and*

1. *for every component F of a fiber $f^{-1}(y)$, $y \in Y$, we have $\dim F \ge n-m$,*
2. *and there is a nonempty open subset $U \subset Y$ with $\dim f^{-1}(y) = n - m$ for all $y \in U$.*

Proof. The assertions are local on Y, so we can assume $Y \subset \mathbb{A}^M$ closed affine. Since $\dim Y = m$, we can find a chain of subvarieties

$$Y^{(0)} \supsetneq Y^{(1)} \supsetneq \cdots \supsetneq Y^{(m)}$$

where $Y^{(i)}$ is purely i-codimensional in Y so that $Y^{(m)}$ is a finite point set, and $Y^{(m)} = Y \cap Z$ where $Z \subset \mathbb{A}^M$ is defined by m equations. We can assume that $y \in Z$. After shrinking the affine open which we work on we can assume $Z \cap Y = \{y\}$. Suppose that Z is defined by $g_1 = \cdots = g_m = 0$; then $f^{-1}(y)$ in X is defined by $f^*(g_1) = \cdots = f^*(g_n) = 0$. Thus (1) follows from Corollary 8.14.

For (2) we can again replace Y by an affine open W and X by an affine open $V \subset f^{-1}(W)$. Since V is dense on $f^{-1}(W)$ and f is surjective, we have that $f(V)$ is dense in W. Hence there is an inclusion

$$f^*\colon \mathcal{O}_W(W) \hookrightarrow \mathcal{O}_V(V)$$

of coordinate rings. Let $\mathcal{O}_W(W) = k[w_1, \ldots, w_M]$, $\mathcal{O}_V(V) = k[v_1, \ldots, v_N]$ where the w_i resp. v_j are k-algebra generators. Now $k(V)$ has transcendence degree $n - m$ over $k(W)$. We can assume after reordering that $v_1, \ldots, v_{n-m}$ are algebraically independent over $k(W)$, and the other v_i are algebraic over $k(W)[v_1, \ldots, v_{n-m}]$. Thus we have equations

$$F_i(v_i; v_1, \ldots, v_{n-m}; w_1, \ldots, w_M) = 0, \quad i = n - m + 1, \ldots, N$$

where F_i is a polynomial in the variables $v_i, v_1, \ldots, v_{n-m}$ with coefficients in $k[w_1, \ldots, w_M]$. Let Y_i the subvariety of W defined by the vanishing of the leading term of F_i. Then $U = W \setminus \bigcup_i Y_i$ is open and nonempty (the leading terms are nonzero). If $y \in U$, then none of the polynomials

$$F_i(T_i; T_1, \ldots, T_{n-m}; w_1(y), \ldots, w_M(y))$$

is identically zero, whence the functions $v_i \mid_{f^{-1}(y)\cap V}$ are algebraically dependent on the functions $v_1 \mid_{f^{-1}(y)\cap V}, \ldots, v_{n-m} \mid_{f^{-1}(y)\cap V}$. Since the $v_1 \mid_{f^{-1}(y)\cap V}, \ldots, v_N \mid_{f^{-1}(y)\cap V}$ generate $\mathcal{O}(f^{-1}(y) \cap V)$, we obtain $\dim f^{-1}(y) \leq n - m$, and with part (1), equality. □

Corollary 8.17. *In the situation of Theorem 8.16,the sets $Y_k := \{y \in Y \mid \dim f^{-1}(y) \geq k\}$ are closed in Y.*

Proof. By (1) of Theorem 8.16, $Y_{n-m} = Y$, and there is a closed subset $Z \subsetneq Y$ such that $Y_k \subset Z$ for $k > n - m$ by (2) of that Theorem. If Z_i are the irreducible components of Z, then $\dim Z_i < \dim Y$, and the claim follows considering $f \mid_{f^{-1}(Z_i)} \colon f^{-1}(Z_i) \to Z_i$ by induction on $\dim Y$. □

The following is an often useful criterion to prove irreducibility.

Theorem 8.18. *Let $f \colon X \to Y$ be a surjective morphism of projective varieties where we assume that Y is irreducible, but a priori we do not assume irreducibility of X; if then for all $y \in Y$ the fibers $f^{-1}(y)$ are irreducible and of the same dimension, then X is irreducible.*

Proof. Let $X = \bigcup X_i$ the decomposition into irreducible components; then $Y = \bigcup f(X_i)$; note that $f(X_i)$ is closed in Y since X_i is projective, and since Y is irreducible, $Y = f(X_i)$ for some i. Put $n := \dim f^{-1}(y)$. By Theorem 8.16, for all i with $f(X_i) = Y$ there is $U_i \subset Y$ open and dense such that $\dim((f \mid_{X_i})^{-1}(y)) = n_i$ for some $n_i \in \mathbb{N}$ and all $y \in U_i$. For indices j with $f(X_j) \neq Y$, put $U_j = Y \backslash f(X_j)$. Let $y \in \bigcap_i U_i$. Since $f^{-1}(y)$ is irreducible, we have $f^{-1}(y) \subset X_{i_0}$ for some i_0. Then

$$f^{-1}(y) \subset \left(f \mid_{X_{i_0}}\right)^{-1}(y)$$

and the reverse inclusion being trivial we get equality

$$f^{-1}(y) = \left(f \mid_{X_{i_0}}\right)^{-1}(y)$$

and $n = n_{i_0}$. By construction, $f \mid_{X_{i_0}}$ is surjective, hence for all $y \in Y$, we have that the subset $\left(f \mid_{X_{i_0}}\right)^{-1}(y) \subset f^{-1}(y)$ is nonempty and of dimension $\geq n_{i_0} = n$ by Theorem 8.16. Thus, since $f^{-1}(y)$ is irreducible and always of dimension $n = n_{i_0}$, we have

$$f^{-1}(y) = \left(f \mid_{X_{i_0}}\right)^{-1}(y) \quad \forall y \in Y.$$

This means $X = X_{i_0}$. □

It is easy to construct examples (exercise!) that show that the hypothesis of equidimensionality of the fibers in Theorem 8.18 cannot be dropped.

Chapter 9

Lines on surfaces, the associated form of Chow and van der Waerden, and degree

Let $X \subset \mathbb{P}^n$ be a hypersurface given by the vanishing of a homogeneous polynomial F of degree d. We consider k-dimensional projective linear subspaces $\Lambda = \mathbb{P}(L) \subset \mathbb{P}^n$ which are contained in X, i.e. $L \in \mathrm{Grass}(k+1, n+1)$ with $\mathbb{P}(L) \subset X$.

Definition 9.1. Put $\mathbb{P}^N = \mathbb{P}(k[x_0, \ldots, x_n]_d)$, $N = \binom{n+d}{d} - 1$, and define

$$\Phi_{n,d,k} = \Phi = \{([F], \Lambda) \in \mathbb{P}^N \times \mathrm{Grass}(k+1, n+1) \mid \Lambda \subset X = \{F = 0\}\}.$$

Lemma 9.2. *The subset $\Phi \subset \mathbb{P}^N \times \mathrm{Grass}(k+1, n+1)$ is a subvariety.*

Proof. Let $U = U_{i_1 \ldots i_{k+1}} \simeq \mathbb{A}^{(k+1)(n-k)}$ be the open subset of $\mathrm{Grass}(k+1, n+1)$ where the Plücker coordinate $\omega_{i_1 \ldots i_k}$ is not equal to zero, some $1 \leq i_1 < \cdots < i_{k+1} \leq n$. We show that $\Phi \cap (\mathbb{P}^N \times U)$ is an algebraic subvariety.

Without loss of generality we can assume that $i_1 = 1, \ldots, i_{k+1} = k+1$. Elements in U are matrices

$$A = \begin{pmatrix} 1 & \ldots & 0 & a_{11} & \ldots & a_{1\, n-k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & a_{k+1\, 1} & \ldots & a_{k+1\, n-k} \end{pmatrix}$$

whose rows are a basis of $L \in \mathrm{Grass}(k+1, n+1)$. If

$$F = \sum_{i_0 + \cdots + i_n = d} c_{i_0 \ldots i_n} X_0^{i_0} \ldots X_n^{i_n}$$

is an equation for X, then $\Lambda = \mathbb{P}(L) \subset X = \{F = 0\}$ is equivalent to the following condition: if a_i denotes the i-th row vector of A and $\lambda_i \in k$, then

$$F(\lambda_1 a_1 + \cdots + \lambda_{k+1} a_{k+1})$$

vanishes identically as polynomial in the λ's. This means that the coefficients of all monomials in the λ's must vanish, and those are polynomials in the a_{ij} and $c_{i_0 \ldots i_n}$. $\square$

We will now consider more closely the case of lines on surfaces in $\mathbb{P}^3$, i.e. $\Phi = \Phi_{3,d,1}$. We have two projections

$$\begin{tikzcd} & \Phi \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^N & & \mathrm{Grass}(2,4) \end{tikzcd}$$

where $N = \binom{d+3}{3} - 1$ and $\mathrm{Grass}(2,4)$ is the Plücker quadric in $\mathbb{P}^5$. We have $q(\Phi) = \mathrm{Grass}(2,4)$ since every line $l \in \mathbb{P}^3$ lies on some (possibly reducible) surface of degree d, for example we could take a union of d planes through l. The next question is: what is $\dim q^{-1}(l)$? After applying a projectivity, we may assume l is given by $X_0 = X_1 = 0$ in $\mathbb{P}^3$. Then any surface of degree d containing l is given by an equation of the form

$$F = X_0 G + X_1 H, \quad G, H \in k[X_0, X_1, X_2, X_3]_{d-1}$$

and any such equation defines a surface containing l; thus the fibers of q are all projective linear subspaces of $\mathbb{P}^N$ of dimension (the same for all l)

$$\dim q^{-1}(l) = \frac{d(d+1)(d+5)}{6} - 1.$$

To see that this is the dimension of $q^{-1}(l)$, use the exact sequence (where $S = k[X_0, X_1, X_2, X_3]$)

$$0 \longrightarrow S_{d-2} \xrightarrow{\beta = \begin{pmatrix} X_1 \\ -X_0 \end{pmatrix}} S_{d-1} \oplus S_{d-1} \xrightarrow{\alpha = \begin{pmatrix} X_0 & X_1 \end{pmatrix}} S_d$$

which gives

$$\begin{aligned} \dim q^{-1}(l) + 1 = {} & \dim \mathrm{im}(\alpha) = 2 \dim S_{d-1} - \dim S_{d-2} \\ = {} & 2\tbinom{d-1+3}{3} - \tbinom{d-2+3}{3} = \tfrac{2(d+2)(d+1)d - (d+1)d(d-1)}{6} \\ = {} & \tfrac{(d+5)d(d+1)}{6}. \end{aligned}$$

By Theorem 8.18 it follows that Φ is irreducible. By Theorem 8.16

$$\begin{aligned} \dim \Phi = & \dim \mathrm{Grass}(2,4) + \dim q^{-1}(l) \\ = & 4 + \tfrac{d(d+1)(d+5)}{6} - 1 \\ = & \tfrac{(d+3)(d+2)(d+1)}{6} - 1 - d + 3 \\ = & N + 3 - d. \end{aligned}$$

Now by Theorem 6.2, $p(\Phi) \subset \mathbb{P}^N$ is closed. Also by Theorem 8.16, clearly $\dim p(\Phi) \leq \dim \Phi$. Thus, if $\dim \Phi < N$, then $\dim p(\Phi) < N$ and by Theorem 8.4, $p(\Phi) \neq \mathbb{P}^N$. This conclusion means that there is a nonempty Zariski open subset in $\mathbb{P}^N$ such that the corresponding surfaces do not contain any lines (in other words, a "general" such surface does not contain lines).

The numerical condition for this, $\dim \Phi < N$ translates into $d > 3$. Hence we have proven:

Theorem 9.3. *If $d > 3$, then a general surface of degree d in $\mathbb{P}^3$ does not contain any lines.*

There remain the cases $d = 1, 2, 3$. The case $d = 1$ corresponds to planes and is trivial, these contain an abundance of lines. For $d = 2$ we get $N = 9$ and $\dim \Phi = 10$. By Theorem 8.16, we have $\dim p^{-1}([Q]) \geq 1$ where Q is a homogeneous quadratic polynomial defining a quadric. This corresponds to the fact that every quadric contains infinitely many lines. However, this case illustrates nicely the jump phenomenon described in Corollary 8.17: the fiber dimension can jump up on closed subsets of the base. Indeed, if the quadric is irreducible, $\dim p^{-1}([Q]) = 1$: the fiber consists of two disjoint $\mathbb{P}^1$'s corresponding to the two rulings if Q is not a cone, and is one $\mathbb{P}^1$ if Q is a cone. But if Q is reducible, it splits as two planes or a double plane, and then $\dim p^{-1}([Q]) = 2$.

