On the regularity for elliptic equations, elliptic systems and harmonic maps

Laura Bradby

Trinity College

University of Oxford

Dissertation

FHS of Mathematics Part C

Hilary Term 2023

Contents

Introduction

In this essay, we will discuss the following question.

If u is the solution to an elliptic equation (or system) with "nice" coefficients, how "nice" does u have to be?

This problem (the *regularity problem*) is very well understood in the case of elliptic equations, when the solution u is a map from $\mathbb{R}^m$ to $\mathbb{R}$. However, methods which work in this case may not translate to more general situations. We will examine this problem for linear elliptic equations, and two generalisations of this, namely elliptic systems (where the solution is a map $u : \mathbb{R}^m \to \mathbb{R}^n$), and harmonic maps (where the solution is a map between Riemannian manifolds, but we restrict to the equation analogous to $\Delta u = 0$).

Our main focus will be on the method of *freezing coefficients*, which can be used to establish regularity results for both equations and systems in almost exactly the same way, and can also be used to give a regularity result for harmonic maps. However, this method relies heavily on having continuous coefficients, so we will examine a second method which works when the coefficients are merely bounded. This method cannot be easily adapted to systems, but still has applications for harmonic maps.

Each of our main sections is dedicated to a particular situation. In §1, we look at the scalar case, giving an introduction to freezing coefficients, and taking a detailed look at our second method. We give a more in-depth exploration of freezing coefficients in §2, which is dedicated to systems, and examine how to adapt the method to give boundary regularity. In §3, we look at examples which show that some of the results we have found do not hold in more general situations. Finally, §4 is about harmonic maps, and we see how both methods can be applied to give regularity and partial regularity results in this case.

1 Elliptic equations

1.1 The problem

In this section, we are interested in equations with the form

$$\begin{aligned} -\partial_j(a_{ij}\partial_i u) + b_i\partial_i u + cu &= f - \partial_j g_j \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $a_{ij}, b_i, c, f, g_j : \Omega \to \mathbb{R}$ are defined on a domain $\Omega \subseteq \mathbb{R}^m$, and the solution is a map $u : \Omega \to \mathbb{R}$. Here we adopt the convention that repeated indices are summed over. We also require the a_{ij} to be *symmetric* (ie. $a_{ij} = a_{ji}$ for all i, j) and *uniformly elliptic*, that is there must be $\lambda, \Lambda \in \mathbb{R}$ such that

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \forall \xi \in \mathbb{R}^m$$

for almost all $x \in \Omega$. Note this means that the a_{ij} are bounded, with $\|a\|_{L^\infty(\Omega)} \leq \Lambda$.

We are interested in *weak solutions* to (1.1), that is functions $u \in H^1(\Omega)$ which satisfy

$$\int_\Omega a_{ij}\partial_i u\partial_j\varphi + b_i\partial_i u\varphi + cu\varphi = \int_\Omega f\varphi + g_j\partial_j\varphi \qquad \forall \varphi \in H_0^1(\Omega),$$

where as usual H^1 is the space of L^2 functions with weak derivatives in L^2, and H_0^1 is the closure of the space C_c^∞ (of compactly supported smooth functions) in H^1. The weak formulation of the boundary condition is $u \in H_0^1(\Omega)$.

To simplify the discussion, we consider only the case where the lower-order coefficients are all zero, giving the equation

$$\begin{aligned} -\partial_j(a_{ij}\partial_i u) &= f - \partial_j g_j \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

1.2 The method of freezing coefficients

We begin with the method of freezing coefficients. These ideas can be extended to systems with minimal changes, so we give a brief discussion here, and postpone the details to §2.

The main requirement of this method is that the coefficients a_{ij} are continuous. Then we see that $a_{ij}(x)$ should not vary too much compared to $\widetilde{a}_{ij} := a_{ij}(x_0)$ on a sufficiently small ball $B_r(x_0) \subseteq \Omega$. It is therefore reasonable to expect a weak solution u of the equation $-\partial_j(a_{ij}(x)\partial_i u) = f - \partial_j g_j$ to be close to a weak solution w of $\widetilde{a}_{ij}\partial_{ij}w = 0$ on $B_r(x_0)$, if we also have $u = w$ on $\partial B_r(x_0)$.

