FIRST YEAR PHD REPORT

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Abstract. In this report we study Hilbert modular surfaces. In Section 2 these surfaces are defined and an attempt is made to find equations for birational models. Section 3 focuses on an application of Runge's method to the Siegel modular variety and possible improvements in the Hilbert case.

Contents

1. Academic activities

Papers studied:

- K3 surfaces and equations for Hilbert modular surfaces, Noam Elkies and Abhinav Kumar [2]
- Hecke and Sturm bounds for Hilbert modular forms over real quadratic fields, Jose Ignacio Burgos Gil and Ariel Pacetti [8]
- A "tubular" variant of Runge's method in all dimensions, with applications to integral points on Siegel modular varieties, Samuel Le Fourn [3]
- Modular Forms and Projective Invariants, Jun-Ichi Igusa [13]

Books studied:

- Hilbert Modular Surfaces, Gerard van der Geer [7]
- Diophantine Geometry, Marc Hindry and Joseph Silverman [11]
- Algebraic Geometry, Robin Hartshorne [10]

Courses for credit: Introduction to Schemes (TCC), Weil Conjectures (TCC), L functions (TCC)
Courses not for credit: Rigid Analytic Geometry (Oxford), Automorphic Forms and the Langlands Program (MSRI Summer School 2017, watched online)
Seminars followed: Number theory seminar (terms 1 and 2), Number theory study group (terms

Date: April 30, 2018.
During this project, the author was a PhD student at the University of Warwick advised by Prof. Samir Siksek.

1 and 2), Number theory group meeting (terms 1 and 2), Logarithmic geometry (term 1)
Talks given: Logarithmic geometry seminar, number theory study group term 2, postgraduate seminar
Teaching: TA for Galois Theory in term 1 and Algebraic Number Theory in term 2.

2. Finding equations for birational models of Hilbert modular varieties

Elkies and Kumar ([2]) computed equations for birational models of all Hilbert modular surfaces (without level structure) for the real quadratic fields $\mathbb{Q}(\sqrt{D})$ with D a discriminant of size at most 100. In this section we attempt to do the same for Hilbert modular surfaces with level structure. We obtain a method that works for sufficiently large D, where, however, the algorithm takes too long to actually do the computations.

2.1. **Parametrizing abelian surfaces with real multiplication.** An *abelian variety* over a field k is a projective algebraic group over k. Such algebraic groups are automatically abelian.

For each line bundle L on A, we define a map

$$\phi_L : A \to \mathrm{Pic}(A),\ x \mapsto T_x^* L \otimes L^{-1},$$

where $\mathrm{Pic}(A)$ is the Picard group and T_x the translation-by-x map. In fact, $\mathrm{Pic}(A)$ can be given the structure of a group variety on A and the image of ϕ_L lands in the identity component $\mathrm{Pic}^0(A)$. Moreover, $\mathrm{Pic}^0(A)$ can uniquely be endowed with the structure of an abelian variety. We call it the *dual abelian variety of* A and denote it by $A^\vee$. As the name suggests, there is a canonical isomorphism $(A^\vee)^\vee \simeq A$. If the line bundle L is ample, the morphism ϕ_L is an *isogeny*: it is surjective with finite kernel. In that case, there is a dual map $\phi_L^\vee : A \simeq (A^\vee)^\vee \to A^\vee$ and if L is also symmetric, meaning that $[-1]^*L = L$, then $\phi_L^\vee = \phi_L$. A *polarization* is a map $\phi_L : A \to A^\vee$ such that L is ample and symmetric, where L is defined on $A/\overline{k}$. The polarization is called *principal* when it is an isomorphism. Note that elliptic curves come with a natural principal polarization $P \mapsto (O) - (P)$.

Definition 2.1. Let F be a totally real number field. A *principally polarized abelian variety with* $\mathcal{O}_F$*-multiplication* is a triple (A, λ, ι), where A is an abelian variety of dimension $[F : \mathbb{Q}]$, λ is a principal polarization on A and $\iota : \mathcal{O}_F \to \mathrm{End}(A)$ an embedding such that for all $\alpha \in \mathcal{O}_F$, we have

$$\lambda \circ \iota(\alpha) = \iota(\alpha)^* \circ \lambda.$$

Again note that elliptic curves trivially have multiplication by $\mathbb{Z}$.

When $k = \mathbb{C}$, the above definitions can be described in a much more concrete way. Each abelian variety over $\mathbb{C}$ is a *torus* $\mathbb{C}^g/\Lambda$, where $\Lambda \subset \mathbb{C}^g$ is a lattice. Conversely, a torus is an abelian variety precisely when it can be embedded (analytically) in projective space. Such embeddings in projective space correspond to (analytic) divisors. When D is an effective divisor on $\mathbb{C}^g/\Lambda$, its pull-back to $\mathbb{C}^g$ must be the divisor of some entire function θ. Being a pull-back, θ must satisfy

$$\theta(z + \lambda) = e^{g_\lambda(z)}\theta(z) \text{ for each } \lambda \in \Lambda,\ z \in \mathbb{C}^g,$$

where g_λ is a holomorphic function. We call this the *functional equation* of θ. If g_λ is *affine* for all λ, meaning that $g_\lambda(z + w) + g_\lambda(0) = g_\lambda(z) + g_\lambda(w)$, then θ is called a *theta function relative to* Λ. All theta functions relative to Λ with the same functional equation together define a morphism

$$(\theta_0 : \ldots : \theta_n) : \mathbb{C}^g/\Lambda \to \mathbb{P}^n(\mathbb{C}).$$

On the other hand, using the affine functions g_λ, one can define a *Riemann form*: a Hermitian form $H : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ such that $H(\Lambda \times \Lambda) \subset \mathbb{Z}$. The Riemann form is positive-definite if and only if the divisor D is ample.

Proposition 2.2. *A torus $\mathbb{C}^g/\Lambda$ is an abelian variety if and only if it admits a positive-definite Riemann form.*

See [11] Section A.5 for a proof.

Moreover, positive-definite Riemann forms are in canonical bijection with the polarizations on the resulting abelian variety. The map is the obvious one: for an ample symmetric line bundle L inducing the polarization, we choose an effective divisor D inducing L and we define the Riemann form via the functional equation of the theta functions. The polarization is principle if and only if the corresponding Riemann form satisfies $H(\Lambda \times \Lambda) = \mathbb{Z}$. This gives us a way of interpreting polarizations: they can be viewed as a choice of (equivalence classes of) embeddings into projective space.

With this concrete description, it is not hard to parametrize the principally polarized abelian varieties over $\mathbb{C}$. We define the *Siegel half space*, denoted $\mathfrak{S}_g$, to be the space of symmetric matrices $g \times g$-matrices with positive-definite imaginary part. For each $\tau \in \mathfrak{S}_g$, the pair (A_τ, H_τ), where

$$A_\tau = \frac{\mathbb{C}^g}{\mathbb{Z}^g \oplus \tau\mathbb{Z}^g} \text{ and } H_\tau(z,w) = z^t(\mathrm{Im}\tau)^{-1}\overline{w},$$

is a principally polarized abelian variety. Two such pairs corresponding to τ, τ' are isomorphic if and only if $\tau = \gamma(\tau')$ for some $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$, acting via fractional linear transformations:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tau = (A\tau + B)(C\tau + D)^{-1}.$$

This gives us the moduli space of g-dimensional principally polarized abelian varieties $A_g := \mathrm{Sp}_{2g}(\mathbb{Z}) \setminus \mathfrak{S}_g$ called the *Siegel modular variety* of dimension g. Moreover, for each τ, we can define a canonical theta function inducing H, namely

$$\theta_\tau(z) = \sum_{n \in \mathbb{Z}^g} e^{\pi i n^t \tau n + 2\pi i n^t z}.$$

Now let F be a number field of discriminant D. For simplicity of exposition, we take $[F : \mathbb{Q}] = 2$.

Proposition 2.3. *The space* $\mathrm{SL}_2(\mathcal{O}_F) \setminus \mathcal{H}^2$ *is a moduli space for the principally polarized abelian surfaces with* $\mathcal{O}_F$*-multiplication.*

Proof. The group $\mathrm{SL}_2(\mathcal{O}_F)$ acts naturally on $\mathcal{H}^2$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau_1, \tau_2) := \left(\frac{\sigma_1(a)\tau_1 + \sigma_1(b)}{\sigma_1(c)\tau_1 + \sigma_1(d)}, \frac{\sigma_2(a)\tau_2 + \sigma_2(b)}{\sigma_2(c)\tau_2 + \sigma_2(d)} \right),$$

using the action of $\mathrm{SL}_2(\mathbb{R})$ via general linear transformations. Here σ_1, σ_2 are the embeddings $F \to \mathbb{C}$. Note that this is where we make crucial use of the fact that F is a totally real number field. In order to obtain a *principal* polarization in the process to come, we make a slight change of groups. Define $\mathcal{O}_F^* = \frac{1}{\sqrt{D}}\mathcal{O}_F$ and

$$\mathrm{SL}_2(\mathcal{O}_F, \mathcal{O}_F^*) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \mid a, d \in \mathcal{O}_F, \ \ b \in (\mathcal{O}_F^*)^{-1} \text{ and } c \in \mathcal{O}_F^* \right\}.$$

This group acts on $\mathcal{H}^2$ in the same way as $\mathrm{SL}_2(\mathcal{O}_F)$ and in fact the map $(\tau_1, \tau_2) \mapsto (\sigma_1(\sqrt{D})\tau_1, -\sigma_2(\sqrt{D}))\tau_2)$ induces a natural bijection between $\mathrm{SL}_2(\mathcal{O}_F) \setminus \mathcal{H} \times \mathcal{H}^-$ and $\mathrm{SL}_2(\mathcal{O}_F, \mathcal{O}_F^*) \setminus \mathcal{H}^2$ when we assume that $\sigma_1(\sqrt{D}) > 0$ and $\sigma_2(\sqrt{D}) < 0$.

Similar to the elliptic curve case, for each $\tau \in \mathcal{H}^2$, we consider the lattice

$$L(\mathcal{O}_F \oplus \mathcal{O}_F^*) := \mathcal{O}_F\tau \oplus \mathcal{O}_F^* \subset \mathbb{C}^2,$$

where we consider $\mathcal{O}_F$ as embedded in $\mathbb{R}^2$ via (σ_1, σ_2) and multiplication is coordinatewise. This yields a complex torus $\mathbb{C}^2/L$, on which $\mathcal{O}_F$ acts as $\alpha(s_1, s_2) = (\sigma_1(\alpha)s_1, \sigma_2(\alpha)s_2)$. In order to define the Riemann form, define E on $(\mathcal{O}_F \oplus \mathcal{O}_F^*)^2$ by

$$E((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = \mathrm{Tr}_{K/\mathbb{Q}}(\alpha_1\beta_2 - \alpha_2\beta_1).$$

Notice that E has image in $\mathbb{Z}$ because $\mathcal{O}_D^*$ is precisely the dual of $\mathcal{O}_F$ with respect to Tr. We now define E on L via its description in terms of $\mathcal{O}_F \oplus \mathcal{O}_F^*$ and we extend this $\mathbb{R}$-linearly to $\mathbb{C}^2$. This becomes a Riemann form H such that $H(\Lambda \times \Lambda) = \mathbb{Z}$ that is compatible with the $\mathcal{O}_F$-multiplication. One can show that we obtain all principally polarized abelian varieties with real multiplication this way. Lastly, it is not hard to check that two abelian-varieties-with-real-multiplication corresponding to $\tau, \tau' \in \mathcal{H}^2$ are isomorphic if and only if there is some element of $\mathrm{SL}_2(\mathcal{O}_F, \mathcal{O}_F^*)$ transforming z into z'. See Theorem 2.17 in [9] for the details. We conclude that $\mathrm{SL}_2(\mathcal{O}_F) \setminus \mathcal{H}^2$ is the sought moduli space. □

Note how this is much more similar to the elliptic curve case than the Siegel modular variety.

When Γ is a subgroup of $\mathrm{SL}_2(\mathcal{O}_F)$ of finite index we can look at $\Gamma \setminus \mathcal{H}^2$ and this can often be given a moduli interpretation as the space parametrizing principally polarized abelian varieties with real multiplication and some additional structure. For example, when $\mathfrak{n} \subset \mathcal{O}_F$ is an ideal, define

$$\Gamma(\mathfrak{n}) := \{\gamma \in \mathrm{SL}_2(\mathcal{O}_F) \mid \gamma \equiv \mathrm{Id} \bmod \mathfrak{n}\}.$$

Then $\Gamma(n\mathcal{O}_F)\backslash\mathcal{H}^2$ is the moduli space of tuples (A,λ,ι,α), where (A,λ,ι) is a principally polarized abelian variety with $\mathcal{O}_F$-multiplication and $\alpha=(\alpha_1,\alpha_2)$ is an $\mathcal{O}_F/n\mathcal{O}_F$-basis of $A[n]$ with Weil pairing

$$w_n(\alpha_1,\alpha_2)=e^{2\pi i/n}.$$

Here the Weil pairing on $A\times A$ is defined as the natural pairing on $A\times A^\vee$ via λ.

As is the case for the modular curves, these spaces $\Gamma\backslash\mathcal{H}^2$ are in general not compact, but there is a natural way of compactifying them by adding finitely many cusps: this is called the *Bailey-Borel compactification*.

Definition 2.4. This compactification X_Γ of $\Gamma\backslash\mathcal{H}^2$ is called the *Hilbert modular surface* associated to Γ.

There is a natural action of $\mathrm{SL}_2(\mathcal{O}_K)$ on $\mathbb{P}^1(K)$ by fractional linear transformations. The set $X_\Gamma\setminus(\Gamma\setminus\mathcal{H}^2)$ of *cusps* of X_Γ is then $\Gamma\setminus\mathbb{P}^1(K)$. We note that the cusps of $X_{\mathrm{SL}_2(\mathcal{O}_K)}$ are in natural bijection with the class group of K.

A priori this X_Γ is merely a complex analytic space, but, using modular forms called Poincaré series, one can embed X_Γ into projective space to find that X_Γ is in fact a projective algebraic surface. However, unlike modular curves, X_Γ has singularities at the cusps and at the elliptic points (those points x where the isotropy group Γ_x is non-trivial), see [4] p. 30–31.

Therefore, in order to develop a sensible theory of these Hilbert modular surfaces, we would like to find a (minimal) desingularization of X_Γ.

2.2. **The explicit desingularization of Hilbert modular surfaces.** Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_F)$, i.e. a subgroup of finite index. In this section we describe the desingularization of a cusp of X_Γ as done in [7].