The case $d = 3$ is that of cubic surfaces. Here $\dim \Phi = N = 19$. Moreover, there exist cubic surfaces with finitely many lines on them, for example it is not hard to find all lines on

$$X_0^3 + X_1^3 + X_2^3 + X_3^3 = 0.$$

Thus there is a point in $\mathbb{P}^{19}$ over which the fiber of p has dimension 0. By Theorem 8.16 we get $\dim p(\Phi) = 19$, whence $p(\Phi) = \mathbb{P}^{19}$. Thus we have proven:

Theorem 9.4. *Every cubic surface in $\mathbb{P}^3$ contains at least one line. There is a nonempty Zariski open subset in $\mathbb{P}^{19}$ such that the corresponding surfaces contain finitely many lines.*

However, there are cubic surfaces with infinitely many lines, e.g. cones over cubics in $\mathbb{P}^2$. Hence here we observe again jumps in the fiber dimension.

Moreover, one can show that for "most" cubic surfaces with finitely many lines, the number and configuration of the lines (i.e., their incidence graph) are independent of the surface. The number is 27. More concerning this in Chapter 11.

We start discussing the *associated form of Chow and van der Waerden*.

Let $X \subset \mathbb{P}^n = \mathbb{P}(V)$, V an $(n+1)$-dimensional vector space, be a projective variety all of whose irreducible components have dimension k (X is purely k-dimensional). Intuitively, we want to describe such X by "coordinates", i.e. parametrize them by the points of another variety (we have already accomplished this in cases of hypersurfaces or if X is a linear subspace). Write $(\mathbb{P}^n)^* = \mathbb{P}(V^*)$. Look at the incidence correspondence

$$\Gamma := \{(p, H_0, \ldots, H_k) \mid p \in H_i \; \forall\, i\} \subset X \times (\mathbb{P}^n)^* \times \cdots \times (\mathbb{P}^n)^*$$

where the $H_i \subset \mathbb{P}^n$ are hyperplanes; here we identify a hyperplane with its defining equation in $(\mathbb{P}^n)^*$. Clearly, Γ is a closed subvariety of $X \times (\mathbb{P}^n)^* \times \cdots \times (\mathbb{P}^n)^*$ equipped with two projections

$$\begin{tikzcd} & \Gamma \arrow[dl, "\psi"'] \arrow[dr, "\varphi"] & \\ X & & \left((\mathbb{P}^n)^*\right)^{k+1} . \end{tikzcd}$$

Clearly, $\psi(\Gamma) = X$, and for $p \in X$ the set of hyperplanes containing p forms an $(n-1)$-dimensional projective linear subspace $\mathcal{H}_p \simeq \mathbb{P}^{n-1} \subset (\mathbb{P}^n)^*$. Thus $\psi^{-1}(p)$ is irreducible of dimension $(k+1)(n-1)$, and isomorphic to $\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}$ ($k+1$ factors). By Theorems Theorem 8.16 and 8.18 Γ is purely $k + (k+1)(n-1) = (k+1)n - 1$ dimensional, with one irreducible component lying over each irreducible component of X. Moreover, there exist points

$$y = (H_0, \ldots, H_k) \in \left((\mathbb{P}^n)^*\right)^{k+1}$$

such that $\varphi(\Gamma) \ni y$ and $\varphi^{-1}(y)$ is a single point: by the same construction as the one employed in the proof of Corollary 8.8, we can find a chain

$$X^{(0)} = X \supsetneq X^{(1)} = X \cap H_0 \supsetneq \cdots \supsetneq X^{(k+1)} = X \cap \bigcap_{i=0}^{k} H_i = \{p\}$$

where $p \in X$ is a point, and the H_i's are hyperplanes; namely we can choose the H_i's such that all of them pass through p, H_i does not contain any irreducible component of $X^{(i)}$ for $i < k$, and H_k contains only $p \in X^{(k)}$ and none of the other finitely many points of $X^{(k)}$. Then, by Theorem 8.16, we obtain

$$\dim \varphi(\Gamma) = \dim \Gamma = (k+1)n - 1,$$

i.e., $\varphi(\Gamma) \subset ((\mathbb{P}^n)^*)^{k+1}$ is purely 1-codimensional. Now Theorem 8.6 remains valid, with the same proof, in the multi-graded set-up:

Theorem 9.5. *If $Y \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is an algebraic subvariety and if all components of Y have dimension $n_1 + \cdots + n_k - 1$, then Y is defined by a single equation $F = 0$ where F is homogeneous in each of the k set of variables separately; F is unique up to a constant factor if we choose it without multiple factors.*

Thus we can make

Definition 9.6. The multi-homogeneous polynomial F_X without multiple factors (unique up to a constant) which defines $\varphi(\Gamma) \subset ((\mathbb{P}^n)^*)^{k+1}$ is called the *Cayley form* of X or the *associated form of Chow and van der Waerden* of X.

Remark 9.7. The variety X is determined by the datum of F_X:

$$\forall\, p \in \mathbb{P}^n : (p \in X) \iff$$
$$(\forall\, H_0, \ldots, H_k \in (\mathbb{P}^n)^* \text{ with } p \in H_0 \cap \cdots \cap H_k \colon F_X(H_0, \ldots, H_k) = 0)\,.$$

Indeed, $\Longrightarrow$ is nothing but the definition of F_X. To see $\Longleftarrow$, notice that if $p \notin X$, then, by the by now standard construction employed in the proof of Corollary 8.8, there are hyperplanes $H_0, \ldots, H_k$ with $p \in \bigcap_i H_i$ and $X \cap \bigcap_i H_i = \emptyset$. Such H_i are not in the image of φ, i.e. $F_X(H_0, \ldots, H_k) \neq 0$.

If $u_0, \ldots, u_n$ are homogeneous coordinates in $\mathbb{P}^n$ and

$$H_i = (\sum_{j=0}^{n} v_j^{(i)} u_j = 0)$$

the equation of a hyperplane in the $i+1$-th copy of $(\mathbb{P}^n)^*$ in $((\mathbb{P}^n)^*)^{k+1}$, then we can take

$$(v_0^{(0)} : \cdots : v_n^{(0)}; \ldots ; v_0^{(k)} : \cdots : v_n^{(k)})$$

as coordinates in $((\mathbb{P}^n)^*)^{k+1}$.

Remark 9.8. The Cayley form

$$F_X = F_X(v_0^{(0)}, \ldots, v_n^{(0)}; \ldots ; v_0^{(k)}, \ldots, v_n^{(k)})$$

is homogeneous of the *same* degree, d say, in each of the sets of variables $v_0^{(i)}, \ldots, v_n^{(i)}$, $i = 0, \ldots, k$. This follows because $\varphi(\Gamma) \in (\mathbb{P}^n)^* \times \cdots \times (\mathbb{P}^n)^*$ is invariant under permutations of the factors.

Definition 9.9. The coordinates of F_X with respect to the basis given by monomials in $k[v_0^{(0)}, \ldots, v_n^{(0)}; \ldots ; v_0^{(k)}, \ldots, v_n^{(k)}]$ are called the *Chow coordinates* of X.

Thus attached to X we get a discrete parameter d and continuous parameters (the coefficients of monomials in F_X) which determine $X \subset \mathbb{P}^n$.

In the rest of this Chapter we assume $\mathrm{char}(k) = 0$ for simplicity, but the results below continue to hold without this assumption.

Theorem 9.10. *Let X be a projective subvariety of $\mathbb{P}^n$ as above, and F_X its Cayley form. Suppose F_X is of degree d in each of the sets of variables $v_0^{(i)}, \ldots, v_n^{(i)}$, $i = 0, \ldots, k$. Then this d is equal to the maximum number of intersection points $X \cap L$ of X with a linear subspace $L \subset \mathbb{P}^n$ of dimension* $\dim L = n - k$ *whenever this number is finite.*

We call this d the **degree** $\deg X$ of the projective subvariety $X \subset \mathbb{P}^n$.

Proof. Let $H_1, \ldots, H_k \in (\mathbb{P}^n)^*$ be hyperplanes with $|X \cap \bigcap_i H_i| < \infty$ and

$$X \cap \bigcap_i H_i = \{x^{(1)}, \ldots, x^{(c)}\}, \quad x^{(l)} = (u_0^{(l)} : \cdots : u_n^{(l)}),\ l = 1, \ldots, c$$

(such hyperplanes exist by the standard construction in the proof of Corollary 8.8). Let H_0 be a further hyperplane and write

$$H_i = \left(\sum_{j=0}^{n} v_j^{(i)} u_j = 0\right).$$

Now we want to think of the $H_1, \ldots, H_k$ as fixed and of H_0 as variable. Then F_X is a polynomial of degree d in $v_0^{(0)}, \ldots, v_n^{(0)}$ which vanishes if and only if

$$\sum_{j=0}^{n} v_j^{(0)} u_j^{(l)} = 0$$

for at least one l. This means that we get a factorization

$$F_X(v_0^{(0)}, \ldots, v_n^{(0)}; H_1, \ldots, H_k) = \alpha \prod_{l=1}^{c} (\sum_{j=0}^{n} v_j^{(0)} u_j^{(l)})^{r_l}.$$

Here α is a constant and $r_j \geq 1$ some integers. In other words, $c \leq d$ and if $F_X(v_0^{(0)}, \ldots, v_n^{(0)}; H_1, \ldots, H_k)$ does not have multiple factors, then $c = d$.

Thus, to conclude the proof, it suffices to show that for suitable choice of $H_1, \ldots, H_k$, the polynomial $F_X(v_0^{(0)}, \ldots, v_n^{(0)}; H_1, \ldots, H_k)$ does not have multiple factors. We apply Lemma 9.11 below which we leave as an exercise. $\square$

Lemma 9.11. *Suppose* $\mathrm{char}(k) = 0$. *If a polynomial*

$$F(v_0^{(0)}, \ldots, v_n^{(0)}; \ldots; v_0^{(k)}, \ldots v_n^{(k)})$$

has no multiple factors (and at least some variable $v_i^{(0)}$ *occurs in* F*), then there are special values*

$$V_0^{(1)}, \ldots, V_n^{(k)}$$

for the $v_0^{(1)}, \ldots, v_n^{(k)}$ *such that*

$$F(v_0^{(0)}, \ldots, v_n^{(0)}; V_0^{(1)}, \ldots, V_n^{(k)})$$

has no multiple factors.

Corollary 9.12. *Under the hypotheses of Theorem 9.10 there exists a nonempty Zariski open subset of* $(H_1, \ldots, H_k)$ *in* $((\mathbb{P}^n)^*)^k$ *such that*

$$|X \cap H_1 \cap \cdots \cap H_k| = d.$$

Proof. The sets of $(H_1, \dots, H_k)$ for which $F_X(v_0^{(0)}, \dots, v_n^{(0)}; H_1, \dots, H_k)$

1. does not vanish identically
2. has no multiple factor

are both Zariski open and nonempty. Namely, let us first consider the set in (1). It is nonempty since there are hyperplanes $H_0, \dots, H_k$ with $X \cap H_0 \cap \dots \cap H_k = \emptyset$, thus $F_X(H_0, \dots, H_k) \neq 0$. It is clear that it is open since otherwise the coefficients of all monomials in $v_0^{(0)}, \dots, v_n^{(0)}$ occurring in $F_X(v_0^{(0)}, \dots, v_n^{(0)}; H_1, \dots, H_k)$ must vanish and this defines a closed subvariety of $(\mathbb{P}^n)^k$.

Now in the course of the proof of Theorem 9.10 we saw that the set in (2) is nonempty. It is Zariski open since the subset of homogeneous polynomials $F \in k[t_0, \dots, t_n]_d$ with a multiple factor is closed since the multiplication map

$$\mathbb{P}(k[t_0, \dots, t_n]_{d-2k}) \times \mathbb{P}(k[t_0, \dots, t_n]_k) \to \mathbb{P}(k[t_0, \dots, t_n]_d)$$
$$(G, H) \mapsto GH^2$$

is a morphism and we can apply Theorem 6.2 to conclude that the image is closed.