Now $\widetilde{a}_{ij}\partial_{ij}w = 0$ is an elliptic equation with constant coefficients, so it has a unique weak solution w with $w = u$ on $\partial\Omega$ [GT01, theorem 8.3], and we have the following result [HL11, Lemma 3.10].

Lemma 1.1. *If $(\widetilde{a}_{ij})$ is a symmetric constant positive definite matrix such that*

$$\lambda|\xi|^2 \leq \widetilde{a}_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \forall \xi \in \mathbb{R}^m$$

for some $0 < \lambda \leq \Lambda$, and $w \in H^1(B_r(x_0))$ is a weak solution of

$$\widetilde{a}_{ij}\partial_{ij}w = 0 \qquad \text{in } B_r(x_0),$$

then there exists $C = C(\lambda, \Lambda)$ such that

$$\|Dw\|^2_{L^2(B_\rho(x_0))} \le C\left(\frac{\rho}{r}\right)^n \|Dw\|^2_{L^2(B_r(x_0))},$$
$$\left\|Dw - \overline{Dw}_{x_0,\rho}\right\|^2_{L^2(B_\rho(x_0))} \le C\left(\frac{\rho}{r}\right)^{n+2} \left\|Dw - \overline{Dw}_{x_0,r}\right\|^2_{L^2(B_r(x_0))}$$

for each $0 < \rho \le r$. Here we define $\overline{u}_{x_0,r} := \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u$ for each $u \in L^1(B_r(x_0))$.

By bounding $u - w$, we can use the first part of this result to control the behaviour of u on $B_r(x_0)$. Then by considering all balls with centres in some subset of Ω, we obtain the following result[1] (see §2.4 for the proof of this in a more general setting).

Theorem 1.2. *Suppose u satisfies (1.2), where the $a_{ij} \in C^0(\Omega)$ are bounded and uniformly elliptic, $f \in L^q(\Omega)$ for some $q \in (\frac{m}{2}, m)$, and $g \in L^{q'}(\Omega)$ for each j, where $q' > m$.*

Then $u \in C^{0,\alpha}_{loc}(\Omega)$ for $\alpha = \min\left\{2 - \frac{m}{q}, 1 - \frac{m}{q'}\right\}$, and for each $\widetilde{\Omega} \Subset \Omega$ there exists $C = C(\lambda, \Lambda, m, a_{ij}, q, q', \widetilde{\Omega})$ such that

$$\|u\|_{C^{0,\alpha}(\widetilde{\Omega})} \le C\left(\|u\|_{H^1(\Omega)} + \|f\|_{L^q(\Omega)} + \|g\|_{L^{q'}(\Omega)}\right).$$

Here, we write $\widetilde{\Omega} \Subset \Omega$ if $\widetilde{\Omega}$ is open with compact closure in Ω, and $u \in C^{0,\alpha}_{loc}(\Omega)$ if $u \in C^{0,\alpha}(\widetilde{\Omega})$ for each $\widetilde{\Omega} \Subset \Omega$. We write $\|g\|_{L^{q'}(\Omega)} := \sum_j \|g_j\|_{L^{q'}(\Omega)}$.

The same technique can be used to give the following result.[2] Note that this gives higher regularity for u, so requires the coefficients to be more regular.

Theorem 1.3. *Suppose u satisfies (1.2), where the $a_{ij} \in C^{0,\alpha}(\overline{\Omega})$ are bounded and uniformly elliptic, $f \in L^q(\Omega)$ for some $q > m$, and $g_j = 0$. Let $\alpha = 1 - \frac{m}{q}$.*

Then $Du \in C^{0,\alpha}_{loc}(\Omega)$, and for each $\widetilde{\Omega} \Subset \Omega$ there exists $C = C(\lambda, \Lambda, m, a_{ij}, q, \widetilde{\Omega})$ such that

$$\|Du\|_{C^{0,\alpha}(\widetilde{\Omega})} \le C\left(\|u\|_{H^1(\Omega)} + \|f\|_{L^q(\Omega)}\right).$$

This requires both parts of the conclusion of lemma 1.1 because we want to bound $\left\|Du - \overline{Du}_{x_0,\rho}\right\|^2_{L^2(B_\rho(x_0))}$, and the extra terms require some additional estimates, but the structure of the proof is otherwise the same as that of theorem 1.2.