A neighbourhood in X_Γ around the cusp ∞ can be described by $\Gamma_\infty\setminus F_\infty$, where F_∞ is the *sphere of influence* of ∞, as defined on p. 8 of [7]. Then the desingularization of ∞ is determined as follows. First we note that

$$\Gamma_\infty=\left\{\begin{pmatrix}\epsilon & \mu\\ 0 & \epsilon^{-1}\end{pmatrix}\in\Gamma\right\}.$$

Note that such a matrix acts the same as the matrix $\begin{pmatrix}\epsilon^2 & \mu\\ 0 & 1\end{pmatrix}$. So as a transformation group, we have

$$\Gamma_\infty=\left\{\begin{pmatrix}\epsilon' & \mu\\ 0 & 1\end{pmatrix}\mid \mu\in M,\ \epsilon'\in V\right\}=M\rtimes V,$$

where M and V are determined by Γ and the above equality. For example, when

$$\Gamma=\Gamma_0(\mathfrak{c},\mathfrak{n}):=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\begin{pmatrix}\mathcal{O}_K & \mathfrak{c}^{-1}\\ \mathfrak{n}\mathfrak{c} & \mathcal{O}_K\end{pmatrix}\mid ad-bc=1\right\},$$

we have $M=\mathfrak{c}^{-1}$ and $V=(\mathcal{O}_K^\times)^2=\{\epsilon^2\mid\epsilon\in\mathcal{O}_K^\times\}$. To resolve the singularity at a cusp, we transform the cusp to ∞, then resolve the singularity of $M\setminus F_\infty$ and finally we quotient by the action of V.

We consider $M\backslash\mathbb{C}^2$ instead, with coordinates denoted by (z_1,z_2). Note that M acts by real translations via the embeddings $F\to\mathbb{C}$. The infinity cusp is situated at the limit $\mathrm{Im}(z_1)\mathrm{Im}(z_2)\to\infty$, which after applying the exponential map $\mathbb{C}^2\to\mathbb{C}^2$, $(z_1,z_2)\mapsto(\exp(2\pi iz_1),\exp(2\pi iz_2))$ corresponds to the two axes $z_1=0$ and $z_2=0$. Of course this exponential map is not M-invariant, but we can make it so. First choose a $\mathbb{Z}$-basis μ_1,μ_2 for M: we can write an element $z=(z_1,z_2)\in M\setminus\mathbb{C}^2$ in terms of this basis as

$$z=u\mu_1+v\mu_2 \bmod M \text{ for some } u,v\in\mathbb{C}.$$

Then we get a well-defined map

$$\phi_{\mu_1,\mu_2}:\ M\setminus\mathbb{C}^2\to\mathbb{C}^2, z=u\mu_1+v\mu_2v\mapsto(e^{2\pi iu},e^{2\pi iv}).$$

The image of ϕ_{μ_1,μ_2} is $(\mathbb{C}^\times)^2$, so we can view $\mathbb{C}^2=(\mathbb{C}^\times)^2\cup\{z_1=0\}\cup\{z_2=0\}$ as a completion at the cusp. The manifold $\mathbb{C}^2$ is certainly non-singular and would serve as a desingularization, but this definition is in no way canonical. Therefore, we will choose a countable number of "natural" bases and glue the resulting copies of $\mathbb{C}^2$ in the appropriate way. We note that the real embeddings of F embed M as a lattice in $\mathbb{R}^2$, and we view M as such. We will be interested in the behaviour of modular forms at the cusps and these have a Fourier expansion (see 1). So it is in some sense natural to look at "minimal positive points" in M, that is, points $\mu \in M_+$ (the totally positive elements of M) such that

$$\min_{\nu \in (M^\vee)_+} \mathrm{Tr}(\nu\mu) = 1,$$

where $M^\vee \subset K$ denotes the $\mathbb{Z}$-dual with respect to the trace. Then such minimal positive points of M correspond precisely to the points of M_+ on the boundary of its convex hull inside $\mathbb{R}^2$. Denote these points by A_k ($k \in \mathbb{Z}$), where the numbering is chosen such that the second coordinate of A_k increases as k increases. Then one can show (see Lemma 2.1 in [7]) that two consecutive points A_{k-1}, A_k form a $\mathbb{Z}$-basis for M. It follows from the definition that $A_{k-1} + A_{k+1}$ is an integral multiple of A_k, say

$$A_{k-1} + A_{k+1} = b_k A_k.$$

Denote by $\mathbb{C}^2_k$ the copy of $\mathbb{C}^2$ viewed as a desingularization of $M \backslash \mathbb{C}^2$ by choosing the basis A_{k-1}, A_k for M. Denote the coordinates of $\mathbb{C}^2_k$ by (u_k, v_k). The identification between $\mathbb{C}^2_k$ and $\mathbb{C}^2_{k+1}$ is given by $(u_k, v_k) \mapsto (u_k^a v_k^d, u_k^b v_k^c)$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the change-of-basis matrix. By the definition of b_k, we have $a = b_k, c = 1, b = -1, d = 0$, so that

$$(u_k, v_k) \mapsto (u_k^{b_k} v_k, u_k^{-1})$$

is the identification. Glueing the copies $\mathbb{C}^2_k$ ($k \in \mathbb{Z}$) in this way, we obtain a smooth manifold Z. Note that the glueing map above is only defined when $u_k \neq 0$, i.e. outside one of the two axes $S_k := \{u_k = 0\}$ that are meant to lie above the cusp. The other axis $\{v_k = 0\}$ is mapped onto $\{u_{k+1} = 0\} = S_{k+1}$. The axes intersect transversally in $\mathbb{C}^2$, so we obtain a chain $\{S_k \mid k \in \mathbb{Z}\}$ of rational curves, where each S_k intersects its neighbours S_{k-1} and S_{k+1} exactly once, transversally, and is disjoint from the other S_m with $|m - k| \geq 2$. By definition of the identifications, the maps $\phi_{A_k, A_{k+1}} : M \setminus \mathbb{C}^2 \to Z$ all coincide to a single map we denote by ϕ. Now we define $Y^+ := \phi(M \setminus \mathcal{H}^2) \cup \cup_k S_k$. It is important to note that we can find the self-intersections of the S_k explicitly by exhibiting functions on $\mathbb{C}^2_k$, see p.33 of [7]. This yields

$$S_k^2 = -b_k$$

for all $k \in \mathbb{Z}$, where we note that $b_k \geq 2$ by definition of b_k.

Lastly, we need to quotient by the action of the units V. As mentioned before, V acts on M by isomorphisms and hence V acts on $M \setminus \mathcal{H}^2$. In fact, it turns out that the action of V on Y^+ is free and properly discontinuous, so that Y^+/V is a smooth complex manifold (see Lemma 3.1 in [7]).

Let us see what happens under this quotient. Note that V always consists of squares, so each $\epsilon \in V$ is totally positive and we have

$$\min_{\nu \in (M^\vee)_+} \mathrm{Tr}(\nu\epsilon\mu) = \min_{\nu \in (\epsilon M^\vee)_+} \mathrm{Tr}(\nu\mu) = \min_{\nu \in (M^\vee)_+} \mathrm{Tr}(\nu\mu),$$

so multiplication by ϵ preserves the set of points $\mu \in M_+$ where this minimum is 1, i.e. $\{A_k \mid k \in \mathbb{Z}\}$. Now V is a subgroup of $(\mathcal{O}_K^\times)^2$, and $\mathcal{O}_K^\times$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ by Dirichlet's unit theorem as K is a totally real quadratic number field; so V is cyclic. Denote by η a generator. The previous display shows that for each k there is an $r \in \mathbb{Z}$ such that $\eta A_k = A_{k+r}$. By continuity, this number r is independent of k. Therefore, what dividing by V comes down to is identifying those glued copies of $\mathbb{C}^2_k$ where the numbers k are in the same residue class modulo r. In particular, we see that we are left with r rational curves $S_1, \ldots, S_r$ lying above the cusp, intersecting as mentioned before when $r \geq 3$. When $r = 2$ the two curves intersect transversally in two points. When $r = 1$, the curve S_1 is singular with self-intersection $-b_1 + 2$, see Lemma II.3.2 in [7]. As each $b_k \geq 2$, there are no exceptional curves and we find that Y^+/V is the unique minimal resolution of the singular point at infinity of $\overline{\Gamma_\infty \setminus \mathcal{H}^2}$ (see [7] II Theorem 3.3). Glueing all the desingularizations at the different cusps we obtain the algebraic surface Z_Γ. Finally, a similar desingularization can be made for the elliptic points, leading to the smooth algebraic surface Y_Γ. This is the desingularization we will be working with.

2.3. **Hilbert modular forms and differentials.**

For modular curves, the canonical morphism to projective space is determined by the line bundle of holomorphic 1-forms. However, as X_Γ is singular, we cannot simply speak of the "holomorphic 1-forms" on X_Γ. Instead, we must turn to the desingularization Y_Γ. We first define Hilbert modular forms. Recall that on $\mathcal{H}^2$ we have an action of $\mathrm{GL}_2^+(F)$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z_1, z_2) = \left(\frac{\sigma_1(a)z_1 + \sigma_1(b)}{\sigma_1(c)z_1 + \sigma_1(d)}, \frac{\sigma_1(a)z_1 + \sigma_1(b)}{\sigma_2(c)z_2 + \sigma_2(d)} \right).$$

For each pair of ideals $\mathfrak{c}, \mathfrak{n} \subset \mathcal{O}_K$, we define the group

$$\Gamma(\mathfrak{c}, \mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{O}_K & \mathfrak{c}^{-1} \\ \mathfrak{cn} & \mathcal{O}_K \end{pmatrix} \mid ad - bc \gg 0 \right\}.$$

Let $\Gamma = \Gamma(\mathfrak{c}, \mathfrak{n})$ for some choice of ideals.

Definition 2.5. A *Hilbert modular form* of weight $k = (k_1, k_2) \in \mathbb{Z}^2$ for Γ is a holomorphic map $f : \mathcal{H}^2 \to \mathbb{C}$ such that $f[\gamma]_k = f$ for all $\gamma \in \Gamma$, where

$$f[\gamma]_k(z_1, z_2) = (\sigma_1(c)z_1 + \sigma_1(d)z_1)^{-k_1}(\sigma_2(c)z_2 + \sigma_2(d))^{-k_2} f(z_1, z_2)$$

when $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Hilbert modular forms turn out to be automatically holomorphic at the cusps. Each Hilbert modular form f has a Fourier expansion

$$f(z) = \sum_{\mu \in (M^\vee)_+} a_\mu e^{2\pi i(\sigma_1(\mu)z_1 + \sigma_2(\mu)z_2)}, \tag{1}$$

at each cusp, where M is as before. For the infinity cusp, for example, $M = \mathfrak{c}^{-1}$. When $a_0 = 0$ at all cusps, we call f a *cusp form.* The space of modular forms of weight k with respect to Γ is denoted $\mathcal{M}_k(\Gamma)$, and the subspace of cusp forms by $\mathcal{S}_k(\Gamma)$.

For classical modular forms, we know that cusp forms correspond to the 1-forms on the compactified modular curve. However, in this case the compactified modular curve is not even smooth (we cannot speak of *holomorphic* forms) and moreover, even if we could, since the Fourier expansion runs over a rank two lattice, cusp forms do not necessarily yield differential forms with a "holomorphic" q-expansion.

When M is a complex manifold, define by ω_M the canonical sheaf $\det \Omega^1_M$ of holomorphic $(\dim M)$-forms. Also let $\mathcal{H}^{2,'}$ be $\mathcal{H}^2$ minus the inverse images of the elliptic points of $\Gamma \setminus \mathcal{H}^2$ under the projection map and define $X_{\mathrm{reg}} := \Gamma \setminus \mathcal{H}^{2,'} = X_\Gamma \setminus \{\text{elliptic points and cusps}\}$. Unlike X_Γ, this is a complex manifold, since locally a chart is given by an inverse of the projection map $\pi : \mathcal{H}^{2,'} \to X_{\mathrm{reg}}$. On X_Γ there is no standard notion of holomorphic differential forms, but since the elliptic points and the cusps of X_Γ are isolated, the following theorem allows us to consider differentials on X_{reg} instead.

Theorem 2.6 (Hartog's phenomenon)**.** *Let $n \geq 2$, $U \subset \mathbb{C}^n$ open and $K \subset U$ compact. When $U \setminus K$ is connected, each holomorphic function on $U \setminus K$ can be extended to a holomorphic function on U.*

Note how untrue this theorem is in $\mathbb{C}^1$, where we have the function $1/z$. The idea is that in higher dimensions, such functions with a simple pole are undefined on codimension 1 subspaces, so extending over smaller sets is okay.

Proposition 2.7. *Suppose $k \in 2\mathbb{Z}$, $\underline{k} = (k, k)$ is a parallel weight and $\Gamma = \Gamma_0(\mathfrak{c}, \mathfrak{n})$. Let $i : X_{\mathrm{reg}} \to X_\Gamma$ be the inclusion. Then the map $f \mapsto (2\pi i)^2 f \mathrm{d}z_1 \wedge \mathrm{d}z_2$ defines an isomorphism between $\mathcal{M}_{\underline{k}}(\Gamma)$ and $i_* \omega_{X_{\mathrm{reg}}}^{\otimes k/2}(X_{\mathrm{reg}})$.*

Proof. Note that $X_{\mathrm{reg}} = \Gamma \setminus \mathcal{H}^{2,'}$ and the projection $\pi : \mathcal{H}^{2,'} \to X_{\mathrm{reg}}$ yields an embedding of sheaves

$$\pi^* : \omega_{X_{\mathrm{reg}}}^{\otimes k/2} \to \pi_* \omega_{\mathcal{H}^{2,'}}^{\otimes k/2}.$$

By Hartog's phenomenon, we have

$$\omega_{\mathcal{H}^{2,'}}^{\otimes k/2}(\mathcal{H}^{2,'}) = \omega_{\mathcal{H}^2}^{\otimes k/2}(\mathcal{H}^2) = \{f(\mathrm{d}z_1 \wedge \mathrm{d}z_2)^{\otimes k/2} \mid f \text{ is holomorphic on } \mathcal{H}^2\}.$$

It is clear that all elements in the image of $\omega_{X_{\mathrm{reg}}}^{\otimes k/2}$ are invariant for the Γ-action. On the other hand, locally on $\mathcal{H}^{2,'}$ the projection π is biholomorphic and any inverse determines a chart, from which we see that locally on $U \subset X_{\mathrm{reg}}$, a Γ-invariant differential form on $\pi^{-1}(U)$ determines a differential on X_{reg}. In other words, the image of π^* is precisely the push-forward of the sheaf on

$\mathcal{H}^{2,'}$ of those elements $\eta = f(\mathrm{d}z_1 \wedge \mathrm{d}z_2)^{\otimes k/2}$ invariant for the γ-action for each $\gamma \in \Gamma$. Considering the global sections and applying Hartog's phenomenon, we see that indeed $\omega_{X_{\mathrm{reg}}}^{\otimes k/2}(X_{\mathrm{reg}})$ can be identified with the Γ-invariant forms $f(\mathrm{d}z_1 \wedge \mathrm{d}z_2)^{\otimes k/2}$, where f is holomorphic on $\mathcal{H}^2$. Lastly, note that such a form is Γ-invariant if and only if $f \in \mathcal{M}_{\underline{k}}(\Gamma)$. □

Note that $Y_\Gamma \to X_\Gamma$ restricts to an isomorphism on the inverse image of X_{reg}, so global differential forms on Y_Γ determine modular forms via the previous proposition. On the other hand, Proposition 3.7 in [7] shows that an element $f(\mathrm{d}z_1 \wedge \mathrm{d}z_2) \in \omega_{X_{\mathrm{reg}}}(X_{\mathrm{reg}})$ can be extended to a differential form on Y_Γ if and only if $f \in \mathcal{S}_{\underline{2}}(\Gamma)$.