Now being in set (1) means that $H_1, \dots, H_k$ intersect X in finitely many points, and being in (2) means that their number is d. $\square$

The set of all forms $F(v_0^{0)}, \dots, v_n^{(0)}; \dots; v_0^{(k)}, \dots, v_n^{(k)})$ of degree d in each set of variables forms a projective space $\mathbb{P}^N$ and by Remark 9.7 we get an injection

$$c\colon \{X \subset \mathbb{P}^n \mid X \text{ purely } k\text{−dimensional, } \deg X = d\} \to \mathbb{P}^N$$
$$X \mapsto [F_X].$$

Remark 9.13. We summarize some properties of the image of c without proof. It turns out that $\operatorname{im}(c) =: C_{n,k,d} \subset \mathbb{P}^N$ is a quasi-projective variety. Hence in this way one can parametrize purely k-dimensional subvarieties X of $\mathbb{P}^n$ of degree d by the points of another variety, similar to the situation for linear subspaces and Grassmannians. We get a closed (projective) subvariety in $\mathbb{P}^N$ if we consider, instead of X's as before, k-dimensional cycles of degree d which are formal linear combinations

$$Z_k = m_1 X_1 + \dots + m_p X_p$$

where $X_i \subset \mathbb{P}^n$ is irreducible of dimension k and $d = m_1 \deg X_1 + \cdots + m_p \deg X_p$. For those one extends the definition of the Cayley form by

$$F_{Z_k} := F_{X_1}^{m_1} \dots F_{X_l}^{m_l}.$$

In this way we obtain exactly the closure $\overline{C}_{n,k,d}$ of $C_{n,k,d}$ in $\mathbb{P}^N$. Then $\overline{C}_{n,k,d}$ is called the *Chow variety* of algebraic cycles of dimension k and degree d in $\mathbb{P}^n$, and $C_{n,k,d}$ is called the *open Chow variety* of purely k-dimensional subvarieties of $\mathbb{P}^n$ of degree d.

Both $C_{n,k,d}$ and $\overline{C}_{n,k,d}$ are thus parameter spaces for certain algebro-geometric objects. In general it is unknown how many irreducible components these Chow varieties have, even for curves in $\mathbb{P}^3$. Here are some known results for $\overline{C}_{3,1,d}$, $d = 1, 2, 3$:

1. For $d = 1$, $\overline{C}_{3,1,1}$ is the Plücker quadric in $\mathbb{P}^5$, which is irreducible of dimension 4.

2. For $d = 2$, $\overline{C}_{3,1,2}$ is reducible with two irreducible components, $\overline{C}_{3,1,2} = C' \cup C''$, with $\dim C' = \dim C'' = 8$. Here C' parametrizes plane conics, and C'' parametrizes 2 lines in $\mathbb{P}^3$. They intersect in the locus corresponding to two intersecting lines.

3. For $d = 3$ there is a decomposition into irreducible components

$$\overline{C}_{3,1,3} = C^{(1)} \cup C^{(2)} \cup C^{(3)} \cup C^{(4)}$$

where all $C^{(i)}$ have dimension 12 and $C^{(1)}$ corresponds to unions of three lines; points in $C^{(2)}$ parametrize unions of a plane conic and a line; $C^{(3)}$ parametrizes plane cubics; and $C^{(4)}$ parametrizes twisted cubics in $\mathbb{P}^3$.

A funny loose end in our approach to degree is that we still have to show that the answer for hypersurfaces is as expected:

Theorem 9.14. *If X is an irreducible hypersurface in $\mathbb{P}^n$ defined by*

$$G(u_0, \dots, u_n) = 0$$

where G is a homogeneous irreducible polynomial of degree d, then $\deg X = d$.

Proof. It is clear that $\deg X \leq d$. Thus it suffices to show (after dehomogenization): if

$$X \cap \mathbb{P}^n_{u_0 \neq 0} =: H \subset \mathbb{A}^n$$

is defined by an irreducible polynomial $g \in k[X_1, \dots, X_n]$ of degree d, then there is a line $l \subset \mathbb{A}^n$ such that $H \cap l$ consists of exactly d points. We show this directly as follows.

We search for such an l in the parameter form

$$l = \{(x_1, \dots, x_n) + \lambda(y_1, \dots, y_n) \mid \lambda \in k\},$$
$$x = (x_1, \dots, x_n) \in \mathbb{A}^n,\ y = (y_1, \dots, y_n) \in \mathbb{A}^n.$$

Let

$$g = g_d + g_{d-1} + \cdots + g_0$$

be the decomposition into homogeneous components, $g_d \neq 0$. Then the condition $g(x + \lambda y) = 0$ can be written as

$$g_d(y)\lambda^d + \left(\sum_{i=1}^n \frac{\partial g_d}{\partial X_i}(y) x_i + g_{d-1}(y) \right) \lambda^{d-1} + \cdots = 0. \tag{9.1}$$

Now we want to determine x and y such that

1. $g_d(y) \neq 0$ (this is O.K. since k is an infinite field)
2. and such that $\tilde{G}(\lambda) = g(x + \lambda y)$ has degree d (O.K. by (1)) and is coprime to
$$\tilde{G}'(\lambda) = \sum_{i=1}^n \frac{\partial g}{\partial X_i}(x + \lambda y) y_i.$$

If we accomplish (2) the proof is complete since then $\tilde{G}(\lambda)$ has d simple roots.

Let $Y_1, \dots, Y_n, U$ be further indeterminates. Expand

$$\varphi := g(X_1 + UY_1, \dots, X_n + UY_n)$$

in powers of U:

$$\varphi = g(X_1, \dots, X_n) + \left(\sum_{i=1}^n \frac{\partial g}{\partial X_i}(X_1, \dots, X_n) \cdot Y_i \right) \cdot U + \cdots + g_d(Y_1, \dots, Y_n) U^d.$$

We cannot have $\partial g / \partial X_i = 0$ for all $i = 1, \dots, n$ (g is nonzero irreducible). Thus choose $x \in \mathbb{A}^n$ such that

$$g(x) \neq 0, \quad \sum_{i=1}^n \frac{\partial g}{\partial X_i}(x) Y_i \neq 0. \tag{9.2}$$

Then
$$\psi(Y_1, \dots, Y_n, U) := g(x_1 + UY_1, \dots, x_n + UY_n)$$
is irreducible in $k[Y_1, \dots, Y_n, U]$. To see this, assume by contradiction $\psi = h_1 h_2$ is a nontrivial decomposition. There are two possibilities now:

(I) At least one of the variables $Y_1, \dots, Y_n$ occurs in h_1 and h_2 (it need not be the same variable though). Then, for suitable $u \in k^*$, there would be a nontrivial decomposition of
$$\psi(Y_1, \dots, Y_n, u) = g(x_1 + uY_1, \dots, x_n + uY_n)$$
in $k[Y_1, \dots, Y_n]$. This is a contradiction to the irreducibility of g in $k[X_1, \dots, X_n]$ since it will remain irreducible after the substitution $X_i \mapsto x_i + uY_i$.

(II) Either h_1 or h_2 (or both) are polynomials in which none of the Y's occurs. Suppose that it is h_1 without loss of generality. But then, since $g(x) \neq 0$, by our choice in (9.2), h_1 is not divisible by U and there exists a $u \in k^*$ such that $h_1(u) = 0$. But then also $g(x_1 + uY_1, \dots, x_n + uY_n) \equiv 0$, contradicting (9.2).

Thus ψ is irreducible and $\partial\psi/\partial U \neq 0$ because of the second inequality in (9.2). Hence
$$\psi, \quad \frac{\partial\psi}{\partial U}$$
are coprime. Thus there exist polynomials $\delta \in k[Y_1, \dots, Y_n] \backslash \{0\}$, $a, b \in k[Y_1, \dots, Y_n, U]$ such that
$$\delta = a\psi + b\frac{\partial\psi}{\partial U}$$
since ψ and $\partial\psi/\partial U$ are coprime also in $k(Y_1, \dots, Y_n)[u]$ by Gauss's Lemma, and then we can use the Euclidean algorithm solved backwards to find a representation as before. Now choose $y \in \mathbb{A}^n$ such that
$$g_d(y) \neq 0, \quad \delta(y) \neq 0. \tag{9.3}$$
Then
$$\tilde{G}(U) = \psi(y_1, \dots, y_n, U) = g(x_1 + y_1 U, \dots, x_n + y_n U)$$
is of degree d since $g_d(y) \neq 0$ and coprime to
$$\tilde{G}'(U) = \frac{\partial\psi}{\partial U}(y_1, \dots, y_n, U)$$

since $\delta(y) \neq 0$. Hence $l = \{x + \lambda y \mid \lambda \in k\}$ is a line with the desired properties. $\square$

One vexing point in the theory of Cayley forms as we have been developing it so far is that we do not know how to compute them explicitly! We can now say what happens in the case of a hypersurface.

Corollary 9.15. *Under the hypotheses of Theorem 9.14 the Cayley form of*

$$X = \{G(u_0, \ldots, u_n) = 0\}$$

is

$$\begin{aligned} F_X(v_0^{(0)}, \ldots, v_n^{(0)}; \ldots; v_0^{(n-1)}, \ldots, v_n^{(n-1)}) \\ = G(\Delta_0, \ldots, \Delta_n) \end{aligned}$$

where $(-1)^i \Delta_i$ *is the minor of the matrix*

$$V = \begin{pmatrix} v_0^{(0)} & \ldots & v_n^{(0)} \\ \vdots & \ddots & \vdots \\ v_0^{(n-1)} & \ldots & v_n^{(n-1)} \end{pmatrix}$$

obtained by deleting the i-th column (numbering the columns starting with zero).

Proof. If $H_0, \ldots, H_{n-1}$ intersect in a point, then it is given by

$$(\Delta_0 : \cdots : \Delta_n).$$

Hence $G(\Delta_0, \ldots, \Delta_n)$ vanishes on a Zariski open dense subset of $F_X = 0$. Thus F_X divides $G(\Delta_0, \ldots, \Delta_n)$, but according to Theorem 9.14, F_X has multi-degree d, the same as G. $\square$

One can make Cayley forms more explicit using resultants, but we don't go into this.

We state the following important result about degree without proof.

Theorem 9.16 ((Weak) Bezout's Theorem)**.** *Let X and Y be two closed subvarieties in $\mathbb{P}^n$ which are pure dimensional. Suppose that X and Y intersect* properly, *i.e.* $\dim(X \cap Y) = \dim X + \dim Y - n$. *Then*

$$\deg(X \cap Y) \leq \deg X \cdot \deg Y.$$

In fact, by assigning appropriate integers, called *multiplicities*, to the irreducible components of $X \cap Y$, one can even turn the inequality in Theorem 9.16 into an equality.

Chapter 10

Regular and singular points, tangent space

Let $X \subset \mathbb{P}^n$ be a quasi-projective variety. For $x \in X$, we want to define the tangent space T_xX of X in x. Clearly, $X \subset \overline{X} \subset \mathbb{P}^n$, and if $x \in X$, we will want to define $T_xX = T_x\overline{X}$, thus we start with a projective $X \subset \mathbb{P}^n$. After reordering the coordinates we can assume that x is in the affine chart $U_0 = \{X_0 \neq 0\}$. Consider $X_0 = X \cap U_0 \subset \mathbb{A}^n$ and let $I(X_0) = (F_1, \ldots, F_m)$, $F_j \in k[x_1, \ldots, x_n]$ be the ideal of X_0. We can also assume, after an affine-linear coordinate transformation, that $x = (0, \ldots, 0) = 0$.

For a line $L \subset \mathbb{P}^n$ through x we want to define the *intersection multiplicity* $\text{mult}_x(L \cap X)$ of L and X in x. Then $L_0 = L \cap U_0$ is given by

$$L_0 = \{\lambda a \mid \lambda \in k\}, \quad a \in U_0 \backslash \{0\}.$$

Put

$$f(\lambda) := \gcd(F_1(\lambda a), \ldots, F_m(\lambda a)) = \prod_i (\lambda - \alpha_i)^{k_i}.$$

Here the $\lambda = \alpha_i$ correspond to the intersection points of X_0 with L_0 if these are finitely many. Then we put $\text{mult}_x(X \cap L) :=$ multiplicity of the zero $\lambda = 0$ of $f(\lambda)$, i.e. the highest power of λ which divides all $F_i(\lambda a)$. If $F_i(\lambda a) \equiv 0$ for all i, we put $\text{mult}_x(X \cap L) = +\infty$.