Suppose we have a stronger version of theorem 1.3, which holds whenever $a_{ij}, g_j \in C^{0,\alpha}_{loc}(\Omega)$ (see [GT01, theorem 8.32]). Then we can apply a bootstrap argument (adapted from [Eva10, §8.3.2]) to show that the solution is "one step more regular" than that of the coefficients.

Suppose u solves

$$\int_\Omega a_{ij}\partial_i u \partial_j \varphi = \int_\Omega f\varphi + g_j \partial_j \varphi \qquad \forall \varphi \in H^1_0(\Omega),$$

with $a_{ij}, g_j \in C^{1,\alpha}_{loc}(\Omega)$, $f \in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$. Then certainly $u \in C^{1,\alpha}_{loc}(\Omega)$. Further, we have that $u \in H^2_{loc}(\Omega)$ by [Eva10, §8.3, theorem 1], so if we test the equation

[1] [HL11, theorem 3.8], case $c = 0$, adapted to a general domain Ω and the case $g \neq 0$, conclusion adapted to give $C^{0,\alpha}$ norm.

[2] [HL11, theorem 3.13], adapted from the textbook in the same way as theorem 1.2.

against $-\partial_k\varphi$ for $\varphi \in C_c^\infty(\Omega)$, then we can integrate by parts to give

$$\begin{aligned}\int_\Omega -a_{ij}\partial_i u \partial_k\partial_j\varphi &= \int_\Omega -f\partial_k\varphi - g_j\partial_k\partial_j\varphi \\ \int_\Omega a_{ij}\partial_i(\partial_k u)\partial_j\varphi + \partial_k(a_{ij})\partial_i u\partial_j\varphi &= \int_\Omega -f\partial_k\varphi + \partial_k g_j\partial_j\varphi \\ \int_\Omega a_{ij}\partial_i(\partial_k u)\partial_j\varphi &= \int_\Omega (\partial_k g_j - f\delta_{ik} - \partial_k(a_{ij})\partial_i u)\partial_j\varphi.\end{aligned}$$

Now $\partial_k g_j - f\delta_{ik} - \partial_k(a_{ij})\partial_i u \in C^{0,\alpha}_{loc}(\overline{\Omega})$, so applying our stronger version of theorem 1.3 to $\partial_k u$ gives that $u \in C^{2,\alpha}_{loc}(\Omega)$.

From here, we can repeat this argument to see that if $a_{ij}, g_j \in C^{k,\alpha}_{loc}(\Omega)$, $f \in C^{k-1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$, $k \in \mathbb{N}$, then $u \in C^{k+1,\alpha}_{loc}(\Omega)$.

1.3 Bounded coefficients

Our second method gives a more powerful result - we have Hölder continuity of solutions in the case of bounded coefficients. We will outline the idea by following [HL11, §4], beginning with the following theorem.[3]

Theorem 1.4. *Suppose u is a subsolution of* (1.2) *on B_r, where the $a_{ij} \in L^\infty(B_r)$ are bounded and uniformly elliptic, and $f = 0$. That is,*

$$\int_{B_r} a_{ij}\partial_i u\partial_j\varphi \le 0 \qquad \forall\varphi \in H_0^1(B_r). \tag{1.3}$$

Then $u^+ := \min\{u, 0\}$ is locally bounded on B_r, and there exists[4] $C = C\left(m, \frac{\lambda}{\Lambda}\right)$ *such that*

$$\left\|u^+\right\|^2_{L^\infty(B_{\theta r})} \le C \cdot \frac{1}{(1-\theta)^m} r^{-m} \left\|u^+\right\|^2_{L^2(B_r)}$$

for each $\theta \in (0,1)$.

Proof for $r = 1$, $\theta = \frac{1}{2}$. Write $v = u^+$, and set $v_k = \min\{v, k\}$ for $k > 0$ (we will later send $k \to \infty$).