Theorem 2.8. *The map $f \mapsto f\mathrm{d}z_1 \wedge \mathrm{d}z_2$ determines an isomorphism between $\mathcal{S}_{\underline{2}}(\Gamma)$ and $\omega_{Y_\Gamma}(Y_\Gamma)$.*

Note how satisfyingly analogous this is to the modular curve case.

2.4. **How to check whether the image of the canonical map is a surface.** From this section onwards we investigate to what extent we can determine the equations for birational models of Hilbert modular surfaces. For modular curves, one obtains equations for isomorphic models via the canonical embedding into projective space using explicit computations of the Fourier expansions of modular forms (see [5]). For Hilbert modular forms with respect to $\Gamma_0(\mathfrak{n})$ it has recently also become possible to determine the Fourier coefficients in Magma, so one might hope for a similar argument to work in this case as well.

Let k be an even positive integer and $\underline{k} = (k, k)$. Suppose we are given a set of cusp forms in $\mathcal{S}_{\underline{k}}(\Gamma)$ for some congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_F)$. These correspond to global sections of ω_{Y_Γ} and hence determine a rational map Φ from Y_Γ to projective space. We would like this map to be birational. But how can we even know the image has the right dimension? It suffices to check this locally, even at the tangent space, since at a smooth point, the dimension of the tangent space equals the dimension of the variety. Suppose that the image of Φ were a curve. Then the image of the tangent space at any point would be 1-dimensional, at least when the image of this point is non-singular. However, intuitively, these tangent spaces should vary smoothly and hence cannot suddenly jump in dimension.

Proposition 2.9. *Suppose that $\Phi : Y \to C$ is a non-constant rational map between a smooth complete surface Y and a (possibly singular) curve C. Then for each $y \in Y$ where Φ is defined, the image $\Phi_*(T_yY) \subset T_{\Phi(y)}C$ is 1-dimensional.*

Proof. Let $\widetilde{C}$ be the normalisation of C. As Y is smooth, the rational map factors via $\widetilde{C} \to C$. Since $\widetilde{C}$ is smooth, its tangent spaces are 1-dimensional. □

So if we want to show that the image of Φ is a surface, it suffices to show that $\Phi_*(T_aY_\Gamma)$ is 2-dimensional for one $a \in Y_\Gamma$. We take $a = (0, 0) \in \mathbb{C}^2_k$. Around zero, the cusp forms have a Fourier expansion

$$f = \frac{1}{u_k v_k} \sum_{\nu \in M_+^\vee} a_\nu(f) u_k^{\mathrm{Tr}\nu A_{k-1}} v_k^{\mathrm{Tr}\nu A_k}.$$

Note that we divide by $u_k v_k$ because $\mathrm{d}z_1 \wedge \mathrm{d}z_2 = C\frac{\mathrm{d}u_k \wedge \mathrm{d}v_k}{u_k v_k}$ for C a constant. Suppose that $f_0, \dots, f_n$ are a basis for $\mathcal{S}_2(\Gamma)$ such that $f_0(a) \neq 0$. Then locally around a, Φ is the map to affine space given by $(f_1, \dots, f_n)$. Then it suffices to show that the matrix

$$\begin{pmatrix} \frac{\partial f_i}{\partial u_k} & \frac{\partial f_i}{\partial v_k} \end{pmatrix}_{i,j}$$

has rank 2 at $(0, 0)$. These derivatives of the f_i correspond by Taylor's theorem to the coefficients for u_k and v_k respectively. So if we can compute the coefficients $a_\nu(f_i)$ for ν such that $\mathrm{Tr}\nu A_{k-1} = 1$ and $\mathrm{Tr}\nu A_k = 2$ or $\mathrm{Tr}\nu A_{k-1} = 2$ and $\mathrm{Tr}\nu A_k = 1$, then we can determine this matrix. Also we would need to determine $a_\nu(f_i)$ for ν with $\mathrm{Tr}\nu A_{k-1} = \mathrm{Tr}\nu A_k = 1$ in order to determine the cusp form f such that $f(a) \neq 0$.

2.5. **An important inequality.** Having found a way to determine whether the image of Φ is a surface, we now assume that indeed the image is a surface. Then Φ is generically finite and we can associate to it its *degree*. How can we determine this degree? A good tool for this is intersection theory.

Theorem 2.10. *When $f : X \to Y$ is a generically finite morphism of non-singular projective surfaces, we have for each pair of divisors D, D' on Y that*

$$f^*D \cdot f^*D' = (\deg f) D \cdot D',$$

where the dot denotes the intersection index.

Now when D is a divisor on Y, its corresponding map Φ_D to projective space is a priori only *rational*, so we would like to extend the above theorem to rational maps. Note that if $\mathcal{O}_Y(D)$ has at least 2 elements (which is the case when the image is a surface), then we see that the basepoints $\mathcal{B}$ have codimension 2.

Theorem 2.11. *Let D be a divisor on a smooth projective surface Y such that the (closure of the) image Σ of Φ_D is a surface. When either Σ is non-singular or Φ_D is birational, we have*

$$D^2 \geq (\deg \Phi_D)(\deg \Sigma)$$

with equality if and only if Φ_D is a morphism.

Remark 2.12. In particular, when $D^2 \leq 0$ this theorem implies that Σ is singular and $\deg \Phi_D > 1$.

Proof. Intuitively, if E, E' are suitably chosen effective divisors on Σ representing $\mathcal{O}_\Sigma(1)$, then $\Phi_D^*(E)$ and $\Phi_D(E')$ intersect transversally in $(\deg \Phi_D)(E \cdot E')$ points where Φ_D is defined, but also in the base points of D.

To make this precise, we use blow-ups in order to reduce to the case where Φ_D is a morphism. By Theorem II.7 (p. 14) in [1], repeatedly blowing up base points of D yields a surface Y' with a birational morphism $\pi : Y' \to Y$ and a *morphism* $Y' \to \Sigma$ induced by $\pi^*D - E$, where E is a sum of exceptional curves created in the blow-up, such that $Y' \to \Sigma$ equals $Y' \to Y \to \Sigma$. By the previous theorem $(\pi^*D)^2 = D^2$. Also, the pull-back of $\mathcal{O}_\Sigma(1)$ to Y' is $\mathcal{O}_{Y'}(\pi^*D - E)$. But we know that $E \cdot \pi^*D = 0$ by properties of exceptional divisors (see II.3(ii) in [1]) and that $E^2 < 0$. Therefore, we find that $D^2 \geq (\pi^*D - E)^2$ and, replacing Y by Y', we may assume that Φ_D is a morphism. So by the previous theorem we are done when Σ is non-singular.

Suppose now that Φ_D is birational, but Σ possibly singular. We will reduce to the case where Σ is non-singular. Let $\widetilde{\Sigma}$ be a minimal desingularization of Σ with $p : \widetilde{\Sigma} \to \Sigma$. Then p is surjective, so $p_*p^*\mathcal{O}_\Sigma(1) = \mathcal{O}_\Sigma(1)$. Also, a push-forward does not change cohomology, so by definition of the degree as 2 times the leading coefficient of the Hilbert polynomial, we see that $\deg \Sigma = \mathcal{O}_{\widetilde{\Sigma}}(1)^2$. (When S is non-singular and embedded in projective space, the equality $\mathcal{O}_S(1)^2 = \deg S$ is a direct application of Riemann-Roch.) Now because Φ_D is a proper birational morphism and Y is smooth, the map $Y \to \Sigma$ is a desingularization, so it factors via the minimal desingularization $\widetilde{\Sigma}$. Hence we may assume that Σ is non-singular and we are done by the previous part. □

Denote by K a divisor on Y_Γ corresponding to ω_{Y_Γ}. This theorem gives us a means of verifying that our morphism $\Phi = \Phi_K$ is birational. Suppose we have shown that the image is a surface. We can compute all equations for the image since we know it must be a surface, hence we can compute the degree of the image and we can check whether it is non-singular. Moreover, there is a formula for K^2 (see p. 64 of [7])

$$K^2 = 4\zeta_F(-1)[\mathrm{SL}_2(\mathcal{O}_F) : \Gamma] - \sum_{\sigma,k}(b_k^\sigma - 2) - A,$$

where A is a non-negative number depending on the elliptic points and their explicit desingularization (which we did not describe in these notes). When Γ acts on $\mathcal{H}^2$ without fixed points, $A = 0$ and in general A can be computed. Finally, if the image is indeed non-singular and its degree equals K^2, then Φ_K must be birational, as desired. Note that we expect Φ_K to be an embedding for large fields and large ideals $\mathfrak{n} \subset \mathcal{O}_K$ because we then have many cusp forms, in which case the image is indeed non-singular and $\deg \Phi_K = 1$.

2.6. **A Hecke bound for Hilbert modular forms.** Being able to verify that Φ_K is birational, we now turn to computing the equations of its image. For this, we need to compute sufficiently many coefficients of Hilbert modular forms. But how many, and which ones? This question was answered by Gil and Pacetti [8]. They give a bound for arbitrary congruence subgroups Γ; in this section we slightly improve this for our choice $\Gamma = \Gamma_0(\mathfrak{n}) := \Gamma_0(\mathfrak{n}, \mathcal{O}_F)$.

We have seen that $\mathcal{O}_{Y_\Gamma}(K) \simeq \mathcal{S}_2(\Gamma)$. By the same reasons, we have $\mathcal{M}_2(\Gamma) \simeq \mathcal{O}_{Y_\Gamma}(K + \sum_{\sigma,k} S_k^\sigma)$, where the sum runs over the rational curves S_k^σ resolving the singularities at the cusps σ. Taking the global sections of the kth tensor powers, we obtain

$$\mathcal{M}_{2k}(\Gamma) \simeq \mathcal{O}_{Y_\Gamma}(kK + k\sum_{j,\sigma} S_j^\sigma).$$

Moreover, if we expand $f \in \mathcal{M}_{2k}(\Gamma)$ around the cusp at infinity, we have

$$f(z) = \sum_{\nu \in M_+^\vee} a_\nu e^{2\pi i \mathrm{Tr}(\nu z)} = \sum_{\nu \in M_+^\vee} a_\nu u_k^{\mathrm{Tr}(\nu A_k)} v_k^{\mathrm{Tr}(\nu A_{k+1})}$$

where u_k, v_k denote the coordinates on the chart $\mathbb{C}_k^2$ in the desingularization of the infinity cusp. Then we define

$$\mathrm{ord}_{S_j^\infty}(f) := \max\{n \in \mathbb{Z} \mid a_\nu = 0 \text{ for all } \nu \text{ s.t. } \mathrm{Tr}\nu A_k = n\}.$$

The order at other cusps can be defined by first transforming to the infinity cusp. We note that, since ν is in the positive cone of $M^\vee$, for each $n \in \mathbb{Z}_{\geq 0}$ there are only finitely many coefficients ν with $\mathrm{Tr}\nu A_k < n$ and in fact these ν can be computed explicitly in terms of the A_k.

Proposition 2.13. *Suppose that the canonical divisor K on Y_Γ is nef and that for each σ, j the modular form $f \in \mathcal{M}_{2k}(\Gamma)$ vanishes to order a_j^σ on S_j^σ. Then $f = 0$ once (one of) the a_j^σ is sufficiently large.*

Proof. Such a modular forms exists if and only if $\mathcal{O}_{Y_\Gamma}(kK + \sum_{j,\sigma}(k - a_j^\sigma)S_j^\sigma) \neq (0)$, i.e. if and only if the corresponding divisor is linearly equivalent to an effecitve divisor. But K is nef, so the existence of such a modular form implies that

$$K \cdot (kK + \sum_{j,\sigma}(k - a_j^\sigma)S_j^\sigma) \geq 0.$$

As each S_j^σ is a rational curve, it has arithmetic genus 1, so $K \cdot S_j^\sigma = -(S_j^\sigma)^2 = b_k \geq 2$. It follows that the above intersection number is negative once the a_j^σ become sufficiently large. In fact, it suffices for one of these to be sufficiently large. Moreover, this "sufficiently large" number is computable if we can compute K^2. □

In order to see when the condition in the above proposition is satisfied, we use the following proposition.

Proposition 2.14. *When Y is a minimal smooth complex projective surface of general type, its canonical divisor K_Y is nef.*

Recall that D is the discriminant of F.

Theorem 2.15. *Let $\Gamma = \Gamma(\mathfrak{n}, \mathcal{O}_F)$. Then Y_Γ is minimal and of general type in the following cases:*

- *When $D \notin \{5, 8, 12, 13, 17, 21, 24, 28, 33, 60\}$ and moreover $\mathfrak{n} = N\mathcal{O}_F$ with*
  - *$N \geq 3$ when either $D \not\equiv 1$ mod 8 or $D \equiv 1$ mod 8 and $D = (m^2 - 8)/n^2$ for some $m, n \in \mathbb{Z}_{>0}$ with $m \equiv 7$ mod 8 or $D \in \{29, 37, 40, 41, 44, 56, 57, 69, 105, 53, 61, 65, 73, 76, 77, 85, 88, 92, 93, 120, 140, 165\}$;*
  - *or*
  $$N \geq \sqrt{3\sum_{\sigma,j}(b_j^\sigma - 2)}.$$
- *When the pair $(D, \mathfrak{n}) \in \{(5, 3\mathcal{O}_F), (8, \mathfrak{p}_7), (13, 2\mathcal{O}_F), (17, 2\mathcal{O}_F), (21, 2\mathcal{O}_F), (24, \mathfrak{p}_2)\}$, where $\mathfrak{p}_7$ and $\mathfrak{p}_2$ are ideals of norms 7 and 2 respectively.*

Moreover, when $Y_{\Gamma(\mathfrak{n},\mathcal{O}_F)}$ is minimal and of general type, then so is $Y_{\Gamma(\mathfrak{nm},\mathcal{O}_F)}$ for each ideal $\mathfrak{m}$.

Remark 2.16. Note that only the fields with discriminants 12, 33 and 60 are missing from the above list.