This is independent of the choice of the F_i since $f(\lambda) = \gcd\{F(\lambda a) \mid F \in I(X_0)\}$, and it is also independent of a choice of affine chart containing x since we can alternatively define $\text{mult}_x(X \cap L)$ as follows: if

$$L = \{[\mu_0 p_0 + \mu_1 p_1] \mid (\mu_0 : \mu_1) \in \mathbb{P}^1,\ [p_0], [p_1] \in \mathbb{P}^n\}$$

and if the homogeneous ideal of X is

$$I(X) = (\Phi_1, \dots, \Phi_M)$$

and $p = [\mu_0^{(0)} p_0 + \mu_1^{(0)} p_1]$, then $\mathrm{mult}_x(X \cap L)$ is the highest power of $(\mu_1^{(0)}\mu_0 - \mu_0^{(0)}\mu_1)$ which divides all Φ's.

Definition 10.1. A line L is called a **tangent** to X in x if

$$\mathrm{mult}_x(X \cap L) \geq 2.$$

Since $X_0 \ni 0$, in the notation above no F_i has a constant term. Let its linear term be L_i and write $F_i = L_i + G_i$, $i = 1, \dots, m$. Then

$$F_i(\lambda a) = \lambda L_i(a) + G_i(\lambda a), \quad \lambda^2 \mid G_i(\lambda a).$$

Thus λ^2 divides $F_i(\lambda a)$ if and only if $L_i(a) = 0$ for all i; thus L_0 is tangent to X_0 in 0 if and only

$$L_1(a) = \cdots = L_m(a) = 0.$$

Definition 10.2. Let us make $\mathbb{A}^n \simeq U_0$ into a vector space with origin $x = 0$ by componentwise addition and scalar multiplication. The sub-vector space of all points $a \in \mathbb{A}^n$ which lie on tangents to X in x is called the *(embedded, affine) tangent space* of X in x, denoted by $T_x X$. Its closure in $\mathbb{P}^n$ (i.e. the locus of points in $\mathbb{P}^n$ which lie on tangents L to X in x) is called *(embedded, projective) tangent space*, denoted by $\mathbb{T}_x X$.

Definition 10.3. The local dimension of X in x is defined by

$$\dim_x X := \max\{\dim Z \mid Z \text{ irreducible component of } X \text{ through } x\}.$$

Definition 10.4. A point $x \in X$ is called *regular* if $\dim_x X = \dim_k T_x X$ $(= \dim \mathbb{T}_x X)$. A point $x \in X$ is called *singular* if $\dim_k T_x X > \dim_x X$. X is called *regular* or *nonsingular* if all points of X are regular.

At the moment it is not a priori clear that X is equal to the union of its regular and singular points, or that the singular points form a proper closed subvariety. We will see this later.

Theorem 10.5. *If $X \subset \mathbb{P}^n$ is a projective variety, $I(X) = (F_\alpha)_{\alpha\in A}$, F_α homogeneous, then $\mathbb{T}_x X \subset \mathbb{P}^n$ is defined by*

$$\sum_{i=0}^{n} \frac{\partial F_\alpha}{\partial X_i}(x) X_i = 0,\ \alpha \in A,$$

where $(X_0 : \cdots : X_n)$ are homogeneous coordinates in $\mathbb{P}^n$.

If $X \subset \mathbb{A}^n$ is affine, $I(X) = (G_\beta)_{\beta\in B}$, then for $x \in X$, the tangent space $T_x X \subset \mathbb{A}^n$ is defined by

$$\sum_{i=1}^{n} \frac{\partial G_\beta}{\partial t_i}(x)(t_i - x_i) = 0,\ \beta \in B,$$

where $(t_1, \ldots, t_n)$ are coordinates in $\mathbb{A}^n$, and $T_x X$ is an affine linear space through $x = (x_1, \ldots, x_n) \in \mathbb{A}^n$, which we can make into a vector space by choosing $x = 0$.

Proof. Let us prove the affine case first. Put $y_i = t_i - x_i$,

$$\tilde{G}_\beta(y_1, \ldots, y_n) := G_\beta(y_1 + x_1, \ldots, y_n + x_n).$$

In the special case when $X \subset \mathbb{A}^n$ is a hypersurface with $I(X) = (\tilde{F})$, $0 \in X$, and if $\tilde{F} = L + G$ with L the linear part, then $T_0 X = \{L = 0\}$. But

$$L = \sum_{i=1}^{n} \frac{\partial \tilde{F}}{\partial y_i}(0) y_i$$

and

$$\frac{\partial \tilde{F}}{\partial y_i}(0) = \frac{\partial F(y_1 + x_1, \ldots, y_n + x_n)}{\partial y_i}(0) = \frac{\partial F(t_1, \ldots, t_n)}{\partial t_n}(x)$$

whence

$$L = \sum_{i=1}^{n} \frac{\partial F}{\partial t_i}(x)(t_i - x_i),$$

so the formula is valid in the hypersurface case.

In the general case, according to Definition 10.1, we have

$$T_x X = \bigcap_{H \supset X \text{ hypersurface}} T_x H$$

which concludes the proof in the affine case.

If $X \subset \mathbb{P}^n$ is projective, we can assume without loss of generality that $x \in \{X_0 \neq 0\} \simeq \mathbb{A}^n \subset \mathbb{P}^n$ where $t_1 = X_1/X_0, \ldots, t_n = X_n/X_0$ are coordinates in $\mathbb{A}^n$. If X is a hypersurface with $I_X = (F)$, then

$$f(t_1, \ldots, t_n) = F(1, t_1, \ldots, t_n)$$

generates the ideal of $X \cap \mathbb{A}^n$, thus by the the first part,

$$\sum_{i=1}^{n} \frac{\partial f}{\partial t_i}(x_1, \ldots, x_n)(t_i - x_i) = 0,$$

where x_i are affine coordinates of x, defines the affine tangent space $T_x X$ and, by definition, $\mathbb{T}_x X = \overline{T_x X}$. Thus

$$\overline{T_x X} = \left\{ (X_0 : \cdots : X_n) \mid \sum_{i=1}^{n} \frac{\partial F}{\partial X_i}(1, x_1, \ldots, x_n)(X_i - x_i X_0) = 0 \right\}.$$

But since F is homogeneous, of degree d say, we have the Euler relation

$$\sum_{i=0}^{n} \frac{\partial F}{\partial X_i} X_i = d \cdot F$$

and $F(1, x_1, \ldots, x_n) = 0$, whence

$$\sum_{i=1}^{n} \frac{\partial F}{\partial X_i}(1, x_1, \ldots, x_n) \cdot (-x_i X_0) = \frac{\partial F}{\partial X_0}(1, x_1, \ldots, x_n) X_0.$$

In other words, if $x = (P_0 : \cdots : P_n)$, $x_i = P_i/P_0$, then

$$\overline{T_x X} = \left\{ (X_0 : \cdots : X_n) \mid \sum_{i=0}^{n} \frac{\partial F}{\partial X_i}(P_0, \ldots, P_n) X_i = 0 \right\}.$$

This settles the hypersurface case, and the general case follows again by remarking that

$$\mathbb{T}_x X = \bigcap_{H \supset X \text{ hypersurface}} \mathbb{T}_x H.$$

$\square$

Remark 10.6. One must be careful that in Theorem 10.5 one has to take the F_α (or G_β) as ideal generators, i.e. defining equations that do not generate the respective ideals may lead to a different answer. In other words, if $X = \bigcap_i H_i$ where the $H_i \subset \mathbb{A}^n$ are hypersurfaces, then in general $T_x X \neq \bigcap T_x H_i$. To see this, take for example

$$H_1 = \{y - x^2 = 0\}, \quad H_2 = \{y = 0\} \text{ in } \mathbb{A}^2.$$

Then $T_0 H_1 \cap T_0 H_2 = x$-axis, but $T_0(H_1 \cap H_2) = \{0\}$.

Now consider $X \subset \mathbb{A}^n$ affine again, $t_1, \dots, t_n$ coordinates on $\mathbb{A}^n$. Any polynomial $F \in k[t_1, \dots, t_n]$ has a formal "Taylor expansion" around a point $x = (x_1, \dots, x_n) \in \mathbb{A}^n$, i.e., we can write

$$F(t_1, \dots, t_n) = F(x) + F_1(t_1 - x_1, \dots, t_n - x_n) + \cdots + F_k(t_1 - x_1, \dots t_n - x_n)$$

where F_i is homogeneous of degree i. Then we call

$$F_1 =: d_x F = \sum_{i=1}^{n} \frac{\partial F}{\partial t_i}(t_i - x_i)$$

the **differential** of F in x. For the differential we have the rules

$$d_x(F + G) = d_x F + d_x G, \tag{10.1}$$
$$d_x(FG) = F(x) d_x G + G(x) d_x F. \tag{10.2}$$

We can view $d_x F$ as a linear form on $\mathbb{A}^n$ if we give it a linear structure with x as origin. Moreover,

$$T_x X = \{d_x F = 0 \mid F \in I(X)\}.$$

If $g \in \mathcal{O}_X(X) = k[t_1, \dots, t_n]/I(X)$, $G \in k[t_1, \dots, t_n]$ such that $G \mid_X = g$, then clearly $d_x G \mid_{T_x X}$ is independent of the choice of G.

Definition 10.7. The linear form

$$d_x g := d_x G \mid_{T_x X} \in T_x X^*$$

is called the **differential** of $g \in \mathcal{O}_X(X)$ in x.

The rules (10.1) and (10.2) continue to hold. Moreover, $d_x c = 0$ for $c \in k \subset \mathcal{O}_X(X)$. Let $\mathfrak{M}_x \subset \mathcal{O}_X(X)$ be the ideal of $x \in X$. Then d_x kills elements in $\mathfrak{M}_x^2$ by (10.2).

Theorem 10.8. *The map*

$$d_x\colon \mathfrak{M}_x/\mathfrak{M}_x^2 \to T^*_{X,x}$$

is an isomorphism of k-vector spaces.

Proof. Clearly, the map d_x is well-defined and k-linear. We have $\mathrm{im}(d_x) = T^*_{X,x}$ since every linear form $l \in T^*_{X,x}$ extends to a linear form $\tilde{l} \in (\mathbb{A}^n)^*$ and $d_x(\tilde{l}\mid_X) = l$.

Let us show that $\ker(d_x) = \mathfrak{M}_x^2$: we choose coordinates such that $x = (0,\ldots,0)$; suppose $g \in \mathfrak{M}_x$ is such that $d_x(g) = 0$ and suppose g is represented by some polynomial $G \in k[t_1,\ldots,t_n]$, $g = G\mid_X$. Thus we assume $d_xG\mid_{T_xX} = 0$, in other words,

$$d_xG = \lambda_1 d_xF_1 + \cdots + \lambda_m d_xF_m, \text{ where } I(X) = (F_1,\ldots,F_m).$$

Put $\tilde{G} := G - \lambda_1F_1 - \cdots - \lambda_mF_m$, thus $d_x\tilde{G} = 0$. This means $\tilde{G} \in (t_1,\ldots,t_n)^2$, whence

$$\tilde{G}\mid_X = g \in (t_1\mid_X,\ldots,t_n\mid_X)^2.$$

But $\mathfrak{M}_x = (t_1\mid_X,\ldots,t_n\mid_X)$. □

Theorem 10.9. *If $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal of x in the local ring $\mathcal{O}_{X,x}$ of $x \in X$, then there is an isomorphism*

$$T_{X,x} \simeq (\mathfrak{m}_x/\mathfrak{m}_x^2)^*.$$

Proof. For $f \in \mathcal{O}_{X,x}$ we can find a representation F/G, with $F, G \in k[t_1,\ldots,t_n]$, $G(x) \neq 0$, and define the differential

$$d_xf := d_x\left(\frac{F}{G}\right)\mid_{T_{X,x}} = \frac{G(x)d_xF - F(x)d_xG}{G(x)^2}\mid_{T_{X,x}}.$$

In just the same way as above one can show that this is well-defined, satisfies the computational rules (10.1) and (10.2), and gives a k-linear map $d_x\colon \mathfrak{m}_x/\mathfrak{m}_x^2 \to T^*_{X,x}$. The same proof that we used for Theorem 10.8 then also shows that it is an isomorphism. □

Corollary 10.10. *The tangent space T_xX to a quasi-projective variety X in a point $x \in X$ is an isomorphism invariant and isomorphic to $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal.*

Proof. If $\varphi\colon X \to Y$ is a morphism, $\varphi(x) = y$, then it induces a pull-back map $\varphi^*\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ such that

$$\varphi^*(\mathfrak{m}_{Y,y}) \subset \mathfrak{m}_{X,x}, \quad \varphi^*(\mathfrak{m}^2_{Y,y}) \subset \mathfrak{m}^2_{X,x}.$$

Hence it induces a k-linear map

$$d_x\varphi\colon \left(\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}\right)^* \to \left(\mathfrak{m}_{Y,y}/\mathfrak{m}^2_{Y,y}\right)^*.$$

If φ has an inverse ψ, this linear map is an isomorphism. □

Definition 10.11. The vector space $(\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x})^*$ is called the **Zariski tangent space** to X in x. Moreover, for $\varphi\colon X \to Y$ a morphism as in the proof of Corollary 10.10, the linear map $d_x\varphi$ is called the **differential** of φ in x.