For $\beta \ge 0$ and $\eta \in C_0^1(B_1)$ to be determined, we will test (1.3) against the H_0^1 function

$$\varphi = \eta^2(v_k^\beta v - k^{\beta+1}).$$

Note that

$$\begin{aligned}D\varphi &= \eta^2(\beta v_k^{\beta-1}Dv_k v + v_k^\beta Dv) + 2\eta D\eta(v_k^\beta v - k^{\beta+1}) \\ &= \eta^2 v_k^\beta(\beta Dv_k + Dv) + 2\eta D\eta(v_k^\beta v - k^{\beta+1}),\end{aligned}$$

since $Dv_k = 0$ in the set $\{v \ne v_k\}$, so $vDv_k = v_k Dv_k$.

[3][HL11, theorem 4.1, method 2], simplified to the case $c = f = 0$, $p = 2$, given for $B_r \subseteq \mathbb{R}^m$ instead of B_1.

[4][HL11] gives that C depends on λ, Λ separately, but the dependence we have stated follows from the given proof and is needed later on.

Now $D\varphi = 0$ outside $\{u > 0\}$, and $Du = Dv$ in $\{u > 0\}$, so

$$\begin{aligned}
0 &\geq \int_{B_1} a_{ij}\partial_i u \partial_j \varphi \\
&= \int_{\{u>0\}} a_{ij}\partial_i u \partial_j \varphi \\
&= \int_{\{u>0\}} a_{ij}\partial_i v(\beta\partial_j v_k + \partial_j v)\eta^2 v_k^\beta + 2\int_{\{u>0\}} a_{ij}\partial_i v \partial_j \eta (v_k^\beta v - k^{\beta+1})\eta \\
&= \int_{\{u>0\}} a_{ij}(\beta\partial_i v_k \partial_j v_k + \partial_i v \partial_j v)\eta^2 v_k^\beta + 2\int_{\{u>0\}} a_{ij}\partial_i v \partial_j \eta (v_k^\beta v - k^{\beta+1})\eta \\
&\geq \lambda \int_{\{u>0\}} (\beta|Dv_k|^2 + |Dv|^2)\eta^2 v_k^\beta - 2\left\|a_{ij}\right\|_{L^\infty(B_1)} \int_{\{u>0\}} |Dv||D\eta|(v_k^\beta v)\eta \\
&\geq \lambda \int_{\{u>0\}} (\beta|Dv_k|^2 + |Dv|^2)\eta^2 v_k^\beta - 2\Lambda \int_{\{u>0\}} (\frac{\lambda}{4\Lambda}|Dv|^2\eta^2 + \frac{\Lambda}{\lambda}|D\eta|^2 v^2) v_k^\beta \\
&= \lambda \int_{\{u>0\}} \left(\beta|Dv_k|^2 + \frac{1}{2}|Dv|^2\right)\eta^2 v_k^\beta - \frac{2\Lambda^2}{\lambda}\int_{\{u>0\}} |D\eta|^2 v^2 v_k^\beta,
\end{aligned}$$

where for the fourth line we used that $\partial_i v_k = \partial_i v$ wherever $\partial_j v_k \neq 0$.

Hence

$$\int_{\{u>0\}} \left(\beta|Dv_k|^2 + \frac{1}{2}|Dv|^2\right)\eta^2 v_k^\beta \leq \frac{2\Lambda^2}{\lambda^2}\int_{\{u>0\}} |D\eta|^2 v^2 v_k^\beta. \tag{1.4}$$

Note that if $w = v_k^{\beta/2} v$, then

$$\begin{aligned}
|Dw|^2 &= \left|\frac{\beta}{2} v_k^{\beta/2-1} v Dv_k + v_k^{\beta/2} Dv\right|^2 \\
&= \frac{\beta^2}{4} v_k^\beta |Dv_k|^2 + \beta v_k^\beta |Dv_k|^2 + v_k^\beta |Dv|^2 \\
&\leq (1+\beta)\left(\beta|Dv_k|^2 + |Dv|^2\right) v_k^\beta.
\end{aligned}$$