Proof. This is a combination of many results in [7] brought together by Gil and Pacetti. □

The idea is now that we have $\mathcal{M}_{2k}(\Gamma_0(\mathfrak{n}, \mathcal{O}_F)) \subset \mathcal{M}_{2k}(\Gamma(N, \mathcal{O}_F))$ when $\mathfrak{n} \mid N$ and choosing N sufficiently large according to the theorem, we know that $K_{Y_{\Gamma(N,\mathcal{O}_F)}}$ is nef and the previous proposition can be applied. Note that the inclusion of modular forms corresponds to the pull-back of the corresponding sheaf along the quotient $\pi : Y_{\Gamma(N,\mathcal{O}_F)} \to Y_{\Gamma_0(\mathfrak{n},\mathcal{O}_F)}$. We can do better than just using the inclusion of modular forms if we take the ramification along the cusp resolution curves into account. In fact, we will express the result in terms of $Y_{\Gamma(\mathcal{O}_F,\mathcal{O}_F)}$, by considering the maps

$$Y_{\Gamma(N,\mathcal{O}_F)} \to Y_{\Gamma_0(\mathfrak{n},\mathcal{O}_F)} \to Y_{\Gamma(\mathcal{O}_F,\mathcal{O}_F)}.$$

We first explain how to express things in terms of $Y_{\Gamma_0(\mathfrak{n},\mathcal{O}_F)}$. Consider the cusp ∞ of $X_{\Gamma_0(\mathfrak{n},\mathcal{O}_F)}$ and let S_j^∞ be one of its resolution curves. This cusp has type (M,V) where $M=\mathcal{O}_F$ and $V=(\mathcal{O}_F^\times)^2$. All of the cusps of $X_{\Gamma(N,\mathcal{O}_F)}$ above ∞ are of the same type $(N\mathcal{O}_F, V_N)$, where V_N is the group of squares of elements $\mathcal{O}_F^\times$ that are 1 modulo N. The two lattices $\mathcal{O}_K$ and $N\mathcal{O}_F$ are isomorphic via multiplication by N, and so they yield the same numbers b_k. Now if E is the resolution of a cusp of $X_{\Gamma(N,\mathcal{O}_F)}$ lying above infinity such that E is mapped onto S_j^σ, we see that in local coordinates the map $E\to S_j^\infty$ corresponds to $(u,v)\mapsto(u^N,v^N)$. So the ramification degree of E/S_j^∞ equals N and we conclude that

$$\pi^* S_j^\infty = \sum_{S\mapsto S_j^\infty} NS,$$

where the sum runs over all the resolution curves of cusps of $Y_{\Gamma(N,\mathcal{O}_F)}$ lying over infinity. Suppose that the length of the resolution cycle of ∞ in $Y_{\Gamma(\mathfrak{n},\mathcal{O}_F)}$ is r. Then the length of the resolution cycle of each cusp of $Y_{\Gamma(N\mathfrak{n},\mathcal{O}_F)}$ is $r[V:V_N]$. Note that the displayed sum can be broken down into a sum over the cusps over ∞ and for each such cusp a sum over the S_i^σ such that $j\equiv i \bmod r$. The only question that remains is: how many cusps of $X_{\Gamma(N,\mathcal{O}_F)}$ lie above ∞? Let d be the index $[\Gamma_0(\mathfrak{n},\mathcal{O}_F):\Gamma(N)]$ and the degree of the map $Y_{\Gamma(N,\mathcal{O}_F)}\to Y_{\Gamma_0(\mathfrak{n})}$. Note that the index $[\Gamma(N)_\infty:\Gamma_0(\mathfrak{n})_\infty]$ of the isotropy groups is $\mathrm{Norm}(N)[V:V_N]=N^2[V:V_N]$. Then the number of cusps of $X_{\Gamma(N)}$ above a given cusp of $X_{\Gamma_0(\mathfrak{n})}$ is

$$\frac{[\Gamma_0(\mathfrak{n}):\Gamma(N)]}{[\Gamma(N)_\infty:\Gamma_0(\mathfrak{n})_\infty]} = \frac{d}{N^2[V:V_N]}.$$

Proposition 2.17. *If $f\in\mathcal{M}_{2k}(\Gamma_0(\mathfrak{n},\mathcal{O}_K))$ vanishes to order a at the resolution curve S_j^∞ of the cusp infinity in $X_{\Gamma_0(\mathfrak{n},\mathcal{O}_F)}$, then f vanishes to order Na at all the resolution curves S_i^σ in $Y_{\Gamma_0(N,\mathcal{O}_F)}$ where σ is one of the $d/N^2[V:V_N]$ cusps lying over ∞ and $i\equiv j$ mod r, where r is the length of the resolution cycle of $\infty\in X_{\Gamma_0(\mathfrak{n},\mathcal{O}_F)}$. Moreover, in this case we have $b_i^\sigma=b_j^\infty$ for all such i,σ and the length of the resolution cycle of σ is $r[V:V_N]$. In particular,*

$$\sum_{\substack{\sigma\mapsto\infty \\ 1\le k\le r[V:V_N]}} (b_k^\sigma-2) = \frac{d}{N^2}\sum_{1\le k\le r}(b_k^\infty-2).$$

Lastly, we consider the map $Y_{\Gamma_0(\mathfrak{n})}\to Y_{\mathrm{SL}_2(\mathcal{O}_F)}$. If τ is a cusp of the former lying above the infinity cusp of the latter, then their types (M,V) are the same, and hence so are the corresponding numbers b_k and the lengths of the resolution cycles. So all we need to know is the number of cusps of $X_{\Gamma_0(\mathfrak{n})}$ above a given cusp of $X_{\mathrm{SL}_2(\mathcal{O}_F)}$. Since the isotropy groups are the same, this is just the degree $[\mathrm{SL}_2(\mathcal{O}_F):\Gamma_0(\mathfrak{n})]$.

Proposition 2.18. *The degree $[SL_2(O_F):\Gamma_0(\mathfrak{n})]$ equals*

$$N(\mathfrak{n})\prod_{\mathfrak{p}|\mathfrak{n}}(1+1/N(\mathfrak{p})).$$

The proof is analogous to the case $K=\mathbb{Q}$.

We say that a cusp form has order a at a cusp, when it vanishes to order at least a at all of the resolution curves of this cusp.

Theorem 2.19 (Hecke bound)**.** *Choose N such that $\mathfrak{n}\mid N$ and $Y_{\Gamma_0(N\mathcal{O}_F)}$ is minimal and of general type. If $f\in\mathcal{S}_{2k}(\Gamma_0(\mathfrak{n}))$ vanishes at the infinity cusp to order $a+1$, where*

$$a > \frac{4kNd\zeta_F(-1)}{\sum_j(b_{0,j}-2)} - \left(1+(d-1)\frac{k}{N}\right)\frac{\sum_{i=1}^h\sum_j(b_{ij}-2)}{\sum_j(b_{0,j}-2)}$$

and $d=[\mathrm{SL}_2(\mathcal{O}_F):\Gamma_0(\mathfrak{n})]=N(\mathfrak{n})\prod_{\mathfrak{p}|\mathfrak{n}}(1+N(\mathfrak{p})^{-1})$, then $f=0$. Here the numbers $b_{i,j}$ correspond to the resolutions of the cusps of $X_{\mathrm{SL}_2(\mathcal{O}_F)}$, with $i=0$ the infinity cusp.

Proof. We reason as in the proof of Proposition 2.13, using Propositions 2.17 and 2.18 to express everything in terms of the numbers $b_{i,j}$ corresponding to the cusps of $X_{\mathrm{SL}_2(\mathcal{O}_F)}$. Finally, we use the equation for K^2 mentioned before. □

Note that the first term in the Hecke bound comes from the vanishing to order a at the infinity cusp and the second term comes from the vanishing to order 1 at the other cusps.

We have now established all the ingredients to possibly compute the equations for birational models of Hilbert modular surfaces.

2.7. **A major downside of $\Gamma_0(\mathfrak{n})$.** We first need to mention a downside of working with $\Gamma_0(\mathfrak{n})$ that became apparent during the computations. Recall that the isotropy group of ∞ of $\mathrm{SL}_2(\mathcal{O}_F)$ are the upper triangular matrices in $\mathrm{SL}_2(\mathcal{O}_F)$, so the isotropy group of ∞ in $\Gamma_0(\mathfrak{n})$ is the same. This means that if σ is a cusp of $X_{\Gamma_0(\mathfrak{n})}$ mapping to $\infty \in X_{\mathrm{SL}_2(\mathcal{O}_F)}$ then the resolutions of σ and ∞ consist of an equal number of curves with equal self-intersections $-b_k^\sigma = -b_k^\infty$. Moreover, again since the isotropy group does not change, there are $d := [\mathrm{SL}_2(\mathcal{O}_K) : \Gamma_0(\mathfrak{n})]$ cusps of $X_{\Gamma_0(\mathfrak{n})}$ lying above any given cusp of $X_{\mathrm{SL}_2(\mathcal{O}_K)}$. Now recall the formula

$$K^2 = 4\zeta_K(-1)d - \sum_{\sigma,k}(b_k^\sigma - 2) - A.$$

It is important to note here that the contribution A from the elliptic points is very small in practice; for small discriminants almost always smaller than 2 (see the table on p.65 and Table 2 at the end of the book of [7]). But the contribution from the cusps now by the previous remarks equals $d\sum_{\sigma,k}(b_k^\sigma - 2)$ where the sum now runs over the cusps of $X_{\mathrm{SL}_2(\mathcal{O}_K)}$ instead. So we obtain

$$K^2_{Y_{\Gamma_0(\mathfrak{n})}} = d\left[4\zeta_K(-1) - \sum_{k,\sigma}(b_k^\sigma - 2)\right] - A$$

and up to the small contribution from the elliptic points, K^2 just gets multiplied with the index compared to the case $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$. Unfortunately, $K^2 + A$ is negative on $Y_{\mathrm{SL}_2(\mathcal{O}_K)}$ for all K of discriminant at most 89 and therefore, K^2 will be negative on $Y_{\Gamma_0(\mathfrak{n})}$ for such fields K as well, regardless of $\mathfrak{n}$. So we know that Φ_K is not birational in all those cases by Theorem 2.11. This problem does not occur for e.g. the principal congruence subgroups $\Gamma(\mathfrak{n})$, because the isotropy groups do change. However, for those subgroups Magma cannot compute Fourier coefficients.

Despite Φ_K not being birational to a surface, it is still interesting to compute the equations of its image. We still have a way of checking whether the image is a surface. However, we do not have an upper bound for the degree of the equations that occur. Unfortunately, we will see that equations indeed have a high degree.

2.8. **Computations with Magma.** Appended to this report is Magma code that can potentially compute equations of Hilbert modular surfaces, based on the previous sections. In this section, we give an overview of the computations that need to be done and we do some examples.

Our first aim is to compute a matrix of Fourier coefficients of a basis of cusp forms for $\mathcal{S}_2(\Gamma_0(\mathfrak{n}))$ such that the cusp forms of some weight k are determined by these Fourier coefficients. In order to obtain the indices of these coefficients, we first need to determine the Hecke bound, for which the numbers b_k^σ need to be computed, at least at the infinity cusp but preferrably at all cusps. Then, in order to apply the Hecke bound to find the indices, we also need to know explicitly the points A_k at the infinity cusp. In order to compute the A_k and b_k (at any cusp), I used the explicit continued fraction method described in Section II.5 of [7]. The totally positive indices of bounded trace can then be enumerated using the dual bases of the bases A_{k-1}, A_k.

Consequently, the Magma package on Hilbert modular forms can be used to compute the Fourier coefficients at these indices of *new*forms. Doing this at all levels dividing the given level and embedding the newforms at lower levels appropriately, one obtains all cusp forms. (There is a similar decomposition of the space of cusp forms in terms of newforms at the levels dividing the given level as for modular forms.) This finally yields the desired matrix of Fourier coefficients. Using the tensor product, this can efficiently be transformed into the corresponding matrices for the degree $d \geq 2$ monomials in the cusp forms. We can then let Magma compute the kernel of the transpose of this matrix to compute the possible equations.

Example 2.20. We consider $F = \mathbb{Q}(\sqrt{10})$. Then the discriminant $D = 40$ and we compute that $\zeta_F(-1) = 7/6$. This field has class number 2, so each element $\mathfrak{a}$ of the class group satisfies $\mathfrak{a}^{-2} = \mathcal{O}_K$. The resolution cycle for each of the two cusps is

$$(8, 2, 2, 2, 2, 2).$$

This yields that

$$4\zeta_F(-1) - \sum_{\sigma,k}(b_k^\sigma - 2) = -\frac{22}{3}$$

so $K^2 < 0$. Therefore, the rational map Φ_K from $Y_{\Gamma_0(\mathfrak{n})}$ to projective space is not birational for any choice of $\mathfrak{n}$. Let us take $\mathfrak{n} = 2\mathcal{O}_F$. Then the adelic cusp forms of level two with respect to $\Gamma_0(2\mathcal{O}_F)$ are 8-dimensional and the narrow class number is 2. We can compute the Hecke bound: $f \in \mathcal{S}_2(\Gamma_0(2\mathcal{O}_F))$ vanishes everywhere once its order of vanishing at all the resolution curves of the infinity cusp is at least 14. The number of totally positive elements μ of $\mathcal{O}_F^\vee$ that are inequivalent with respect to the action of the unit group and that satisfy $\mathrm{Tr}\mu A_k \leq 14$ for some k, is 646. We compute that $\dim \mathcal{S}_2(\Gamma_0(2\mathcal{O}_F)) = 4$. These four cusp forms must satisfy at least one equation. At all the intersection points of the resolution curves at infinity, we find that the Jacobian matrix of Φ_K has rank 1, so one would even expect two equations. However, we are not in posession of an upper bound for the degree of such equations. Our programme did not find any equations of degree ≤ 18. Note that for *not* finding equations, one need not look for all coefficients up to the Hecke bound, but it suffices to verify that no equation exists when considering any number of coefficients.

If we take an example with larger discriminant, we do find a positive K^2.

Example 2.21. Consider $K = \mathbb{Q}(\sqrt{101})$. Then the discriminant $D = 101$. The class number is 1, and the resolution cycle at infinity for $\Gamma_0(\mathfrak{n})$ (any $\mathfrak{n}$) is

$$(11, 3, 2, 2, 2, 2, 2, 2, 2, 2, 3).$$

We compute that

$$4\zeta_K(-1) - \sum_k (b_k^\infty - 2) = 5/3.$$

Consider $\mathfrak{n} = 2\mathcal{O}_K$. Then $[\mathrm{SL}_2(\mathcal{O}_K) : \Gamma_0(\mathfrak{n})] = 5$, so the canonical self-intersection on $Y_{\Gamma_0(\mathfrak{n})}$ is slightly smaller than 25/3. The Hecke bound is 21 and the dimension of the space of adelic cusp forms is 12. However, for a field as large as this, the number of coefficients up to the Hecke bound is huge and the algorithm that computes equations takes too long.

3. Applying Runge's method to the Siegel modular variety

In this section we study an application of Runge's method to the Siegel modular variety introduced by Le Fourn [3], who proves the following theorem.