Note that for morphisms $\varphi\colon X \to Y$, $\psi\colon Y \to Z$ we have

$$d_x(\psi \circ \varphi) = d_x\psi \circ d_x\varphi, \quad d_x(\mathrm{id}_X) = \mathrm{id}_{T_xX}.$$

Corollary 10.10 sometimes gives an answer to certain embedding problems:

Corollary 10.12. *The union X of the three coordinate axes in $\mathbb{A}^3$*

$$X = \{x = y = 0\} \cup \{x = z = 0\} \cup \{y = z = 0\}$$

with $I(X) = (xy, yz, xz)$ is not isomorphic to any subvariety of $\mathbb{A}^2$ (in particular not isomorphic to the union of three lines through a point in $\mathbb{A}^2$).

Proof. Indeed,

$$\dim T_0X = 3.$$

□

We now want to study the loci of regular and singular points in a variety X.

Theorem 10.13. *Let X be an irreducible variety. The set of regular points $x \in X$, defined by $\dim T_xX = \dim_x X$, is open and Zariski dense in X. The set of singular points, by definition those where $\dim T_xX > \dim_x X$, is exactly the complement of the regular points. In particular, we always have $\dim T_xX \geq \dim_x X$.*

Proof. The proof will be divided into several steps.

Step 1. Without loss of generality we can assume $X \subset \mathbb{A}^n$ affine and irreducible. Look at the incidence correspondence

$$\begin{array}{c} \mathcal{T} := \{(x,a) \in X \times \mathbb{A}^n \mid a \in T_x X\} \subset X \times \mathbb{A}^n \,. \\ \downarrow{\scriptstyle \pi} \\ X \end{array}$$

This is an algebraic subvariety of $X \times \mathbb{A}^n$ by Theorem 10.5, and $\pi(\mathcal{T}) = X$, $\pi^{-1}(x) = T_x X$. For this reason one calls $\mathcal{T}$ the (total space of the) *tangent bundle* in the case when X consists entirely of regular points. In general one may call it the *tangent fiber space*.

By Theorem 8.16 and Corollary 8.17, there is an $s \in \mathbb{N}$ such that $\dim T_x X \geq s$ for all $x \in X$ and

$$\{y \in X \mid \dim T_y X > s\} \subset X$$

is a proper closed algebraic subvariety; here $s = \min_{x \in X} \dim T_x X$. Thus it is sufficient to show $s = \dim X$.

Step 2. We show that $s = \dim X$ holds for a hypersurface $X \subset \mathbb{A}^n$ with ideal $I(X) = (F)$. In this case,

$$\sum_{i=1}^{n} \frac{\partial F}{\partial t_i}(x)(t_i - x_i) = 0$$

defines $T_x X \subset \mathbb{A}^n$, and $s = \dim X = n-1$ if and only if not all the partials $\partial F / \partial t_i$ vanish identically on X. Now if $\mathrm{char}(k) = 0$, the contrary means that F is constant, a contradiction, or if $\mathrm{char}(k) = p > 0$, this means that we can write $F = G(t_1^p, \ldots, t_n^p)$; since k is algebraically closed and in $\mathrm{char}(k) = p$ the binomial theorem takes the form $(a+b)^p = a^p + b^p$, we conclude that F is then a p-th power, a contradiction because then $I(X)$ would not be generated by F.

Step 3. We reduce to the case of a hypersurface. We claim: for $X \subset \mathbb{A}^n$ irreducible, there exists a hypersurface $Y \subset \mathbb{A}^m$ and open, nonempty subsets $U \subset X$, $V \subset Y$ and an isomorphism $\varphi\colon U \to V$. Indeed, this claim follows from Theorem 10.14 and Theorem10.15 below. Now the subset $Y_{\mathrm{reg}} \subset Y$ of regular points is open by Step 2, and

$$\dim T_y Y = \dim Y = \dim X \quad \forall\, y \in Y_{\mathrm{reg}}$$

whence, by Corollary 10.10, we have for all points $x \in \varphi^{-1}(Y_{\mathrm{reg}} \cap V) \subset X$ (which is open) that $\dim T_x X = \dim X$. □

Theorem 10.14. *Every irreducible subvariety $X \subset \mathbb{A}^n$ is birational to a hypersurface.*

Proof. We can find a transcendence basis $x_1, \dots, x_r$ of $k(X)$ such that $k(X) \supset k(x_1, \dots, x_r)$ is a separable algebraic extension (this needs k perfect, which is O.K. since our k is algebraically closed). Then the Theorem of the Primitive Element says that there is a $y \in k(X)$ such that $k(X) = k(y, x_1, \dots, x_r)$. Now y satisfies a polynomial equation over $k(x_1, \dots, x_r)$. Hence there is an irreducible nonzero polynomial

$$f(x_1, \dots, x_r, y) = 0$$

which defines a hypersurface in $\mathbb{A}^{r+1}$ with function field $k(X)$. □

Theorem 10.15. *Let X, Y be irreducible and affine. Suppose that X and Y are birational to each other. Then there exist a nonempty open $U \subset X$ and a nonempty open $V \subset Y$ such that U and V are isomorphic.*

Proof. Suppose that $X \subset \mathbb{A}^m$ with coordinates x_i and $Y \subset \mathbb{A}^n$ with coordinates y_j. Without loss of generality, we can assume that X and Y are not contained in any hyperplane. Let

$$i \colon k(Y) \simeq k(X)$$

be an isomorphism. Viewing y_j as an element of $\mathcal{O}_Y(Y) \subset k(Y)$, we can consider $i(y_j)$, $j = 1, \dots, n$. These are not identically zero on X, and there is an affine open subset $W_1 \subset X$ on which all the $i(y_j)$ are regular. Then i defines an inclusion $\mathcal{O}_Y(Y) \hookrightarrow \mathcal{O}_{W_1}(W_1)$ corresponding to a dominant morphism

$$\varphi \colon W_1 \to Y.$$

Applying the same reasoning to i^{-1}, we also get a dominant morphism

$$\psi \colon W_2 \to X,$$

$W_2 \subset Y$ affine open, such that $\psi \circ \varphi = \mathrm{id}$, $\varphi \circ \psi = \mathrm{id}$ (as rational maps). Now consider

$$\begin{gathered} \varphi^{-1}(\psi^{-1}(W_1)) \subset \varphi^{-1}(W_2) \subset W_1 \subset X, \\ \psi^{-1}(\varphi^{-1}(W_2)) \subset \psi^{-1}(W_1) \subset W_2. \end{gathered}$$

Then $\psi \circ \varphi$ is defined on $\varphi^{-1}(W_2)$ and $\varphi \circ \psi$ is defined on $\psi^{-1}(W_1)$,

$$\psi \circ \varphi \mid_{\varphi^{-1}(W_2)} = \mathrm{id}, \quad \varphi \circ \psi \mid_{\psi^{-1}(W_1)} = \mathrm{id}.$$

If we put $U := \varphi^{-1}(\psi^{-1}(W_1))$, $V := \psi^{-1}(\varphi^{-1}(W_2))$, then $U \simeq V$ via φ and ψ. □

As a corollary of Theorem 10.5 and Theorem 10.13 we get

Corollary 10.16. *Let $X \subset \mathbb{P}^n$ be an irreducible projective variety. Then it singular locus* $\mathrm{Sing}(X)$ *can be described as*

$$\mathrm{Sing}(X) = \left\{ \mathrm{rk} \left(\frac{\partial F_i}{\partial X_j} \right) \leq \mathrm{codim}(X) - 1 \right\} \cap X.$$

Chapter 11

Cubic surfaces and their lines

Let $S \subset \mathbb{P}^3$ be a nonsingular cubic surface given by $f(x, y, z, w) = 0$ where $f = f_3$ is an irreducible homogeneous polynomial of degree 3, and x, y, z, w are homogeneous coordinates in $\mathbb{P}^3$.

By Theorem 9.4 we know that S contains at least one line. The next Proposition states some facts about lines on S which use that S is nonsingular.

Proposition 11.1. *Let $S \subset \mathbb{P}^3$ be a nonsingular cubic surface as before.*

1. *There are at most three lines passing through a point $p \in S$. If they are three, they lie in a plane.*

2. *Every plane $E \simeq \mathbb{P}^2 \subset \mathbb{P}^3$ intersects S in one of the following:*

   (a) an irreducible cubic curve in E;

   (b) the union of an irreducible conic and a line in E;

   (c) three distinct lines in E.

   In other words, the intersection $E \cap S$ does not contain a line counted with multiplicity > 1.

Proof. To show (1) notice that if $l \subset S$ is a line, then $T_p l = l \subset T_p S$, hence all lines through p lie in the plane $T_p S$ (this is a plane since S is nonsingular). It will follow from the proof of (2) that there are at most three such lines.

For (2) we can assume without loss of generality that $E = \{w = 0\}$ and $l = \{z = w = 0\} \subset E$. We have to show: $f\,|_E$ does not have the equation of

l as a double or triple factor. If so, we would have

$$f = z^2 L(x,y,z,w) + wQ(x,y,z,w)$$

with L linear and Q quadratic. But then S would be singular in a point where $z = w = Q = 0$, i.e. in the zeroes of Q on the line l. $\square$

The following Theorem gives us some first picture of the configuration of the lines on S.

Theorem 11.2. *Given a line $l \subset S$, there are exactly five pairs of lines (l_i, l_i'), $i = 1, \ldots, 5$, on S which intersect l and such that*

1. *for all $i = 1, \ldots, 5$, $l \cup l_i \cup l_i'$ lie in a plane;*

2. *for all $i \neq j$ one has $(l_i \cup l_i') \cap (l_j \cup l_j') = \emptyset$.*

Proof. Let $E \supset l$ be a plane in $\mathbb{P}^3$. Then by Proposition 11.1 (2), $E \cap S$ is one of the following:

(A) a nonsingular conic union a line;

(B) a triangle of lines;

(C) three planar lines through a point.

Now Theorem 11.2 (2) follows from Proposition 11.1 (1) once the remaining assertions have been proven. Thus we only have to show: there are exactly five planes $E_i \supset l$ in which the intersection with S looks like (B) or (C). After a coordinate change we can assume $l = \{z = w = 0\}$ whence

$$f = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

where $A, B, C \in k[z,w]_1$, $D, E \in k[z,w]_2$, $F \in k[z,w]_3$. This defines a conic with coefficients in $k[z,w]$. It is singular if and only if the discriminant

$$\Delta(z,w) = 4ACF + BDE - AE^2 - B^2F - CD^2 = 0.$$

In other words: every plane E through l is of the form $E = \{\mu z = \lambda w\}$, $(\mu : \lambda) \in \mathbb{P}^1$. If $\mu \neq 0$, then we can scale μ to be 1, and the equation for E becomes $z = \lambda w$. We can take $(x : y : w)$ as homogeneous coordinates on E and write

$$f\,|_E = w(A(\lambda,1)x^2 + B(\lambda,1)xy + C(\lambda,1)y^2 + D(\lambda,1)wx + E(\lambda,1)wy + F(\lambda,1)w^2).$$

In the same way, if $\lambda \neq 0$, then we scale λ to 1, $w = \mu z$, and use $(x : y : z)$ as coordinates on E and have

$$f\mid_E = z(A(1,\mu)x^2 + B(1,\mu)xy + C(1,\mu)y^2 + D(1,\mu)zx + E(1,\mu)zy + F(1,\mu)z^2).$$

Thus we see that $E \cap S$ splits as a union of three lines exactly for zeroes of Δ, which is a homogeneous quintic; we have to prove (using the nonsingularity of S) that $\Delta(z, w)$ has no multiple zeroes.

Suppose, after a change of coordinates, that $z = 0$ is a zero of Δ, and $E = \{z = 0\}$ the corresponding plane. We have to show that z^2 does not divide Δ. Now $E \cap S$ consists of three lines such that (i) they do not all pass through one point, (ii) or they do pass through one point.

