Modularity of elliptic curves over certain totally real quartic fields

Linfoot Seminar in Bristol

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This theorem is part of a work in progress. The ultimate goal is to extend the result to all quartic fields.

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On this curve that is the map

$$w_5 : \ (u : v : w) \mapsto (u : v : -w).$$

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Modular form computations (and Kolyvagin–Logachev) show that $\mathrm{rk} J_X(\mathbb{Q}) = \mathrm{rk} J_C(\mathbb{Q}) = 2$.

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and the intersection $X(\mathbb{Q}_p) \cap \overline{J_X(\mathbb{Q})}$. If $g \geq r + 1$, i.e. $r < g$, then $X(\mathbb{Q}_p) \cap \overline{J_X(\mathbb{Q})}$ is expected to be finite.

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If $r < g$ then there exists ω such that $\int_D \omega = 0$ for all $D \in J_X(\mathbb{Q})$.

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An unknown point can map to a limited set of $\mathrm{Ker}(\mathrm{red}_p)$-cosets.

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An unknown point can map to a limited set of $\mathrm{Ker}(\mathrm{red}_p)$-cosets. Now intersect cosets for different primes for which the kernels $\mathrm{Ker}(\mathrm{red}_p)$ intersect.

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and $\mathrm{Sym}^d X(\mathbb{Q}_p) \cap \overline{J_X(\mathbb{Q})}$. If $r < g - (d-1)$ then we "expect" $\mathrm{Sym}^d X(\mathbb{Q}_p) \cap \overline{J_X(\mathbb{Q})}$ to be finite.

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If Q_i has multiplicity m in $\mathcal{Q}$, you need to look at $a_0(\omega_j, t_{Q_i}), \ldots, a_{m-1}(\omega_j, t_{Q_i})$.

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But so far I have not seen examples where the infinite set was not due to a map $X \to C$ of degree at most d.

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Now $C = X/w_5$ implies that $\rho_* = 1 + w_5^*$, so we need to compute

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C has finitely many rational points *but* $r_C = g_C$ so C does not satisfy the Chabauty assumption for $d = 1$.

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with $P_1 + P_2 \in \rho^* C(\mathbb{Q})$ and $Q_1 + Q_2 \in \mathrm{Sym}^2 X(\mathbb{Q})$. Suppose that $\omega_1, \dots, \omega_s \in H^0(\mathcal{X}, \mathbb{Z}_p)$ are linearly independent differentials such that $\int_D \omega_i = 0$ for all i and $D \in J_X(\mathbb{Q})$ and $\rho_* \omega_i = 0$ for all i. If the mod v reduction $(v \mid p)$ of the matrix with rows

$$\left(\frac{\omega_i}{\mathrm{d}t_{Q_1}}(Q_1) \quad \frac{\omega_i}{\mathrm{d}t_{Q_2}}(Q_2) \quad \frac{\omega_i}{\mathrm{d}t_{P_1}}(P_1) \right), \quad i \in \{1, \dots, s\}$$

has rank 3, then all points in $\mathrm{Sym}^4 X(\mathbb{Q})$ in the residue class of $\mathcal{Q}$ are of the form $Q_1 + Q_2 + R_1 + R_2$ with $R_1 + R_2 \in \rho^* C(\mathbb{Q})$.

Partially relative Chabauty

Theorem (Siksek, ■–Gajovic–Goodman)

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This theorem is part of (ongoing) joint work with Stevan Gajovic and Pip Goodman.

Conclusion

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But that field is not totally real.

Thank you for listening.