Theorem 3.1 (Le Fourn)**.** *Suppose that K is $\mathbb{Q}$ or an imaginary quadratic number field. Consider a principally polarized abelian surface (A, λ) such that (A, λ) and its 2-torsion are defined over K, and having potentially good reduction at all finite places of K. If the semi-stable reduction of (A, λ) is a product of elliptic curves at most at 3 finite places, then*

$$h_F(A) \leq 1070,$$

where h_F denotes the Faltings height. In particular, there are finitely many such abelian varieties.

The aim of this project was to improve this result for principally polarized abelian surfaces with real multiplication, but so far we have not managed to do so.

3.1. An introduction to Runge's method.

The following example illustrates the simplest case of Runge's method for solving Diophantine equations.

Example 3.2. Suppose we want to solve the Diophantine equation

$$y^2 = x^4 + ax^3 + bx^2 + cx + d, \quad a, b, c, d \in \mathbb{Z}$$

in integers $x, y \in \mathbb{Z}$. By completing the square, we can rewrite the equation as

$$y^2 = (x^2 + a'x + b')^2 + c'x + d', \quad a', b', c', d' \in \mathbb{Q}$$

which yields an equation of the form

$$C(y - (x^2 + \alpha x + \beta))(y + x^2 + \alpha x + \beta) = \gamma x + \delta, \quad C, \alpha, \beta, \gamma, \delta \in \mathbb{Z}.$$

If either $y - (x^2 + \alpha x + be) = 0$ or $y + x^2 + \alpha x + \beta = 0$ then $\gamma x + \delta = 0$. Otherwise, we find that y is x-linearly close to both $x^2 + \alpha x + \beta$ and $-(x^2 + \alpha x + \beta)$, which is impossible when x is large. So this yields an upper bound on $|x|$, and we have thus found a method to solve the given Diophantine equation.

We now show how this idea can be extended to curves over $\mathbb{Q}$. Suppose that C is a smooth projective irreducible curve over $\mathbb{Q}$ and $\phi \in \mathbb{Q}(C)$ has poles $Q = Q_1, \dots Q_n$ in $C(\mathbb{C})$. We say that a point $P \in C(\mathbb{Q})$ is *ϕ-integral* when $\phi(P) \in \mathbb{Z}$.

Theorem 3.3. *Suppose that $U \subset C(\mathbb{C})$ is an open neighbourhood (in the Euclidean topology) of the Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates of Q. Then the number of ϕ-integral points of $C(\mathbb{Q})$ outside of U is finite.*

Proof. For n sufficiently large, we see from the Riemann-Roch theorem that there exists a non-constant $f \in \overline{\mathbb{Q}}(C)$ with $\operatorname{div} f \geq -nQ$. Since f is defined over a number field, the number of Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates of f is finite. Let g be their product. Then the poles of g are precisely $Q_1, \dots, Q_n$ and $g \in \mathbb{Q}(C)$. We claim that g is integral over $\mathbb{Q}[\phi]$. By commutative algebra, the integral closure of $\mathbb{Q}[\phi]$ in $\mathbb{Q}(C)$ is the intersection of all valuation rings of $\mathbb{Q}(C)$ containing $\mathbb{Q}[\phi]$. But each such valuation ring contains a valuation ring determined by a point of $C(\overline{\mathbb{Q}})$ that is not a pole of Y. Indeed, $\mathbb{Q}(C)$ is the fraction field of $\mathbb{Q}[X,\phi]/h(X,\phi)$ for some $h \in \mathbb{Q}[X,\phi]$ irreducible over $\overline{\mathbb{Q}}$. The non-poles of ϕ are then precisely the (affine) solutions of h. As the poles of g are a subset of the poles of Y, we find that g is a regular function of the affine curve determined by $\mathbb{Q}[X,\phi]/h(X,\phi)$, i.e. $g \in \mathbb{Q}[X,\phi]/h(X,\phi)$ and therefore it is in the valuation ring determined by each such affine point.

Now, multiplying g by a constant multiple we find that g is in fact integral over $\mathbb{Z}[\phi]$. In particular, if $P \in C(\mathbb{Q})$ is ϕ-integral, then $g(P)$ is integral over $\mathbb{Z}[\phi(P)] = \mathbb{Z}$. Since also $g(P) \in \mathbb{Q}$ we find that $g(P) \in \mathbb{Z}$. But g is continuous on the compact set $C(\mathbb{Q}) \setminus U$ and therefore bounded. So $g(C(\mathbb{Q}) \setminus U)$ is finite and therefore, so is $C(\mathbb{Q}) \setminus U$ since g has finite degree. ☐

A priori, the previous theorem is a concentration result: the integral points are concentrated near the Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit of Q. However, if ϕ has two (disjoint) Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$ orbits of poles, then the integral points cannot be concentrated near both orbits, so we find that the set of all ϕ-integral points is finite. To be precise, we should take the open sets around the two orbits to be disjoint.

Example 3.4. How does this relate to Example 3.2? First of all, note that the curve in the example is singular at infinity. Let $T = y/x^2$ and $S = 1/x$. This yields a birationally equivalent model

$$T^2 = 1 + aS + bS^2 + cS^3 + dS^4$$

where the points $Q_1 : S = 0, T = 1$ and $Q_2 : S = 0, T = -1$ are mapped onto the singular point at infinity. Here these two points are non-singular, and we see (x, y) in the original equation approaches these two points at infinity when $y \to \infty$ and $y \to -\infty$ respectively. The poles of the function x are precisely these two points at infinity, whereas $y - (x^2 + Xx + X)$ has only one pole Q_2 and $y + x^2 + Xx + X$ has one pole Q_1. So both of these can play the role of g in the above proof and therefore we indeed get finitely many x-integral solutions. We even obtain an explicit bound on the number of solutions because we have explicitly determined the Riemann-Roch functions g. Note that with this choice of $\phi = x$, we see that ϕ-integral points are precisely the integral points.

The crucial property of the curve in the above example is that its normalization has two points at infinity.

Let us now see what happens when we try to generalise the previous theorem to arbitrary number fields K and their rings of integers. The proof of the theorem then remains the same after we choose an embedding $\sigma : K \to \mathbb{C}$. However, there can be infinitely many $x \in \mathcal{O}_K$ with $\sigma(x)$ bounded. What we want is to bound the height of x, for which we need to bound $|x|_v$ for each place v. For the finite places, we have $|x|_v \leq 1$ when we assume $x \in \mathcal{O}_K$. For the infinite place, we need ϕ to have more orbits of poles. Suppose that s is the number of infinite places of K and ϕ has $\geq s + 1$ pole orbits. Let $g_1, \dots, g_{s+1}$ be the functions corresponding to the orbits via Riemann-Roch. For each embedding $\sigma : K \to \mathbb{C}$, we find open neighbourhoods U_i around the orbit of Q_i. For each $P \in C(K)$, there is one i (we assume we chose the U_i disjointly) such that $\sigma(P) \in U_i$. Then we have an upper bound for $\sigma(g_j(P))$ for all $j \neq i$. Since we have more than s functions $g_1, \dots, g_{s+1}$, we find that for each P there must be a j such that $\sigma(g_j(P))$ is bounded *for all* σ, as desired.

Another way to view this, is that for each point P we are looking for an orbit of the poles of Y that is far away from P in every norm. For each place v, a given point P can be v-adically close to only one orbit. If there is one, we remove it. After this process, we are left with at least one more orbit, say of Q_j, from which P is far away in every norm. Then the height $h(g_j(P))$ is bounded. From this, we see that we can even be more flexible.

Theorem 3.5 (Runge over number fields). *Suppose that C is a smooth irreducible curve over a number field K. Let S_K be any finite set of places containing all infinite places and $\phi \in \mathbb{C}(K)$ such that it's poles consist of r Galois orbits. If the* Runge condition

$$S_K < r$$

holds, the number of S_K, ϕ-integral points is finite.

We continue to increase generality. Suppose now instead that we start with a normal projective variety X over K of dimension possibly greater than 1. The zeros of a function are now not points, but codimension 1 subvarieties. Instead of a function, we start with the data of a number of non-Galois conjugate effective Weil divisors $D_1, \ldots, D_r$. There are now two adjustments that need to be made with respect to the 1-dimensional case.

Firstly, each of these divisors yields functions in the Riemann-Roch spaces $\mathcal{L}(nD_i)$ for large n. However, in order to be able to get a bound on the Weil height of integral points, we need a bound on all the "coordinates". If each D_i is ample, the functions in $\mathcal{L}(nD_i)$ for n sufficiently large embed X into projective space. After the embedding, these functions are just the coordinate maps and thus bounds on $\phi(P)$ for $\phi \in \mathcal{L}(D_i)$ would suffice for bounding the Weil height.

Secondly, the Galois orbits of these divisors are no longer disjoint, so each point P can be v-adically close to many such orbits. So we let m be the maximal size of a subset of $\{D_1, \ldots, D_r\}$ with non-empty common intersection. Then each point P can be close to at most m divisors D_i, so instead of the condition $S_K < r$, we need

$$mS_K < r.$$

Let $D = \sum D_i$. To describe the result, we need to say what D-integral points are. Let S_K be a finite set of places of K. We say that $R \subset X \setminus D$ is a *set of D, S_K-integral points*, when for every $f \in K(X \setminus D)$ there is a constant $c_f \in K^*$ such that $c_f f(P) \in \mathcal{O}_{K,S_K}$ for all $P \in R$ ("bounded denominators").

Theorem 3.6 (Levin [14]). *With the definitions as above, if the Runge condition $mS_K < r$ holds then every set of D, S_K-integral points is finite.*

In fact Levin's theorem is stronger than this: the union of sets of D, S_L-integral points over number fields $L \supset K$ and finite sets S_L of places of L satisfying the Runge condition is finite (see [14] for the precise statement).

3.2. **Application to abelian varieties.** The main drawback of Runge's method is the Runge condition, which is rather strong. Samuel Le Fourn [3] set out to make this condition more flexible, as follows. If we already know that the integral points we are looking for lie in a certain region of X where less of the D_i intersect, then the value of m in the Runge condition may be lowered. See Theorem 1 in [3] for the precise statement. Following Le Fourn, we refer to this improved Runge method as the *tubular Runge method*. This method is particularly useful when dealing with moduli spaces, as Le Fourn illustrated on the Siegel modular variety. We describe his result below.

The Siegel modular variety $A_2(2)$ over $\mathbb{C}$ is the moduli space of principally polarized abelian varieties with a symplectic basis for their 2-torsion (with respect to the Weil pairing). This has a *Satake compactification* $A_2(2)^S$, whose boundary $A_2(2)^S \setminus A_2(2)$ has a moduli-interpretation of non-smooth objects. The compactification $A_2(2)^S$ has an integral model $\mathcal{A}_2(2)^S$ over $\operatorname{Spec}\mathbb{Z}[1/2]$ containing as open dense subscheme the moduli scheme $\mathcal{A}_2(2)$ parametrizing principally polarized abelian varieties with level 2 structure over $\operatorname{Spec}\mathbb{Z}[1/2]$-schemes.

Therefore if K is a number field and v a finite place not above 2, a triple $(A, \lambda, \alpha_2) \in \mathcal{A}_2(2)(K)$ is v-adically far from the boundary if A has potentially good reduction at v.

The divisors used by Le Fourn on $\mathcal{A}_2(2)$ are theta divisors, to be defined later, whose union consists of the triples (A, λ, α_2), where (A, λ) is a product of elliptic curves (with the product principal polarization). There are 10 such divisors and they are disjoint on $A_2(2)$, but on the boundary there is a common intersection of at most 6 of them. A triple $(A, \lambda, \alpha_2) \in \mathcal{A}_2(2)(K)$ is *integral* at a place v with respect to these divisors when A has potentially good reduction at v and this reduction is not a product of elliptic curves. Informally, this means (A, λ, α_2) is v-adically far away from the divisors. As the integral triples have everywhere potentially good reduction, we know they are v-adically far away from the boundary at the finite places v, and hence far away from the common intersection of 6 divisors. This idea allowed Le Fourn to weaken his Runge condition and prove his theorem, which we restate here (see Theorem 4 in [3]).

Theorem 3.7 (Le Fourn). *Suppose that K is $\mathbb{Q}$ or an imaginary quadratic number field. Consider a principally polarized abelian surface (A, λ) such that (A, λ) and its 2-torsion are defined over K, and having potentially good reduction at all finite places of K. If the semi-stable reduction of (A, λ) is a product of elliptic curves at most at 3 finite places, then*

$$h_F(A) \leq 1070,$$

where h_F denotes the Faltings height. In particular, there are finitely many such abelian varieties.

Remark 3.8. The number 3 in this theorem is the largest integer smaller than $10 - 6$. The number 10 is the number of divisors and 6 is the common intersection number at the boundary for the one infinite place. By Le Fourn's "tubular" approach, we are then allowed to add 3 non-integral finite places because the divisors are disjoint away from the boundary. In particular, without this tubular idea we would have obtained the same theorem with the stronger condition that the semi-stable reduction is *never* a product of elliptic curves.

In the remainder of this subsection we sketch Le Fourn's proof more finely. Recall that the principally polarized abelian surfaces over $\mathbb{C}$ can be parametrized by a quotient of the Siegel upper half space $\mathfrak{S}_2$. For each $\tau \in \mathfrak{S}_2$, we have $A_\tau = \mathbb{C}^2/\mathbb{Z}^2 \oplus \tau\mathbb{Z}^2$, $H_\tau(z, w) = z^t(\mathrm{Im}(\tau)^{-1}\overline{w}$ and

$$\theta_\tau(z) = \sum_{n \in \mathbb{Z}^2} e^{\pi i n^t \tau n + 2\pi i n^t z}.$$

If we decide to let τ vary instead of z, this becomes a function on $\mathfrak{S}_2$. Define $\theta(\tau) := \theta_\tau(0)$ and for each $a, b \in \frac{1}{2}\mathbb{Z}^2$ set $\theta_{a,b}(\tau) := e^{-i\pi a^t b}\theta_\tau(a\tau + b)$. Let $\Gamma_2(2)$ be the subgroup of $\mathrm{Sp}_4(\mathbb{Z})$ consisting of those matrices congruent to I_4 modulo 2. We have seen that $\mathrm{Sp}_4(\mathbb{Z}) \setminus \mathfrak{S}_2$ is the moduli space of principally polarized abelian varieties over $\mathbb{C}$. The reader familiar with modular curves will not be surprised that $\Gamma_2(2) \setminus \mathfrak{S}_2$ is precisely $A_2(2)$.

Proposition 3.9. *For each $a, b \in \frac{1}{2}\mathbb{Z}^2$, $\theta_{a,b}$ is independent of the values of a, b modulo $\mathbb{Z}^2$ and moreover, $\theta_{a,b}^{16}$ is a Siegel modular form of degree g, level 2 and weight 8.*

See Definition-Proposition 6.14 in [3] for a proof. I will not define a Siegel modular form here, but, as expected, this means the $\theta_{a,b}^{16}$ satisfy a transformation property with respect to the action of $\Gamma_2(2)$. Moreover, since the Siegel modular forms together define an embedding into projective space, the line bundle of Siegel modular forms of degree g, level 2 and weight 8, is ample. Note that we now have 16 theta functions $\theta_{a,b}$ in total.