Then we can choose coordinates $(x : y : w)$ in E such that in case (i) the lines are

$$l = \{w = 0\},\ l_1 = \{x = 0\},\ l_1' = \{y = 0\}$$

and in case (ii) the lines are

$$l = \{w = 0\},\ l_1 = \{x = 0\},\ l_1' = \{x = w\}.$$

In case (i) the equation f of S takes the form $f = xyw + zg$, g quadratic. Since

$$f = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

we get $B = w + az$, $a \in k$, and z divides A, C, D, E, F. Hence looking back at the equation for Δ we find

$$\Delta \equiv -w^2 F \pmod{z^2}.$$

But $p = (0 : 0 : 0 : 1) \in S$ and the nonsingularity of S in p means that F contains zw^2 with coefficient $\neq 0$. Thus $z^2 \nmid F$ and $z^2 \nmid \Delta$.

In case (ii) we can write

$$f = x(x - w)w + z\tilde{g}$$

with $\tilde{g}$ quadratic. Then it follows that

$$A = w + \tilde{a}z, \quad D = -w^2 + z\tilde{l}$$

where $\tilde{a} \in k$, $\tilde{l}$ is linear. Thus $z \mid B, C, E, F$, $z \nmid D$. The nonsingularity of S in $(0 : 1 : 0 : 0)$ implies $C = \tilde{c}z$, $\tilde{c} \neq 0$. But

$$\Delta(z, w) \equiv -CD^2 \pmod{z^2}$$

in this case, so we are done since $z^2 \nmid \Delta$ here as well. $\square$

Corollary 11.3. *There are two disjoint lines on S and S is birational to $\mathbb{P}^2$.*

Proof. The first assertion follows from Theorem 11.2 (2).

For the second assertion take two disjoint lines $l, m \subset S$ and define

$$\varphi\colon S \dashrightarrow l \times m \simeq \mathbb{P}^1 \times \mathbb{P}^1,$$
$$\psi\colon l \times m \simeq \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow S$$

as follows: if $p \in \mathbb{P}^3 \backslash (l \cup m)$, then there is a unique line n through p which intersects l, m. This defines φ. If conversely $(Q, R) \in l \times m$, $n = \overline{QR} \subset \mathbb{P}^3$ the line passing through them, then Theorem 11.2 (1) implies that there are only finitely many lines on S intersecting l, and the same holds for m, in other words, in general, n intersects S in three points $\{P, Q, R\}$. One then defines $\psi(Q, R) = P$. Then φ and ψ are clearly dominant rational maps inverse to each other. □

Let us now try to describe the number of lines on S and their configuration more precisely. We need

Lemma 11.4. *Let $l_1, l_2, l_3, l_4 \subset \mathbb{P}^3$ be disjoint lines. Then*

1. *either all of the l_i lie on a nonsingular quadric Q and then there exist infinitely many lines intersecting $l_1, \dots, l_4$*
2. *or the l_i do not all lie on a quadric and there are exactly one or two lines which intersect $l_1, \dots, l_4$.*

Proof. It is easy to see as an exercise that l_1, l_2, l_3 always lie on a nonsingular quadric $Q \subset \mathbb{P}^3$. In suitable coordinates, $Q = \{xw - yz =\} \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Every line which intersects all of l_1, l_2, l_3 lies on Q. If l_4 does not lie on Q, then it has 1 or 2 intersection points with Q. The lines intersecting $l_1, \dots, l_4$ in this case are those of the ruling on Q to which l_1, l_2, l_3 do not belong and which pass through $l_4 \cap Q$.

If $l_4 \subset Q$, then all of $l_1, \dots, l_4$ belong to one ruling since they are disjoint, and then the lines of the opposite ruling give infinitely many lines intersecting all four. □

Now let l be a line on S, and let (l_i, l_i'), $i = 1, \dots, 5$, be as in Theorem 11.2. Every other line $n \subset S$ intersects exactly one of the lines l_i, l_i' for all $i = 1, \dots, 5$: namely, n intersects $E_i = \langle l_i, l_i' \rangle$, and $E_i \cap S = l \cup l_i \cup l_i'$. Because of Proposition 11.1 (1), n cannot intersect both l_i and l_i'.

Now let l, m be disjoint lines on S. Then by the preceding m intersects exactly one of the lines l_i, l_i' for all $i = 1, \ldots, 5$. Let us number the lines such that m always intersects l_i, $i = 1, \ldots, 5$. We denote the five pair of lines intersecting m by (l_i, l_i''), $i = 1, \ldots, 5$. Remark that l_i and l_j do not intersect for $i \neq j$ so that in each of the pairs of lines which meet m, exactly one l_i occurs.

Now Theorem 11.2 (2) applied to m gives: for $i \neq j$, the line l_i'' does not intersect l_j. But as we just saw, every line on S intersects l, l_j or l_j'. Thus l_i'' intersects l_j' for $i \neq j$.

Thus we now have 17 distinct lines on S: l_i, l_i', l_i'', l, m.

Lemma 11.5. *1. If $n \subset S$ is a line which is distinct from these 17 lines, then n meets exactly three out of the lines $l_1, \ldots, l_5$.*

2. Conversely, for every choice of 3 indices $\{i, j, k\} \subset \{1, 2, 3, 4, 5\}$ there is a unique line $l_{ijk} \subset S$ which meets l_i, l_j, l_k.

Proof. We prove (1) first. To begin with, four disjoint lines on S do not all lie on a quadric Q since otherwise $Q \subset S$, contradicting the irreducibility of S: indeed, the equation of S gives a polynomial of bidegree $(3, 3)$ on $\mathbb{P}^1 \times \mathbb{P}^1 \simeq Q$ and if this vanishes identically for four values of one of the sets of variables, it is identically zero. Now if n intersects ≥ 4 of the l_i, then we can conclude by Lemma 11.4 that $n = l$ or $n = m$, a contradiction.

If, on the other hand, n intersected ≤ 2 of the l_i, then it would intersect ≥ 3 of the l_i' since each line $\neq l$ intersects one of l_i or l_i'. That is, we could assume without loss of generality that l_2', l_3', l_4', l_5' or l_1, l_3', l_4', l_5'. But l and l_1'' are then two lines which intersect the disjoint lines l_2', l_3', l_4', l_5' and l_1. Thus under our assumption that n intersects ≥ 4 of those, Lemma 11.4 implies $n = l$ or $n = l_1''$, a contradiction.

To prove (2), we note that by Theorem 11.2 there are 10 lines on S which intersect l_1 only four of which we have given names to so far, namely l, l_1', m, l_1''. By part (1), each of the remaining 6 lines must intersect two out of $l_2, \ldots, l_5$. Moreover, every line n (not equal to l, l_i or l_j') is uniquely determined by which of the lines l_i or l_j' it intersects by Lemma 11.4 (2). Thus there are exactly $6 = \binom{4}{2}$ choices, and all of them occur. $\square$

Thus we now know all the lines on S (even by name!):

$$\{l, m, l_i, l_i', l_i'', l_{ijk}\},$$

in total

$$1 + 1 + 5 + 5 + 5 + \binom{5}{3} = 27.$$

We can summarize what we have seen so far in

Theorem 11.6. *Every nonsingular cubic surface $S \subset \mathbb{P}^3$ contains exactly 27 lines $\{l, m, l_i, l_i', l_i'', l_{ijk}\}$, $i \in \{1, \dots, 5\}$, $i, j, k \in \{1, \dots, 5\}$ with $i < j < k$, as above. The incidence graph Γ of those, whose vertices are lines and edges denote intersections, can be described as follows:*

1. *l intersects $l_1, \dots, l_5, l_1', \dots, l_5'$;*
2. *l_1 intersects l, m, l_1', l_1'' and l_{1jk} for 6 choices of $\{j, k\} \subset \{2, 3, 4, 5\}$.*
3. *l_1' intersects l, l_1, l_j'' (for 4 choices of $j \neq 1$) and l_{ijk} (for 4 choices of $\{i, j, k\} \subset \{2, 3, 4, 5\}$).*
4. *l_1'' intersects m, l_1, l_j' (for 4 choices of $j \neq 1$) and l_{ijk} (for 4 choices of $\{2, 3, 4, 5\}$).*
5. *l_{123} intersects $l_1, l_2, l_3, l_{145}, l_{245}, l_{345}, l_4', l_5', l_4'', l_5''$.*

The other intersection relations follows by symmetry.

One can treat the lines on S a bit more symmetrically, and we explain this now.

Definition 11.7. Define a $\mathbb{Z}$-module $A(S)$ by picking as generators the 27 lines on S and relations

$$l' + l'' + l''' = m' + m'' + m'''$$

whenever l', l'', l''' and m', m'', m''' are coplanar lines, i.e. they form triangles on S.

Proposition 11.8. *The module $A(S)$ is free of rank 7, i.e., a lattice $A(S) \simeq \mathbb{Z}^7$. As a basis we can choose $l_1, \dots, l_4, l_5', l_5'', l_5$.*

Proof. The triangles $l + l_i + l_i'$, $i = 1, \dots, 5$, contain l, thus $l + l_i + l_i' = l + l_5 + l_5'$ for $i = 1, \dots, 5$, and we get

$$l_i' = l_5 + l_5' - l_i, \quad i = 1, \dots, 5. \tag{11.1}$$

Considering the triangle which contain m yields in the same way

$$l_i'' = l_5 + l_5'' - l_i, \quad i = 1, \dots, 5. \tag{11.2}$$

Now $l_i' + l_j'' + l_{kop}$ is a triangle if $\{ij, k, o, p\}$ is a permutation of $\{1, 2, 3, 4, 5\}$, hence

$$l_{kop} = l + l_i - l_j''. \tag{11.3}$$

But $l_1 + l_{123} + l_{145}$ is also a triangle; thus

$$\begin{aligned} l + l_1 + l_1' &= l_1 + l_{123} + l_{145} \\ &= l_1 + 2l + l_4 - l_5'' + l_2 - l_3'' \end{aligned}$$

and thus

$$l = l_1' - l_4 + l_5'' - l_2 + l_3'' = 2(l_5 + l_5' + l_5'') - l_1 - l_2 - l_3 - l_4 - l_5'. \tag{11.4}$$

Then (11.1), (11.2) and (11.3) show that the elements in the statement of Proposition 11.8 generate. That they are independent will be shown in Proposition 11.10 below. □

Before we can finish the proof of Proposition 11.8 we need a further definition.

Definition 11.9. Define a bilinear pairing

$$A(S) \times A(S) \to \mathbb{Z}$$

in the following way:

1. for two different lines l, l' we put $ll' = 0$ or $ll' = 1$ depending on whether they are disjoint or intersect;
2. for all lines l we put $l^2 = -1$;
3. we put $l(m+m'+m'') = 1$ for every line l and every triangle $m+m'+m''$.

This is well-defined by (1) and (2) alone: $A(S)$ is generated by lines, and we know that if l is distinct from m, m', m'', it intersects exactly one of m, m', m'', hence (3) follows; if on the other hand $l = m$, say, then $lm' = lm'' = 1$ and $l^2 = lm = -1$ by (2), hence (3) follows as well. So the bilinear pairing descends to the quotient and is well-defined.

Proposition 11.10. *If*

$$e_0 = l_5 + l_5' + l_5'', \quad e_1 = l_1,\ e_2 = l_2,\ e_3 = l_3,\ e_4 = l_4,$$
$$e_5 = l_5', \quad e_6 = l_5''$$

then

$$e_0^2 = 1, \quad e_i^2 = -1,\ i = 1, \ldots, 6, \quad e_ie_j = 0\ i \neq j.$$

In particular, $e_0, \ldots, e_6$ are a basis of $A(S)$.

Proof. Remark that $e_1 = l_1, \ldots, e_4 = l_4, e_5 = l_5', e_6 = l_5''$ are six disjoint lines, and the first four are also disjoint from l_5; this implies everything except $e_0^2 = 1$. To see this remark that

$$e_0e_5 = e_0l_5' = (l_5 + l_5' + l_5'')l_5' = 1 - 1 + 0 = 0$$

and similarly $e_0e_6 = e_0l_5'' = 0$. Thus

$$\begin{aligned} e_0^2 &= e_0(l_5 + l_5' + l_5'') = e_0l_5 \\ &= (l_5 + l_5' + l_5'')l_5 = -1 + 1 + 1 = 1. \end{aligned}$$

□

Proposition 11.11. *If h is the class of a triangle in $A(S)$, then $h^2 = 3$ and $hx \equiv x^2$ (mod 2) for all $x \in A(S)$*

Proof. If $h = l + l' + l''$ is a triangle, then $h^2 = h(l + l' + l'') = 1 + 1 + 1 = 3$.