Proposition 3.10. *The theta function $\theta_{a,b}$ for $a, b \in \frac{1}{2}\mathbb{Z}^2/\mathbb{Z}^2$ vanishes identically if and only if $4a^t b = 1$, i.e. when (a, b) is one of*

$$\left(\frac{1}{2}, 0, \frac{1}{2}, 0\right), \left(0, \frac{1}{2}, 0\frac{1}{2}\right), \left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right).$$

Moreover, no quotient of two of the ten non-vanishing $\theta_{a,b}$s is constant.

See Definition-Proposition 7.7 of [3] for a proof by computing the Fourier coefficients. We call the six pairs above *singular*; the ten remaining ones are *regular*. Because the theta functions $\theta_{a,b}$ are Siegel modular forms, they have a well-defined divisor of zeros $\Theta_{a,b}$ on $A_2(2)^S$ when (a, b) is regular. As the line bundle of Siegel modular forms is ample, so are these 10 theta divisors.

In order to study these divisors on the moduli space, note that $\theta_{a,b}(\tau) = 0$ if and only if $\theta_\tau(a\tau + b) = 0$, so the union of the ten theta divisors restricted to the non-compactified $A_2(2)$ consists of those τ such that θ_τ vanishes at a 2-torsion point of A_τ. Le Fourn used this to show that the theta divisors are disjoint on $A_2(2)$ and that their union are those triples $(A_\tau, H_\tau, \alpha_{2,\tau})$ such that (A_τ, H_τ) is a product of elliptic curves.

We need more, however, because all of this a priori only holds over the field of complex numbers. In order to deal with the semi-stable reduction of abelian varieties we need a similar description of these divisors over fields of arbitrary characteristic. Since we have a Siegel moduli *scheme* $\mathcal{A}_2(2)$ over $\mathrm{Spec}\mathbb{Z}[1/2]$, we may hope to generalise the theta functions to arbitrary fields of characteristic unequal to 2.

Let k be any field of characteristic unequal to 2. Instead, the right objects to generalise are their divisors $\Theta_{a,b}$. For any triple (A, λ, α_2), we are given a principal polarization λ. This is induced by a line bundle L whose global sections are 1-dimensional because λ is principal. So given L, there is a unique effective divisor in its linear equivalence class. However, there are multiple ample symmetric line bundles inducing λ. In fact, they are precisely

$$T_x^*L \text{ for } x \in A[2].$$

So in order to canonically choose a divisor on A, it suffices to make a canonical choice for L inducing λ.

Proposition 3.11 (Igusa correspondence)**.** *Suppose that (A,λ) is a principally polarized abelian variety over k. Then each choice of symplectic $\mathbb{F}_2$-basis for $A[2]$ determines canonically a line bundle L inducing λ.*

This was first proved by Igusa [13]; see the next section for a survey of Igusa's proof. Igusa correspondence is the reason we added the symplectic basis for the 2-torsion to the moduli space. Over $\mathbb{C}$, it turns out that the line bundle associated to the standard basis $\alpha_{2,\tau} = (\frac{1}{2}, \frac{1}{2}\tau)$ by Igusa correspondence is indeed $\mathrm{div}\theta_\tau$. This correspondence allows us to define the theta divisors $\Theta_{a,b}(k)$ over arbitrary fields of characteristic unequal to 2, as follows.

Definition 3.12. The *theta divisor* $\Theta_{a,b}(k)$ (for (a,b) regular) on $\mathcal{A}_2(2)^S(k)$ is the divisor whose restriction to $\mathcal{A}_2(2)(k)$ consists of those triples $(A,\lambda,\alpha_2) \in \mathcal{A}_2(k)$ such that the unique effective divisor $\Theta_{A,\lambda,\alpha_2}$ on A associated to α_2 by Igusa correspondence contains the 2-torsion point of α_2-coordinates (a,b).

Of course it needs verification that this is a well-defined divisor; see Definition 7.10 in [3].

Again, the union of the ten theta divisors $\Theta_{a,b}(k)$ on $\mathcal{A}_2(2)(k)$ is the set of products of elliptic curves, which should explain the nature of the condition in Theorem 3.7. In characteristic 2 one does not simply have a moduli space and more effort is needed; see Section 8.3 of [3].

All of the above explains how Le Fourn could obtain a finiteness result, but to obtain an explicit upper bound on the Faltings height, we need a witness for the ampleness of the theta divisors: Van der Geer [6] showed that the map

$$\psi : A_2(2) \to \mathbb{P}^9,\ \tau \mapsto (\theta_{a,b}(\tau)^4)_{(a,b) \text{ regular}}$$

is an embedding such that $A_2(2)^S$ is identified with the closure of the image, for which he computed explicit equations. This map allowed Le Fourn to obtain his explicit bound; see [3] for more details.

3.3. **Igusa correspondence.**

In this Section we prove Proposition 3.11, following Igusa's original proof in [13].

3.3.1. *Line bundles as 1-cocycles.* Let $\ell \in 2\mathbb{Z}$ be an even integer and (A,λ) a principally polarized abelian variety. Then λ equals the map

$$x \mapsto T_x^*L \otimes L^{-1}$$

for some ample symmetric line bundle L. However, there can be multiple such line bundles inducing λ, and these line bundles are precisely

$$T_x^*L \text{ such that } x \in A[2].$$

Here x needs to be in $A[2]$ to make the line bundle symmetric. Choose such an L. Then $h^0(L,A) = (\deg\lambda)^2 = 1$, so there is a unique effective divisor X whose associated line bundle is L, and X is also symmetric. We then define for each $u \in A[\ell]$ a function ψ_u on A by the equality

$$\mathrm{div}(\psi_u) = (\ell \cdot \mathrm{Id})^*(T_u^*X - X).$$

Note that T_u^*L and L are isomorphic line bundles, hence such a function exists. Of course ψ_u is only defined up to a constant.

Because we defined ψ_u via the pull-back of multiplication by ℓ, the divisor $\mathrm{div}(\psi_u)$ is invariant under translation by elements of $A[\ell]$. In particular, for each $v \in A[\ell]$, we have

$$\psi_u(z+v) = e_\ell(u,v)\psi_u(z)$$

for some $e_\ell(u,v) \in \mu_\ell$. This defines the *Weil pairing* e_ℓ on $A[\ell]$. It is a non-degenerate, alternating, multiplicative bilinear form. Here alternating means that it is trivial on the diagonal.

From now on, we consider only the case $\ell = 2$ and we ease notation by setting $b := e_2$. Then note that $\mathrm{div}(\psi_u)$ is symmetric for each $u \in A[2]$ because X is symmetric. Hence there exists a $c_X(u) \in \{\pm 1\}$ such that

$$\psi_u(-z) = c_X(u)\psi_u(z).$$

If $r' \in A[4]$ is such that $2r' = r$, then one check that

$$\psi_{r+s}(z) = C \cdot \psi_r(z)\psi_s(z - r').$$

Using both this identity and the same identity with z replaced by $r - z$, we can use the definition of c_X to deduce that

$$b(r, s)c_X(r + s) = c_X(r)c_X(s). \tag{2}$$

So c_X is not a homomorphism, but we can view it as a 1-cocycle for group cohomology on $A[2]$ and the above identity means that c_X is mapped to b by the boundary map. So let T be the set of all 1-cocycles mapped to b by the boundary map, i.e. satisfying the above identity.

Now $A[2]$ acts on T via $t \cdot c_X = c_{T_t^* X}$. We would like to express the right-hand side in terms of c_X. To that end, we consider $\psi_u(z - t')$ for $t' \in A[4]$, which has divisor $(2 * \mathrm{Id})^*(T_{t+u}^* X - T_t^* X)$. Then

$$\psi_u(-z - t') = c_X(u)\psi_u(z + t') = c_X(u)b(r, t)\psi_u(z - t'),$$

from which we find

$$c_{T_t^* X}(u) = b(u, t)c_X(u).$$

In particular, since b is non-degenerate, the map $X \mapsto c_X$ is injective. Also, each element of T is determined by its values on $2g$ generators due to (2), so $|T| = 2^{2g}$. On the other hand, the line bundles inducing λ are $T_u^* L$ for $u \in A[2]$ and each line bundle determines a unique divisor, so we also have 2^{2g} such divisors X. Therefore we may identify T with these divisors, and the (transitive) action of $A[2]$ on T is then given by $t \cdot X = T_t^* X$. This re-interpretation of the line bundles is the key fact from which the Igusa correspondence follows by abstract non-sense.

3.3.2. *Symplectic bases and 1-cocycles.* In this subsection we use the notation from the previous subsection, but we note that we can work in greater generality. So define $A[2]$ to be any abelian group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2g}$, let $b : A[2] \times A[2] \to \{\pm 1\}$ be a non-degenerate, alternating, bilinear multiplicative pairing. Now b, considered as a function on $B[2] \times B[2]$, can be viewed as a μ_2-valued 2-cocycle for group cohomology. Since $A[2]$ is a free group, its 2nd cohomology group vanishes, so b is the image of some 1-cochain $c_X : A[2] \to \mu_2$ under the boundary map. Define T to be the set of all 1-cocyles $c_X : A[2] \to \{\pm 1\}$ mapping to b, i.e. satisfying (2). Again $A[2]$ acts transitively on T as follows: for $s \in B[2]$ and $c_X \in T$ we set

$$s \cdot c_X(r) = b(r, s)c_X(r).$$

We denote the action of $A[2]$ on T by $s \cdot c_X := s + c_X$. Now T is not quite a group (there is not even an identity element), but T does admit a notion of difference: for $c_X, c_Y \in T$, we say $c_X - c_Y = s$ when $c_Y + s = c_X$. On the disjoint union $A[2] \sqcup T$ we thus obtain a group law, given on T by $c_X + c_Y := c_X - c_Y$. We denote this group by $\mathcal{X}[2]$.

Lemma 3.13. *For each $r \in A[2]$ and $c_X \in T$ the following identities hold:*

$$\left(\sum_{r \in A[2]} c_X(r)\right)^2 = 2^{2g} \quad \textit{and}$$

$$\sum_{c_X \in T} c_X(r) = \begin{cases} 0 & \textit{if } r \neq 0 \textit{ and} \\ 2^{2g} & \textit{if } r = 0. \end{cases}$$

Proof. We first prove the second identity. It follows from the coboundary-identity that $c_X(0) = 1$ for all X, so now assume $r \neq 0$. We choose $c_X \in T$ and rewrite the sum:

$$\sum_{c_X \in T} c_X(r) = \sum_{s \in B[2]} c_{X+s}(r) = \sum_{s \in B[2]} b(s, r)c_X(r) = c_X(r) \sum_{s \in B[2]} b(s, r).$$

So it suffices to show that $\sum_{b \in A[2]} b(s, r) = 0$, which is easily seen by multiplying the sum with $b(s', r) = -1$.

For the first identity, we compute the square to obtain

$$\sum_{r,s} c_X(r)c_X(s).$$

We substitute $c_X(r)c_X(s) = b(r, s)c_X(r + s)$ in this term and sum over $p = r + s$ and r instead to obtain

$$\sum_p \sum_r b(r, p)c_X(p) = \sum_p \sum_r c_{X+r}(p) = 2^{2g}$$

by the second identity. □

For $c_X \in T$, we now define $e(c_X)$ to be the sign of $\sum_r c_X(r)$. This gives rise to a notion of *even* and *odd*. In fact, the c_Xs are determined by these signs, in the following way:

$$e(c_X)e(c_Y) = \frac{1}{2^{2g}} \sum_{r,t} c_X(r+t)c_Y(r) = \frac{1}{2^{2g}} \sum_r \sum_t b(t, r+X-Y)c_X(r) = c_X(X+Y).$$

We define a *symplectic basis* for $A[2]$ to be an $\mathbb{F}_2$-basis such that b is given by a matrix with only -1 on the diagonal and 1 everywhere else. Choose such a symplectic basis $\mathcal{B}$. This yields an isomorphism $A[2] \simeq \mathbb{F}_2^{2g}$. For $u \in A[2]$, we denote its coordinates in $\mathbb{F}_2^{2g}$ by $m = (m', m'')$, where $m', m'' \in \mathbb{F}_2^g$. Now suppose we choose a $c_Z \in T$. This yields a "projection along c_Z" map

$$T \to A[2] \to \mathbb{F}_2^{2g}, \;\; c_Z + u \mapsto u \mapsto m$$

that allows us to assign coordinates to all elements of T.

Lemma 3.14. *The choice of symplectic basis $\mathcal{B}$ yields a unique* even $c_Z \in T$ *such that*

$$e(c_X) = (-1)^{m'^t m''},$$

where $c_X \in T$ has coordinates $m = (m', m'') \in \mathbb{F}_2^{2g}$ under the projection along c_Z.

Proof. First choose any even $c_Z \in T$. Consider any $c_X \in T$ and write it as $c_X = c_Z + r$ for some $r \in A[2]$ with coordinates $m = (m', m'')$. The key to prove this lemma is to realise that the map

$$d: \; r \mapsto (-1)^{m'^t m''}$$

is an element of T. Indeed, the identity (2) follows from the fact that

$$b(r, s) = (-1)^{m'^t n'' - m''^t n'}$$

if s has coordinates $n = (n', n'')$, which is easily verified. Then since c_Z is even, we have

$$e(c_X) = e(c_X)e(c_Z) = c_Z(r).$$

By transitivity of the action of $A[2]$ on T, there exists $s \in A[2]$ such that

$$e(c_X) = c_Z(r) = b(r, s)d(r) = (-1)^{m'^t n'' - m''^t n' + m'^t m''}.$$

To show that there is a unique choice of even c_Z corresponding $n = 0$, we must verify which $n \in \mathbb{F}_2^{2g}$ correspond to even $c_Z \in T$. From the previous lemma, one deduces that

$$\sum_{c_Y} e(c_Y) = 2^g,$$

while by counting (or because it is an element of T), we also have $\sum_m (-1)^{m'^t m''} = 2^g$. But

$$\sum_{c_Y} e(c_Y) = \sum_m (-1)^{(m+n)'^t (m+n)'' - n'^t n''} = (-1)^{n'^t n''} \sum_m (-1)^{m' m''},$$

so we must have $n'^t n'' = 0$. Now because $e(c_X) = c_Z(r)$, the map $c_Z \mapsto n$ is injective. But two similar induction arguments show that both the number of even elements of T and the number of $n \in \mathbb{F}_2^{2g}$ with $n'^t n'' = 0$ equal $2^{g-1}(2^g+1)$, so the map is a bijection. We conclude that there is a unique $c_Z \in T$ corresponding to $n = 0$. □

We say that this symplectic basis $\mathcal{B}$ and the line bundle Z are *related by Igusa correspondence*. This finishes the proof of Proposition 3.11.

3.4. **Symmetry of the theta divisors.** Let k be an algebraically closed field of characteristic unequal to 2. Recall that we were originally only interested in principally polarized abelian surfaces, but we had to add a level 2 structure in order to define the theta divisors over k using Igusa correspondence. We thus have a group action of $\mathrm{Sp}_4(\mathbb{F}_2)$ on $\mathcal{A}_2(2)(k)$ given by

$$\gamma(A, \lambda, \alpha_2) = (A, \lambda, \gamma\alpha_2) \text{ for } \gamma \in \mathrm{Sp}_4(\mathbb{F}_2).$$

If we understand this group action, we may be able to exploit this symmetry and our freedom of choosing the level 2 structure we fancy. When $k = \mathbb{C}$, we have $\mathcal{A}_2(2)(\mathbb{C}) = \Gamma_2(2) \setminus \mathfrak{S}_2$ which comes with an action of

$$\mathrm{Sp}_4(\mathbb{Z})/\Gamma_2(2) = \mathrm{Sp}_4(\mathbb{F}_2).$$

It turns out these two actions for $k = \mathbb{C}$ are related via $\gamma \mapsto \gamma^{-1}$.

The purpose of this section is to describe the action of $\mathrm{Sp}_4(\mathbb{F}_2)$ on the theta divisors $\Theta_{a,b}(k)$.

Proposition 3.15. *Suppose that $(a,b) \in (\mathbb{Z}/2\mathbb{Z})^4$ is a regular pair. For each $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{F}_2)$ we have*

$$\gamma\left(\Theta_{a,b}\right) = \Theta_{[(a,b)-(n',n'')]\gamma^t},$$

where n' and n'' are the pairs of diagonal entries of B^tD and A^tC respectively. This action is transitive on the set of the ten regular theta divisors, and

$$\mathrm{Stab}_{\mathrm{Sp}_4(\mathbb{F}_2)}(\Theta_{0000}) = \Gamma_{1,2} := \{\gamma \in \mathrm{Sp}_4(\mathbb{F}_2) \mid Q((x,y)\gamma^t) = Q(x,y)\},$$

where $Q(x,y) = x^ty$ for $x, y \in (\mathbb{Z}/2\mathbb{Z})^2$.

Remark 3.16. The action of $\mathrm{Sp}_4(\mathbb{F}_2)$ should preserve the regular/singular distinction and hence the form Q. So $(n', n'') = 0$ (i.e. γ stabilizes Θ_{0000}) should indeed only happen when γ leaves Q invariant.

Proof. Suppose that $\gamma \in \mathrm{Sp}_4(\mathbb{F}_2)$ transforms α_2 into α_2'. We assume that $(a,b) \in \Theta_{A,\lambda,\alpha_2}$, where (a,b) denote the α_2-coordinates of an element in $A[2]$. Then for some $u = u(\gamma, \alpha_2) \in A[2]$, we have by Proposition 6.12 of [3] that

$$\Theta_{A,\lambda,\alpha_2'} = T_u^*\Theta_{A,\lambda,\alpha_2}.$$

We compute the α_2-coordinates of u using Igusa correspondence: $\Theta = \Theta_{A,\lambda,\alpha_2}$ satisfies by definition

$$c_\Theta(r) = (-1)^{m'^tm''} \quad (r \in A[2]),$$

where $r = (m', m'')$ in α_2-coordinates. Write $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Similarly, $\Theta' = \Theta_{A,\lambda,\alpha_2'}$ satisfies

$$c_{\Theta'}(r) = (-1)^{\left(\begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} m' \\ m'' \end{pmatrix}\right)^t \begin{pmatrix} C & D \end{pmatrix}\begin{pmatrix} m' \\ m'' \end{pmatrix}}.$$

Using the that $A^tD - C^tB = I$, we find that the exponent of (-1) is equivalent to

$$m'^tA^tCm' + m''^tB^tDm'' + m'^tm'' \bmod 2.$$

Lastly, using that A^tC and B^tD are symmetric, this is seen to equal

$$m'^t\begin{pmatrix} (A^tC)_{11} \\ (A^tC)_{22} \end{pmatrix} + m''^t\begin{pmatrix} (B^tD)_{11} \\ (B^tD)_{22} \end{pmatrix} + m'^tm'' \bmod 2.$$

In particular, we have shown that

$$c_{\Theta'}(r) = b(r,u)c_\Theta(r),$$

where $u \in A[2]$ has α_2-coordinates $(n', n'') := ((B^tD)_{11}, (B^tD)_{22}, (A^tC)_{11}, (A^tC)_{22})$. This means that $\Theta' = T_u^*\Theta$.

So given that $(a,b) \in \Theta$, we find, now in α_2'-coordinates, that

$$\left((a,b) - (n', n'')\right)\gamma^t \in \Theta'.$$

As $\mathrm{Sp}_4(\mathbb{F}_2)$ is finite, one can now compute this action explicitly and verify the transitivity.

As mentioned in the remark, it is clear that the group fixing Θ_{0000} is contained in $\Gamma_{1,2}$. The converse can be proved by finding an explicit set of generators for $\Gamma_{1,2}$, see the appendix to Section II.5 of [15]. □

This agrees with the functional equation for these theta functions in the complex case $k = \mathbb{C}$, which says that for all $\gamma \in \Gamma_{1,2}$ we have

$$\theta_{(a,b)}(\gamma(\tau)) = \zeta \det(C\tau + D)\theta_{(a,b)\gamma^t},$$

where ζ is a root of unity depending only on a, b and γ. See Section II.5 of [15] for a proof. In fact, as the Proposition suggests, there is indeed an analogous transformation formula for theta functions with respect to the entire symplectic group $\mathrm{Sp}_4(\mathbb{Z})$, see page 226 of [12].

3.5. **When principally polarized abelian surfaces with real multiplication are a product of elliptic curves.** When commencing this project at the start of Term 2, the aim was to try and (im)prove Le Fourn's theorem for Hilbert modular surfaces. Indeed, theta functions can be generalised to the Hilbert setting, though there appears to be less knowledge about theta functions over real quadratic fields. The main obstruction we found, however, is that Igusa correspondence does not appear to generalise to abelian varieties with an $\mathcal{O}_F/2\mathcal{O}_F$-basis for their 2-torsion. Essentially, the problem is that an $\mathcal{O}_F/2\mathcal{O}_F$-basis for the 2-torsion gives much less information than a $\mathbb{Z}/2\mathbb{Z}$-basis and hence does not allow us to canonically choose a line bundle.

This is why we decided to look at abelian varieties with real multiplication inside the Siegel modular variety. An additional benefit of this is that possible results may be uniform over all real quadratic fields. We initially hoped that the $\mathcal{O}_F$-multiplication would impose a condition on the theta divisors, e.g. that some theta divisors never contained any triples (A, λ, α_2) such that (A, λ) has $\mathcal{O}_F$-multiplication, but by the symmetry of the previous Section that cannot be the case. On the other hand, a principally polarized abelian variety with real multiplication cannot be a product of any two elliptic curves.

Proposition 3.17. *Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic number field and suppose that (A, λ) is a principally polarized abelian surface over an algebraically closed field k with real multiplication by $\mathcal{O}_F$. If (A, λ) is isomorphic to a product of elliptic curves $E_1 \times E_2$ (with product principal polarization), then E_1 and E_2 are isogenous. Moreover, if $\mathrm{End}E_1 = \mathbb{Z}$ then D is of the form $D = a^2 + eb^2$, where e is the degree of an isogeny between E_1 and E_2 and $a, b \in \mathbb{Z}$.*

Proof. If an abelian surface has real multiplication, then it is either simple, or isogenous to a product of an elliptic curve with itself. The reason is that, when E is an elliptic curve, $\mathrm{End}^0(E) := \mathrm{End}E \otimes \mathbb{Q}$ does not contain a real quadratic field. When E_1 is not isogenous to E_2, we have $\mathrm{End}^0(E_1 \times E_2) \simeq \mathrm{End}^0(E_1) \times \mathrm{End}^0(E_2)$, which hence does not contain a real quadratic field either. So suppose that $A = E_1 \times E_2$, where $\phi : E_1 \to E_2$ is an isogeny of degree e. Then $\phi \times \mathrm{Id} : E_1 \times E_2 \to E_2 \times E_2$ yields an isomorphism

$$\mathrm{End}^0(A) \simeq \mathrm{End}^0(E_2 \times E_2) = \mathrm{Mat}_{2\times 2}(\mathrm{End}^0(E_2)).$$

This means $\sqrt{D} \in \mathcal{O}_F$ is mapped to a matrix

$$\alpha = \begin{pmatrix} a & b \circ \widehat{\phi} \\ \phi \circ c & d \end{pmatrix} \text{ with } a, b, c, d \in \mathrm{End}(E),$$

where a hat denotes the dual isogeny. Solving the equation $\alpha^2 = D \cdot \mathrm{I}_2$ we find $a = -d$ and $a^2 + bce = D$ (using that $D \neq a^2$ as D is positive and square-free and a is either in $\mathbb{Z}$, an order in an imaginary quadratic field or a quaternion algebra). We now compute implications of the equality $\alpha = \lambda^{-1} \circ \alpha^* \circ \lambda$, where $\lambda = \lambda_1 \times \lambda_2$ is the product principle polarization given by

$$\lambda : \ (P, Q) \mapsto (O_1) \times E_2 - (P) \times E_2 + E_1 \times (O_1) - (Q) \times E_2.$$

Note that λ is indeed surjective because $\mathrm{Pic}^0(E_1 \times E_2) \simeq \mathrm{Pic}^0(E_1) \times \mathrm{Pic}^0(E_2)$. We write $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and $\alpha_2 = \begin{pmatrix} 0 & b \circ \widehat{\phi} \\ \phi \circ c & 0 \end{pmatrix}$ and we compute that

$$\begin{aligned} \lambda^{-1} \circ \alpha_1^*(O_1) \times E_2 - \lambda^{-1} \circ \alpha_1^*(P) \times E_2 &= \lambda^{-1} \sum_{R \in \ker a} (R) \times E_2 \text{ and } - (\widetilde{P} + R) \times E_2 \\ &= \lambda^{-1} \circ \deg a \left[(O_1) \times E_2 - (\widetilde{P}) \times E_2 \right] \\ &= \lambda^{-1} \left[(O_1) \times E_2 - (\widehat{a}P) \times E_2 \right] = (\widehat{a}P, O_2), \end{aligned}$$

where $a\widetilde{P} = P$, and similarly

$$\lambda^{-1} \circ \alpha_2^* \left[(O_1) \times E_2 - (P) \times E_2 \right] = \lambda^{-1}(E_1 \times (O_2) - E_1 \times (\phi \circ \widehat{c})) = (O_1, \widehat{b} \circ \widehat{\phi}(P)).$$

The analogous equations hold for $\alpha_i^*[E_1 \times (O_2) - E_1 \times (Q)$ and we obtain

$$\lambda^{-1} \circ \alpha^* \circ \lambda(P, Q) = (\widehat{a}P + \widehat{c} \circ \widehat{\phi}Q, \widehat{b} \circ \phi P + \widehat{d}Q).$$

In other words, the equation $\alpha = \lambda^{-1}\alpha^*\lambda$ comes down to

$$\begin{pmatrix} a & b \circ \widehat{\phi} \\ \phi \circ c & d \end{pmatrix} = \begin{pmatrix} \widehat{a} & \widehat{c} \circ \widehat{\phi} \\ \phi \circ \widehat{b} & \widehat{d} \end{pmatrix},$$

so $a = \widehat{a}$, $d = \widehat{d}$ and $\widehat{b} = c$. In particular, if E_1, E_2 have endomorphism ring $\mathbb{Z}$, then $D = a^2 + eb^2$. $\square$

The above proof also shows us how to construct products of elliptic curves with real multiplication. Given D, we find a $u \in \mathbb{Z}_{>0}$ and an elliptic curve E with complex multiplication by $\mathbb{Q}(\sqrt{-u})$ such that $D = a^2 + (x^2 + uy^2)$ for some $x, y \in \mathbb{Z}$. Then $E \times E$ has real multiplication by the ring of integers of $\mathbb{Q}(\sqrt{D})$ by mapping $\sqrt{D}$ to the matrix

$$\begin{pmatrix} a & x + uy \\ x - uy & -a \end{pmatrix} \in \mathrm{End}(E \times E).$$

Another way is to simply find two D-isogenous elliptic curves. This is possible since there is no restriction on the field the isogeny should be defined over.

So it is possible for triples (A, λ, α_2), where (A, λ) has real multiplication by $\mathcal{O}_F$ for any real quadratic F, to be in one of the theta divisors. In fact, by the previous Section, such triples are in every theta divisor. There are restrictions on which products of elliptic curves can occur, but we have not managed to link this to the geometry of the theta divisors. Moreover, the author does not expect a result such as "all products of elliptic curves with real multiplication have v-adic distance at least x to the boundary of $A_2(2)^S$" to hold for any place v, as the j-invariant of a CM elliptic curve can have arbitrarily large v-valuation.