Now for all lattices the function $A \to \mathbb{F}_2$, $x \mapsto x^2$ (mod 2) is linear because $(x + y)^2 \equiv x^2 + y^2$ (mod 2). Thus it suffices to prove the second assertion for generators of $A(S)$. But $A(S)$ is generated by lines l, and for those $hl = 1$, $l^2 = -1$. □

To summarize: S has an associated lattice $A(S) \simeq \mathbb{Z}^7$ with a nondegenerate bilinear pairing which we can diagonalize (in the basis $e_0, \ldots, e_6$) to $(1, -1, \ldots, -1)$. Moreover there is a class $h \in A(S)$ with $h^2 = 3$ and $hx \equiv x^2$ (mod 2) for all $x \in A(S)$. Indeed, one can show that these conditions characterize the pair A, h uniquely up to isomorphism.

One also calls $A(S)$ the *Picard group* of S and the bilinear pairing the *intersection pairing* for reasons that will become clearer in the next Chapter.

The advantage of $A(S)$ is that one can show that the lines on S are exactly the solutions of the equations $hl = 1$, $l^2 = -1$ in $A(S)$. This is a bit messy to do directly by hand; it is easy to see once one knows adjunction and the genus formula. This gives a *symmetric description* of the lines.

We also mention without proof that there is a striking connection to the root system of type E_6: namely, $h^\perp \subset A(S)$ is the (negative of the) root lattice, $(-E_6)$.

Chapter 12

Local parameters, power series methods, divisors

Let X be a (not necessarily irreducible) variety, $x \in X$ a regular point, $\dim_x X = n$.

Definition 12.1. Elements $p_1, \dots, p_n \in \mathcal{O}_{X,x}$ are called *local parameters* (or local coordinates) in x if $p_i \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ and the residue classes $\bar{p}_1, \dots, \bar{p}_n$ form a basis of the k-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2 = T^*_{X,x}$. An equivalent condition is that the differentials $d_x p_1, \dots, d_x p_n$ are independent linear forms on $T_x X$.

Choose $U \ni$ open affine such that $p_1, \dots, p_n \in \mathcal{O}_U(U)$. Let $H_i := \{u \in U \mid p_i(u) = 0\}$ and let $I(H_i) \subset \mathcal{O}_U(U)$ its ideal. Assume that $U \subset \mathbb{A}^N$ closed and let $P_i(t_1, \dots, t_N)$ be a polynomial in the coordinates on $\mathbb{A}^N$ such that $P_i \mid_U = p_i$. Then $(P_i, I(U)) \subset I(H_i)$, thus

$$T_x H_i \subset \{v \in T_x U \mid d_x P_i(v) = 0\}.$$

Since the $d_x p_1, \dots, d_x p_n$ are linearly independent, we get that $\dim T_x H_i \leq n-1$. But because of Theorem 8.11 and Theorem 10.13 we have

$$\dim T_x H_i \geq \dim_x H_i \geq n-1,$$

hence $\dim T_x H_i = n-1$ and x is regular on H_i. In a neighborhood of x the intersection of the H_i consists of x alone, for otherwise $d_x p_1 = \cdots = d_x p_n = 0$ wouldn't have 0 as its only solution (but would vanish on the tangent space of a component of the intersection of the H_i's passing through x). Thus we obtain

Theorem 12.2. *Let $p_1, \dots, p_n$ be local parameters around x. Then x is a regular point on every $H_i = \{p_i = 0\}$ and $\bigcap_i T_x H_i = \{0\}$*

One also says that the H_i's intersect *transversally* at x.

Theorem 12.3. *Local parameters generate the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$.*

Proof. The module $M = \mathfrak{m}_x$ over $\mathcal{O}_{X,x}$ is finite. Now the residue classes of the p_i in $\mathfrak{m}_x/\mathfrak{m}_x^2$ generate that vector space, hence the submodule $M' \subset M$ generated by the p_i's satisfies $M' + \mathfrak{m}_x M = M$, in other words

$$\mathfrak{m} \cdot (M/M') = (M/M').$$

By Nakayama's Lemma 7.6 we have $M' = M$. □

We will now consider *power series expansions*, also called *Taylor expansions* of elements $f \in \mathcal{O}_{X,x}$, given local parameters $p_1, \dots, p_n$. AMong other things, this will allow us to understand regular and singular points on X better, in particular prove that the intersection of two irreducible components must consist entirely of singular points.

Put

$$f(x) =: x_0.$$

Then

$$f_1 := f - x_0 \in \mathfrak{m}_x.$$

Since $\bar{p}_1, \dots, \bar{p}_n$ generate $\mathfrak{m}_x$, there are elements $c_1, \dots, c_n \in k$ such that

$$f_2 := f_1 - \sum_{i=1}^{n} c_i p_i = f - x_0 - \sum_{i=1}^{n} c_i p_i \in \mathfrak{m}_x^2.$$

Now $f_2 \in \mathfrak{m}_x^2$, thus we can write $f_2 = \sum_j g_j h_j$ with $g_j, h_j \in \mathfrak{m}_x$, thus there exist elements $d_{ij}, e_{ij} \in k$ such that $g_j - \sum_i d_{ij} p_i \in \mathfrak{m}_x^2$, $h_j - \sum_i e_{ij} p_i \in \mathfrak{m}_x^2$. Defining elements f_{lk} in k by $\sum_{lk} f_{lk} p_l p_k := \sum_j (\sum_i d_{ij} p_i)(\sum_i e_{ij} p_i)$ we obtain

$$f_3 = f_2 - \sum f_{lk} p_l p_k = f - x_0 - \sum_i c_i p_i - \sum_{l,k} f_{lk} p_l p_k \in \mathfrak{m}_x^3.$$

Continuing in this way we inductively obtain polynomials

$$F_i,\ i = 0, 1, 2, \dots, \quad F_i \in k[t_1, \dots, t_n]_i$$

with

$$f - \sum_{i=0}^{k} F_i(p_1, \dots, p_n) \in \mathfrak{m}_x^{k+1}, \quad \forall\, k \geq 0.$$

Definition 12.4. The *ring of formal powers series* in variables $t_1, \dots, t_n$, denoted by $k[[t]] = k[[t_1, \dots, t_n]]$, is the ring whose elements are formal sums

$$F = F_0 + F_1 + F_2 + \dots$$

(in other words, simply sequences $(F_0, F_1, F_2, \dots))$ where F_i is a homogeneous polynomial of degree i in the $t_1, \dots, t_n$; and addition is defined componentwise:

$$F + G := (F_0 + G_0) + (F_1 + G_1) + \dots$$

and multiplication

$$F \cdot G := H_0 + H_1 + H_2 + \dots$$

is defined by formally imitating the Cauchy product rule:

$$H_q := \sum_{i+j=q} F_i \cdot G_j.$$

The leading term of a powers series F, $\mathrm{lt}(F)$ is F_{i_0} where $i_0 = \min\{i \mid F_i \neq 0\}$.

It then follows that $\mathrm{lt}(F \cdot G) = \mathrm{lt}(F) \cdot \mathrm{lt}(G)$, whence $k[[t]]$ is an integral domain.

Definition 12.5. An element $F \in k[[t]]$ is called a *Taylor series* for $f \in \mathcal{O}_{X,x}$ (with respect to local parameters $p_1, \dots, p_n$ if

$$f - \sum_{i=0}^{k} F_i(p_1, \dots, p_n) \in \mathfrak{m}_x^{k+1}, \quad \forall\, k \geq 0.$$

We have seen above that *every $f \in \mathcal{O}_{X,x}$ has at least one Taylor series.*

Theorem 12.6. *If $x \in X$ is a regular point and $p_1, \dots, p_n$ are local parameters around x, then every $f \in \mathcal{O}_{X,x}$ has a* unique *Taylor series.*

Proof. It suffices to show that if $f = 0$ in $\mathcal{O}_{X,x}$, then $F = 0$ in $k[[t]]$ for every Taylor series F for f; in other words, it suffices to show that for $F_k \in k[t_1, \dots, t_n]_k$

$$F_k(p_1, \dots, p_n) \in \mathfrak{m}_x^{k+1} \implies F_k(t_1, \dots, t_n) = 0 \text{ in } k[t_1, \dots, t_n]_k.$$

Suppose by contradiction that $F_k(t_1, \dots, t_n) \neq 0$. The coefficient of t_n^k in F_k is $F_k(0, \dots, 0, 1)$ and there is a $(c_1, \dots, c_n) \in k^n$ such that $F_k(c_1, \dots, c_n) \neq 0$. Thus after a linear change of coordinates with $(c_1, \dots, c_n) \mapsto (0, \dots, 0, 1)$, we can assume that the coefficient of t_n^k is nonzero. Thus without loss of generality

$$F_k(t_1, \dots, t_n) = \alpha t_n^k + G_1(t_1, \dots, t_{n-1})t_n^k + \cdots + G_k(t_1, \dots, t_{n-1})$$

with $\alpha \neq 0$ and $G_i \in k[t_1, \dots, t_{n-1}]_i$. Since the $p_1, \dots, p_n$ generate $\mathfrak{m}_x$ by Theorem 12.3 we can write $F_k(p_1, \dots, p_n) \in \mathfrak{m}_x^{k+1}$ as a homogeneous polynomial of degree k in $p_1, \dots, p_n$ with coefficients in $\mathfrak{m}_x$ whence we get an equality

$$\begin{aligned} &\alpha p_n^k + G_n(p_1, \dots, p_{n-1})p_n^{k-1} + \cdots + G_k(p_1, \dots, p_{n-1}) \\ &= m p_n^k + \tilde{G}_1(p_1, \dots, p_{n-1})p_n^{k-1} + \cdots + \tilde{G}_k(p_1, \dots, p_{n-1}) \end{aligned}$$

with $m \in \mathfrak{m}_x$ and $\tilde{G}_i$ a homogeneous polynomial of degree i in $n-1$ variables with coefficients in $\mathfrak{m}_x$. Then $(\alpha - m)p_n^k \in (p_1, \dots, p_{n-1})$ and $\alpha - m$ is invertible in $\mathcal{O}_{X,x}$ whence $p_n^k \in (p_1, \dots, p_{n-1})$. But this means, putting $H_i = (p_i = 0)$, that

$$H_n \supset H_1 \cap \cdots \cap H_{n-1}$$

and thus

$$T_x H_n \supset T_x H_1 \cap \cdots \cap T_x H_{n-1}.$$

Then

$$T_x H_1 \cap \cdots \cap T_x H_n \neq \{0\}$$

contradicting Theorem 12.2. □

Thus if $x \in X$ is regular, then we get a well-defined map

$$\mathcal{T} \colon \mathcal{O}_{X,x} \to k[[t_1, \dots, t_n]]$$

by associating to an f its unique Taylor series. Moreover, this is clearly a homomorphism. We want to determine its kernel: $\mathcal{T}(f) = 0$ means $f \in \mathfrak{m}_x^k$ $\forall\, k \geq 0$, i.e. $f \in \bigcap_{k \geq 0} \mathfrak{m}_x^k$. We then use the following commutative algebra result.

Theorem 12.7 (Krull's Intersection Theorem)**.** *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Then*

$$\bigcap_{k\geq 0} \mathfrak{m}^k = (0).$$

The proof is not overly complicated, but we omit it to continue with the geometric consequences. Notice that Krull's Intersection Theorem can become false without the Noetherian hypothesis: for example it does not hold for germs of infinitely differentiable functions around $0 \in \mathbb{R}^1$ since there are functions all of whose derivatives in 0 vanish, but the germ is nonzero.

Theorem 12.8. *If $x \in X$ is a regular point on a variety, then we have an inclusion*

$$\mathcal{O}_{X,x} \hookrightarrow k[[t_1, \dots, t_n]]$$

defined by associating to a germ its Taylor series with respect to some set $p_1, \dots, p_n$ of local parameters.

As a consequence we can now prove a result that complements Theorem 10.13.

Theorem 12.9. *Let $x \in X$ be a regular point on a variety X. Then x lies on a unique irreducible component of X. In other words, intersections of two irreducible components consist entirely of singular points.*

Proof. Replacing X by some affine open subset containing x we can assume that X is affine and all irreducible components of X pass through x. Then $\mathcal{O}_X(X) \subset \mathcal{O}_{X,x} \subset k[[t_1, \dots, t_n]]$. Since $k[[t_1, \dots, t_n]]$ is an integral domain, $\mathcal{O}_X(X)$ cannot have any zero divisors, hence X is irreducible. □

We will assume the following result without proof to proceed further.