Appendix A. Magma code for computing equations for birational models of Hilbert modular surfaces

```
RemoveSquares:=function(d)
Fact:=Factorization(d);
for i in Factorization(d) do
        if i[2] gt 1 then
            if IsOdd(i[2]) then d:=d/i[1]^(i[2]-1);
            else d:=d/i[1]^(i[2]);
            end if;
        end if;
end for;
return d;
end function;

disc:=function(d)
d:=IntegerRing()!RemoveSquares(d);
if d mod 4 eq 1 mod 4 then return d;
else return 4*d;
end if;
end function;

RemoveDoubles:=function(List)
newList:=[];
for s in List do
    if not s in newList then
    Append(~newList,s);
    end if;
end for;
return newList;
end function;

ZBasis:=function(F,OF,y,d,I) /*This computes a Z-module basis 1,w0 of an ideal */
gens:=Generators(I);/*strictly equivalent to I such that w0_1>2 and 0<w0_2<1 */
A<[r]> := AbelianGroup([0,0]);
denom:=1;
for g in gens do
    denom:=denom*Denominator(g[1])*Denominator(g[2]);
end for;
groupgens:=[];
Z:=IntegerRing(); Q:=RationalField();
```

```
for g in gens do
    if disc(d) mod 4 ne 1 then
        Append(~groupgens,(Z!(denom*g[1]))*r[1]+(Z!(denom*g[2]))*r[2]);
        Append(~groupgens,d*(Z!(denom*g[2]))*r[1]+(Z!(denom*g[1]))*r[2]);
    else Append(~groupgens,(Z!(denom*g[1]))*r[1]+(Z!(denom*g[2]))*r[2]);
        Append(~groupgens,(Z!((d-1)/4))*(Z!(denom*g[2]))*r[1]+
        (Z!(denom*g[1])+Z!(denom*g[2]))*r[2]);
    end if;
end for;
B<t>:=sub<A|groupgens>;
grpgens:=Generators(B);
basis:=[(OF!Eltseq(A!g))/denom : g in grpgens];
if Floor(Conjugate(basis[1],1)*Conjugate(basis[1],2)
-Conjugate(basis[1],2)*Conjugate(basis[2],1)) gt 0 then
    Reverse(~basis);
end if;
G:=Automorphisms(F);
a:=Q!Norm(basis[1]); c:=Q!Norm(basis[2]);
b:=Q!(G[2](basis[1])*basis[2]+basis[1]*G[2](basis[2]));
den:=Denominator(a)*Denominator(b)*Denominator(c);
a:=Z!(a*den); b:=Z!(b*den); c:=Z!(c*den);
NM:=Gcd([a,b,c]);
a:=a/NM; b:=b/NM; c:=c/NM;
Delta:=b^2-4*a*c;
sqrtDelta:=(Z!SquareRoot(Delta/d))*y;
w0:=(b+sqrtDelta)/(2*c);
if Floor(Conjugate(w0,1)) lt 0 then w0:=-w0; end if;
w0:= w0-Floor(Conjugate(w0,2));
if Floor(Conjugate(w0,1)) ge 2 then return w0;
else return "error";
end if;
end function;


ResolutionCycle:=function(F,OF,w0,d)
Z:=IntegerRing();
/*Complexy:=Conjugates(y)[1];
if (Z!d) mod 4 ne 1 mod 4 then
w0:=Ceiling(Complexy)+y;
else if IsOdd(Ceiling(Complexy)) then w0:=Ceiling(Complexy)/2+y/2;
    else w0:=Ceiling(Complexy)/2+1/2+y/2;
    end if;
end if;*/
B:=[w0-1,1];
A:=[];
U,m:=UnitGroup(OF);
if Floor(Conjugate(m(U.2),1)) lt 1 then
    u:=m(U.2)^2;
else u:=m(U.2)^(-2);
end if;
while u ne B[# B] do
    ak:=Floor(Conjugate(B[# B -1]/B[# B],1));
    Append(~B,B[#B -1]-ak*B[#B]);
    Append(~A,ak);
end while;
ResolutionCycle:=[];
for i in [1..#A] do
    if IsEven(i) then /*Note that our set A is [a_0,...,a_{2l-1}] in
```

```
                        VdGeer's notation */
        cyclepart:=[];
        for j in [1..A[i]-1] do
        Append(~cyclepart,2);
        end for;
        ResolutionCycle:=ResolutionCycle cat cyclepart;
    else Append(~ResolutionCycle,A[i]+2);
    end if;
end for;

Aklist:=[];
for i in [2..#B-2] do /*the list B is B-1, B0,...B2l */
    if IsEven(i) then Append(~Aklist,B[i]);
    if A[i] gt 1 then
        for j in [1..A[i]-1] do
            Append(~Aklist,B[i]-j*B[i+1]);
        end for;
    end if;
    end if;
end for;
Append(~Aklist,B[#B]);
return ResolutionCycle, Aklist; /*Aklist is actually [A0,...Ar] where Ar=ep*A0*/
end function;

/*The results of this function do not always match the O_K-column of the
table on p41 of VdGeer because VdGeer uses the full totally positive part of
the unit group which may differ from the squares of the units. Also, we find
that the number of 2's appearing in the cycle when D = alpha^2+4 is
alpha -1 instead of 2*alpha - 1. */

DualBasisOf:=function(F,OF,y,Basis)
A1:=F!Basis[1];
A2:=F!Basis[2];
conj:=Automorphisms(F)[2];
sqdelta:=A1*conj(A2)-conj(A1)*A2;
A2dual:=-conj(A1)/sqdelta;
A1dual:=conj(A2)/sqdelta;
return [A1dual,A2dual];
end function;

SurfaceCheckCoefficients:=function(F,OF,y,d)
ResCycle,Aklist:=ResolutionCycle(F,OF,ZBasis(F,OF,y,d,1*OF) ,d);
mu1:=Aklist[1];
mu2:=Aklist[2];
DBasis:=DualBasisOf(F,OF,y,[mu1,mu2]);
mu1dual:=DBasis[1]; mu2dual:=DBasis[2];
List:=[mu1dual+2*mu2dual,2*mu1dual+mu2dual,mu1dual+mu2dual];
for i in [1..2] do
    if not IsTotallyPositive(List[i]) then
        Remove(~List,i);
        Insert(~List,i,1/3);
    end if;
end for;
if not IsTotallyPositive(List[3]) then
    return "All forms map zero to zero";
end if;
return List;
end function;
```

```
NumberN:=function(d) /*Apart from the special cases 12,28,33,60 the only */
Z:=IntegerRing();    /* D up to 100 this function does not cover are 89,97*/
D:=disc(d);          /*for that would need to compute b_k's at other cusps*/
if not D in [5,8,12,13,17,21,24,28,33,60] then
    if D mod 8 ne 1 mod 8 or D in [41,105,65,73] then
        return 3;
    else for e in Divisors(D) do
        if e mod 8 ne 1 mod 8 then return 3;
        end if;
        end for;
    end if;
    if IsSquare(D+8) then
        if Z!SquareRoot(D+8) mod 8 eq 7 mod 8 then return 3;
        end if;
    end if;
else if D eq 5 then return 3;
    end if;
    if D eq 8 then return 7;
    end if;
    if D eq 13 or D eq 17 or D eq 21 or D eq 24 then return 2;
    end if;

end if;
return 10; /*This is just to not make it give an error for the D we still miss*/
end function;

zeta:=function(F,d);
zetaminone:=0;
sqrtD:=Floor(SquareRoot(disc(d)));
index:=[x : x in [-sqrtD..sqrtD] | x^2 mod 4 eq disc(d) mod 4];
for x in index do
    y:=(disc(d)-x^2)/4;
    y:=IntegerRing()!y;
    zetaminone:=zetaminone+SumOfDivisors(y);
end for;
return zetaminone/60;
end function;

degree:=function(F,OF,nn)
primes:=[p : p in Divisors(nn) | IsPrime(p)];
deg:=Norm(nn);
for p in primes do
    deg:=deg*(1+1/Norm(p));
end for;
return deg;
end function;

SturmBound:=function(F,OF,y,d,nn,k)
N:=NumberN(d);
ResCycle,Aklist:=ResolutionCycle(F,OF,ZBasis(F,OF,y,d,1*OF),d);
zetaF:=zeta(F,d);
b:=0;
for bk in ResCycle do
    b:=b+bk-2;
end for;
gens:=Generators(nn);
if #gens eq 1 and gens[1][2] eq 0 then
```

```
    n:=gens[1][1];
else n:=Norm(nn);
end if;
if n ge N then level:=n;
    else level:=n*Ceiling(N/n);
end if;
deg:=degree(F,OF,nn);
bound:=2*k*deg*level*zetaF/b; /*Note that our k is often denoted k/2 */
NCG,f:=NarrowClassGroup(F);
for g in NCG do
    I:=f(g)^(-2);
    ResCycle,Aklist:=ResolutionCycle(F,OF,ZBasis(F,OF,y,d,I),d);
    for bk in ResCycle do
        bound:=bound-(1+(deg-1)*k/(2*N))*(bk-2)/b;
    end for;
end for;
return Floor(bound+1);
end function;

CoefficientList:=function(d,SturmBound);
Z:=IntegerRing();
R<x> := PolynomialRing(IntegerRing());
d:=Z!RemoveSquares(d);
F<x>:= NumberField(x^2-d);  OF:= MaximalOrder(F); RR:=RealField();
y:=OF ! x;
ResCycle, Aklist:=ResolutionCycle(F,OF,ZBasis(F,OF,y,d,1*OF),d);
List:=[];
for i in [1..# Aklist -1] do
    DBasis:=DualBasisOf(F,OF,y,[Aklist[i],Aklist[i+1]]);
    lbd1:=Conjugate(Aklist[i+1]/Aklist[i],1);
    ubd1:=Conjugate(Aklist[i+1]/Aklist[i],2);
    for m in [1..SturmBound] do
        lbd:=(RR!m)*lbd1;
        ubd:=(RR!m)*ubd1;
        List cat:= [m*DBasis[1]+n*DBasis[2] : n in [Ceiling(lbd)..Floor(ubd)]];
    end for;
end for;
U,m:=UnitGroup(OF);
u:=m(U.2)^2;
/*for s in List do
    if u*s in List then Exclude(~List,u*s);
    end if;
end for;*/
return Setseq(Set(List));
end function;

R<x> := PolynomialRing(IntegerRing());
F<x>:= NumberField(x^2-10);  OF:= MaximalOrder(F);
y:=OF ! x;


RemoveDoubles2:=function(List)
return Setseq(Set(List));
end function;

EigenformFourierCoefficients:=function(f,K,L,d,y)
B:=[];
ChangeUniverse(~B,K);
```

```
if disc(d) mod 4 eq 1 then
    dif:=y;
else dif:=2*y;
end if;
for ksi in L do
    if IsIntegral(dif*ksi) then /*the ksi-th Fourier coefficient is the */
        A:=[];                  /*   eigenvalue of T_{dif*ksi*OF} */
        Fact:=Factorization(dif*ksi);
        for i in [1..# Fact] do
        P:=Fact[i][1];
        ap:=HeckeEigenvalue(f,P);
        AP:=[1,ap];
        if Fact[i][2] gt 1 then
            for j in [1.. Fact[i][2]-1] do
            Append(~AP,ap*AP[# AP]-Norm(P)*AP[# AP-1]);
            end for;
        end if;
        Append(~A,AP[# AP]);
        end for;
        a:=1;
        for b in A
         do a:=b*a;
        end for;
        Append(~B,a);
    else Append(~B,0);
    end if;
end for;
return B;
end function;

IrredEigenspaceFourierMatrix:=function(N,L,d,y);
KK:=HeckeEigenvalueField(N);
if KK eq RationalField() then KK:=QNF(); end if;
K:=SplittingField(KK);
boolean,f:=IsSubfield(KK,K);
Embed(KK,K,f(KK.1));
G:=Automorphisms(K);
B:=EigenformFourierCoefficients(Eigenform(N),K,L,d,y);
Q:=[[sigma(x) : x in B] : sigma in G];
return Matrix(Q);
end function;

NewspaceFourierMatrix:=function(M,L,d,y);
Dec:=NewformDecomposition(M);
Q:=Matrix(RationalField(),[]);
KK:=RationalField();
for N in Dec do
    list:=IrredEigenspaceFourierMatrix(N,L,d,y);
    if RowSequence(Q) eq [] then Q:=list;
    else
    K:=Compositum(BaseRing(list),BaseRing(Q));
    Q:=ChangeRing(Q,K);
    list:=ChangeRing(list,K);
    Q:=VerticalJoin(Q,list);
    end if;
end for;
return Q;
end function;
```

```
CuspFormBasisMatrix:=function(F,OF,I,L,d,y,k)
Levels:=Divisors(I);
T:=[];
for nn1 in Levels do
    Divs:=Divisors(I/nn1);
    KK:=RationalField();
    for nn2 in Divs do
        LL:=[ksi*OF/nn2 : ksi in L];
        N:=NewSubspace(HilbertCuspForms(F,nn1,[k,k]));
        if Dimension(N) ge 1 then
        Matrixpart:=NewspaceFourierMatrix(N,LL,d,y);
        if #Eltseq(T) eq 0 then T:=Matrixpart;
        else
        KK:=Compositum(BaseRing(Matrixpart),BaseRing(T));
        T:=ChangeRing(T,KK);
        Matrixpart:=ChangeRing(Matrixpart,KK);
        T:=VerticalJoin(T,Matrixpart);
        end if;
        end if;
    end for;
end for;
T:=RowSequence(RemoveZeroRows(EchelonForm(Matrix(T))));
return Matrix(RationalField(),T);
end function;

LatticeSymmProduct:=function(A,B,Acolindex,Bcolindex)
ABcolindex:=RemoveDoubles([ksi+eta : ksi in Acolindex, eta in Bcolindex]);
N:=Ncols(A);
M:=Ncols(B);
remainingcolumns:=[1..N*M];
product:=TensorProduct(A,B);
for nu in ABcolindex do
    pairs:=[[i,j] :  j in [1..M],i in [1..N]
     | Acolindex[i]+Bcolindex[j] eq nu];
    firstpair:=pairs[1];
    fpindex:=(firstpair[1]-1)*M+firstpair[2];
    for p in [2..#pairs] do
        i:=pairs[p][1];
        j:=pairs[p][2];
        index:=M*i-M+j;
        AddColumn(~product,1,index,fpindex);
        Exclude(~remainingcolumns,index);
    end for;
end for;
return
Matrix(RemoveDoubles2(RowSequence(Submatrix(product,[1..Nrows(product)],remainingcolumns)))),
        ABcolindex;
end function;
/* Removing double rows ensures we treat fi*fj and fj*fi as equal. This would only
    kill something in the kernel if there are cusp forms f_i,f_j,f_k,f_m such
    that f_i*f_j=f_k*f_m. We can verify this by checking whether the number of
    rows of the returned matrix equals the number of rows of the symmetric square,
    which is n*(n+1)/2 where n is the nr of rows of the original matrix */

/*The index of the rows of the monomial matrix the above function creates
has the following order: when f1,...fn is a basis of the cusp forms, the order is
```

```
f1^d,f1^(d-1)f_2,f1^(d-1)f3,...f1^(d-1)fn,f1^(d-2)f2f1 (omitted), f1^(d-2)f2f2
etc. We need a function that computes the monomial from the row number. */

MonomialFromRowIndex:=function(index,d,P,n)
names:=[];
for i in [1..n] do
    Append(~names,"x" cat IntegerToString(i-1));
end for;
AssignNames(~P,names);
A:=[[0,0]];
while d gt 0 do
    sum:=0;
    a:=0;
    while sum lt index do
        sum:=sum+Binomial(n+d-2,d-1);
        n:=n-1;
        a:=a+1;
    end while;
    sum:=sum-Binomial(n+d-1,d-1);
    n:=n+1;
    Append(~A,[a,A[#A][2]+a-1]);
    index:=index - sum;
    d:=d-1;
end while;
p:=1;
for i in [2..#A] do
    j:=A[i][2]+1;
    p:=p*P.j;
end for;
return p;
end function;

EquationsFromMonomialMatrix:=function(matrix,d,n)
P:=PolynomialAlgebra(RationalField(),n);
V:=Basis(Kernel(matrix));
I:=ideal<P|0>;
for v in V do
    eqn:=0;
    w:=Eltseq(v);
    for i in [1..#w] do
        if w[i] ne 0 then eqn:=eqn+w[i]*MonomialFromRowIndex(i,d,P,n);
        end if;
    end for;
    I:=I+ideal<P|eqn>;
end for;
return I;
end function;
```

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