Theorem 12.10. *Let $x \in X$ be a regular point on a variety X. Then the local ring $\mathcal{O}_{X,x}$ is factorial.*

A proof can be found in almost any textbook on commutative algebra. One approach is to use power series methods and the Weierstrass Preparation Theorem to prove that $k[[t_1, \dots, t_n]]$ is a UFD, and then show that the UFD property is inherited from $\mathcal{O}_{X,x} \subset k[[t_1, \dots, t_n]]$. This would be perfectly feasible with our current background, but is a bit lengthy and messy, so we omit it.

We will assume that varieties X are irreducible in the sequel.

Definition 12.11. Germs of functions $f_1, \dots, f_m \in \mathcal{O}_{X,x}$ are called local equations for a subvariety $Y \subset X$ if there is an affine open neighborhood $U \ni x$ with $f_i \in \mathcal{O}_X(U)$ for all i and $I(Y \cap U) = (f_1, \dots, f_m) \subset \mathcal{O}_X(U)$.

Let $I_{Y,x} \subset \mathcal{O}_{X,x}$ be the ideal of germs of functions in $\mathcal{O}_{X,x}$ which vanish on Y in an open neighborhood of x; if X is affine we have

$$I_{Y,x} = \left\{ f = \frac{u}{v} \mid u, v \in \mathcal{O}_X(X),\ u \in I(Y),\ v(x) \neq 0 \right\}.$$

Theorem 12.12. *The elements $f_1, \dots, f_m \in \mathcal{O}_{X,x}$ are called local equations for a subvariety $Y \subset X \iff I_{Y,x} = (f_1, \dots, f_m)$.*

Proof. Clearly $\implies$ holds since if in an affine open neighborhood U of x we have $I(Y \cap U) = (f_1, \dots, f_m)$, then $I_{Y,x} = (f_1, \dots, f_m)$ by what we remarked immediately before the statement of the Theorem.

For $\impliedby$ assume $I_{Y,x} = (f_1, \dots, f_m)$, and write $I(Y \cap U) = (g_1, \dots, g_s)$, $g_i \in \mathcal{O}_U(U)$ for some affine open U containing x. Since $g_i \in I_{Y,x}$ we have equations

$$g_i = \sum_{j=1}^{m} h_{ij} f_j, \quad h_{ij} \in \mathcal{O}_{X,x},\ i = 1, \dots, s.$$

All f_i, h_{ij} are regular on some small principal open subset $V \ni x$, $V = U - \{g = 0\}$, $g \in \mathcal{O}_U(U)$. Thus in $\mathcal{O}_V(V)$ we have

$$(g_1, \dots, g_s) = I(Y \cap U) \cdot \mathcal{O}_V(V) \subset (f_1, \dots, f_m).$$

Let us show that $I(Y \cap U) \cdot \mathcal{O}_V(V) = I(V \cap Y)$. This will imply $I(V \cap Y) \subset (f_1, \dots, f_m)$, hence the Theorem since $f_i \in I(V \cap Y)$ so that the reverse inclusion is obvious.

Now it is clear that $I(Y \cap U) \cdot \mathcal{O}_V(V) \subset I(V \cap Y)$ Now let $v = u/g^l \in I(V \cap Y)$, $u \in \mathcal{O}_U(U)$, so $u = vg^l$. Then $u \in I(Y \cap U)$ and since $1/g^l \in \mathcal{O}_V(V)$ it follows that $v \in I(Y \cap U) \cdot \mathcal{O}_V(V)$. □

Theorem 12.13. *An irreducible subvariety $Y \subset X$ of codimension* 1 *has a local equation around any nonsingular point x of X.*

Proof. We can assume that X is affine. Let $f \in \mathcal{O}_{X,x}$ be a germ with $f \in I_{Y,x}$. Note that $\mathcal{O}_{X,x}$ is factorial by Theorem 12.10. Decompose f into prime factors in $\mathcal{O}_{X,x}$. Since Y is irreducible, one of the prime factors g must vanish on Y. We show that $I_{Y,x} = (g)$. After shrinking X we can assume that g is regular on X. Then $V(g) \supset Y$, $V(g) = Y \cup Y'$. If $x \in Y'$, then there exist regular functions $h, h' \in \mathcal{O}_X(X)$ with $hh' = 0$ on $(g = 0)$, but h and h' are both not identically zero on $(g = 0)$. By the Nullstellensatz 3.2, g divides $(hh')^r$ for some r where the divisibility holds in $\mathcal{O}_X(X)$; thus also $g \mid (hh')^r$ in $\mathcal{O}_{X,x}$. Now since $\mathcal{O}_{X,x}$ is *factorial*, we get $g \mid h$ or $g \mid h'$ since g is prime. Hence h or h' is zero on $V(g)$ in a neighborhood of x. Shrinking X we can assume from the beginning that all irreducible components of $V(g)$ pass through x. Thus we have a contradiction because we obtained that h or h' is identically zero on $V(g)$ but assumed the contrary at the beginning.

Thus $V(g) = Y$. If $u \in \mathcal{O}_X(X)$ vanishes on Y, then the Nullstellensatz implies that $g \mid u^s$, some s, in $\mathcal{O}_X(X)$. Hence $g \mid u^s$ in $\mathcal{O}_{X,x}$ and since the latter is factorial, $g \mid u$. Thus $I_{Y,x} = (g)$. □

Recall that a rational map $f\colon X \dashrightarrow Y$ is an equivalence class of morphisms $f_U\colon U \to Y$, $U \subset X$ Zariski open dense; two being considered equivalent if they coincide when they are both defined. It follows that there is a largest open set $\mathrm{dom}(f) \subset X$ on which f is a morphism, namely the union of all open sets U for representatives (f_U, U) of f. The complement $Z_f = X \backslash \mathrm{dom}(f)$ is called the *indeterminacy locus* of f. It is a closed subset of X.

Theorem 12.14. *Let X be a nonsingular variety, and $f\colon X \dashrightarrow \mathbb{P}^n$ a rational map. Then the indeterminacy locus $Z_f \subset X$ has codimension ≥ 2.*

Proof. It suffices to prove the assertion in an affine neighborhood of a nonsingular point $x \in X$. We can then find a representative of f in the form

$$f = (f_0 : \cdots : f_n)$$

where $f_i \in k(X)$, $f_i \in \mathcal{O}_{X,x}$ for all i (we can clear denominators because the target is a projective space) and the f_i without common factor in $\mathcal{O}_{X,x}$ (we can divide by any common factor since the target is projective). Then no irreducible codimension 1 subvariety Y can be contained in $f_0 = \cdots = f_n = 0$ since by Theorem 12.13 we would have $I_{Y,x} = (g)$ and if f_i vanishes on $(g = 0)$, then $g \mid f_i$ for all i, a contradiction. □

The following are immediate consequences of Theorem 12.14.

Corollary 12.15. *Every rational map $f\colon C \dashrightarrow \mathbb{P}^n$ from a nonsingular curve C to $\mathbb{P}^n$ is a morphism.*

Corollary 12.16. *Two birationally equivalent nonsingular projective curves are isomorphic.*

We proceed to discuss divisors of zeroes and poles of rational functions. Let X be an irreducible nonsingular projective variety, $f \in k(X)$ a rational function, $Y \subset X$ an irreducible subvariety of codimension 1. We want to define the order of zero or pole of f along Y. Let $U \subset X$ be affine open, $U \cap Y \neq \emptyset$ such that $I(Y \cap U) = (\pi)$ with $\pi \in \mathcal{O}_X(U)$, which is possible by Theorem 12.13.

For $g \in \mathcal{O}_X(U)$, $g \neq 0$, there is a $k \geq 0$ with $g \in (\pi^k)$, but $g \notin (\pi^{k+1})$; namely, if we had $g \in \bigcap_{k\geq 0}(\pi^k)$, then we would have $g \in \bigcap_{k\geq 0} \mathfrak{m}^k$ where $\mathfrak{m}$ is the maximal ideal of the local ring of $Y \cap U$, i.e. the localization $\mathcal{O}_U(U)_\mathfrak{p}$ where $\mathfrak{p} = I(Y \cap U)$. But then Krull's intersection theorem would imply $g = 0$ in the local ring of $Y \cap U$, hence g would be identically zero on U.

Now we put $k := v_Y(g)$. This is independent of the choice of an affine U above: if $V \subset U$ is affine open, $V \cap Y \neq \emptyset$, then π gives a local equation of $Y \cap V$ and $v_Y^U(g) = v_Y^V(g)$. In general, if U, V are open affine with $U \cap Y \neq \emptyset$, $V \cap Y \neq \emptyset$, then there is an affine open $W \subset U \cap V$ with $W \cap Y \neq \emptyset$.

We call $v_Y(g)$ the *valuation* of g in Y. We have

$$v_Y(g_1 g_2) = v_Y(g_1) + v_Y(g_2), \tag{12.1}$$

$$v_Y(g_1 + g_2) \geq \min\{v_Y(g_1), v_Y(g_2)\} \text{ if } g_1 + g_2 \neq 0. \tag{12.2}$$

If $f \in k(X)$, we can write

$$f = \frac{h_1}{h_2}, \quad h_1, h_2 \in \mathcal{O}_U(U).$$

Now (12.1) and (12.2) imply that if $f \neq 0$, then

$$v_Y(f) := v_Y(h_1) - v_Y(h_2)$$

is independent of the choice of representative h_1/h_2 and (12.1), (12.2) remain valid for all $g_1, g_2 \in k(X)\backslash\{0\}$.

If $v_Y(f) = k > 0$, then we say f has a *zero of order k along Y*. If $v_Y(f) = k < 0$, then we say that f has a *pole of order $-k$ along Y*.

There are only finitely many Y's with $v_Y(f) \neq 0$: indeed, if U is affine, $f \in \mathcal{O}_U(U)$, then $v_Y(f) = 0$ if Y is not component of $(f = 0)$. If $f = g_1/g_2$, then $v_Y(f) = 0$ if Y is not a component of $(g_1 = 0)$ or $(g_2 = 0)$. In general, X is a union of finitely many such U_i.

Definition 12.17. For smooth irreducible projective X let $\mathrm{Div}(X)$ the free abelian group on all codimension 1 irreducible subvarieties. We call $\mathrm{Div}(X)$ the group of *(Weil) divisors* on X. Any element in it is called a *(Weil) divisor*. For $f \in k(X)^*$ we put

$$\mathrm{div}(f) := \sum_{Y \subset X} v_Y(f) Y \in \mathrm{Div}(X)$$

where the sum runs over all irreducible codimension 1 subvarieties. A divisor of the form $\mathrm{div}(f)$ (or the zero divisor 0) is called a *principal divisor*. The principal divisors form a subgroup $\mathrm{PrincDiv}(X) \subset \mathrm{Div}(X)$. The quotient

$$\mathrm{Pic}(X) := \mathrm{Div}(X)/\mathrm{PrincDiv}(X)$$

is called the *Picard group* (or more accurately, Weil divisor class group) of X.

The group $\mathrm{Pic}(X)$ is an important invariant of X.

Example 12.18. We have $\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$. Every irreducible codimension 1 subvariety is defined by an irreducible homogeneous polynomial of some degree k by Theorem 8.6. We have for $U_i = \{X_i \neq 0\} \subset \mathbb{P}^n$ that $I(Y \cap U) = (F/X_i^k)$. If

$$f = \frac{F}{G} \in k(\mathbb{P}^n)^*$$

is a rational function, F, G homogeneous of the same degree k, $F = \prod_i H_i^{k_i}$, $G = \prod_j L_j^{m_j}$ with H_i and L_j irreducible, then

$$\mathrm{div}(f) = \sum_i k_i(H_i = 0) - \sum_j m_j(L_j = 0)$$

which has degree 0 as an $n-1$ cycle. If conversely, $D = \sum k_i Y_i$ with $\sum k_i \deg Y_i = 0$, and Y_i is defined by an irreducible H_i, then $f = \prod H_i^{k_i}$ is an element in $k(X)$ with $\mathrm{div}(f) = D$.

Example 12.19. Using the Segre embedding, one can show that

$$\mathrm{Pic}(\mathbb{P}^n \times \mathbb{P}^m) = \mathbb{Z} \oplus \mathbb{Z}.$$

The proof is similar to the one in Example 12.18.

By these simple examples one should not be tempted to think that Picard groups are always finitely generated or torsion-free: this fails in very simple examples already, like that of plane cubic curves.

Instead of with codimension 1 subvarieties, leading to $\mathrm{Pic}(X)$, one can work with higher-codimensional subvarieties as well, leading to the so-called *Chow groups* of X. But these are generally much harder to compute.

We also point out that the power series methods we have started to develop in this Chapter have much more far reaching consequences and developments, in particular in Zariski's theory of formal functions and deformation theory.

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