Combining this with (1.4) gives

$$\begin{aligned}
\int_{\{u>0\}} |D(w\eta)|^2 &\leq \int_{\{u>0\}} |Dw|^2\eta^2 \\
&\leq (1+\beta)\int_{\{u>0\}} \left(\beta|Dv_k|^2 + |Dv|^2\right)\eta^2 v_k^\beta \\
&\leq 2(1+\beta)\int_{\{u>0\}} \left(\beta|Dv_k|^2 + \frac{1}{2}|Dv|^2\right)\eta^2 v_k^\beta \\
&\leq C\left(\frac{\Lambda}{\lambda}\right)\cdot(1+\beta)\int_{\{u>0\}} |D\eta|^2 v^2 v_k^\beta \\
&= C\cdot(1+\beta)\int_{\{u>0\}} w^2|D\eta|^2.
\end{aligned}$$

Write $\chi = \frac{m}{m-2}$ if $m > 2$ and take arbitrary $\chi > 2$ if $m \leq 2$. Then the Sobolev inequality gives $\left\|w\eta\right\|_{L^{2\chi}} \leq C\left\|D(w\eta)\right\|_{L^2}$, so

$$\left\|w\eta\right\|_{L^{2\chi}(B_R)}^2 \leq C\left\|D(w\eta)\right\|_{L^2(B_R)}^2 \leq C\cdot(1+\beta)\left\|D\eta\right\|_{L^2(B_R)}^2 \left\|w\right\|_{L^2(B_R)}^2.$$

Then if $0 < \rho < R \leq 1$, we take η to be a cut-off function with

$$\eta \equiv 1 \text{ in } B_\rho, \qquad \eta \equiv 0 \text{ outside } B_R, \qquad |D\eta| \leq \frac{2}{R-\rho},$$

giving

$$\|w\|^2_{L^{2\chi}(B_\rho)} \leq \|w\eta\|^2_{L^{2\chi}(B_R)} \leq C \cdot (1+\beta) \left(\frac{2}{R-\rho}\right)^2 \|w\|^2_{L^2(B_R)}.$$

Now $v_k \leq v$ and $w = v_k^{\beta/2} v$, so

$$\left(\int_{B_\rho} v_k^{(\beta+2)\chi}\right)^{\frac{1}{\chi}} \leq \|w\|^2_{L^{2\chi}(B_\rho)} \leq C\frac{1+\beta}{(R-\rho)^2} \|w\|^2_{L^2(B_R)} \leq C\frac{1+\beta}{(R-\rho)^2} \int_{B_R} v^{\beta+2},$$

provided the final integral is bounded. Writing $\gamma = \beta + 2 \geq 2$ and sending $k \to \infty$ then gives

$$\|v\|_{L^{\gamma\chi}(B_\rho)} \leq \left(\frac{C\gamma}{(R-\rho)^2}\right)^{\frac{1}{\gamma}} \|v\|_{L^\gamma(B_R)}, \tag{1.5}$$

where we allow $\|v\|_{L^\gamma(B_R)} = \infty$.

For the final step, we iterate by setting

$$\gamma_i = 2\chi^i, \qquad r_i = \frac{1}{2} + \frac{1}{2^{i+1}}$$

for $i = 0, 1, \cdots$, so (1.5) gives

$$\|v\|_{L^{\gamma_{i+1}}(B_{r_{i+1}})} \leq \left(2^{i+3}\chi^i C\right)^{\frac{1}{2}\chi^{-i}} \|v\|_{L^{\gamma_i}(B_{r_i})}.$$

Now $\sum(i+3)\chi^{-i}$, $\sum i\chi^{-i}$, $\sum \chi^{-i}$ are all finite, so by iteration we have

$$\|v\|_{L^{\gamma_i}(B_{\frac{1}{2}})} \leq \|v\|_{L^{\gamma_i}(B_{r_i})} \leq C \|v\|_{L^2(B_1)}$$

for each i. Sending $i \to \infty$ gives the result for $r = 1$, $\theta = \frac{1}{2}$.

□

Proof for general r, θ. Suppose $u : B_r \to \mathbb{R}$ is a subsolution of $-\partial_j(a_{ij}\partial_i u) = 0$, and $y \in B_{\theta r}$. Then $B_{(1-\theta)r}(y) \subseteq B_r$, so if

$$v : B_1 \to \mathbb{R}, \qquad v(x) = u((1-\theta)rx + y),$$

then v is a subsolution of $-\partial_j(\widetilde{a}_{ij}\partial_i u) = 0$ in B_1, where $\widetilde{a}_{ij}(x) := a_{ij}((1-\theta)rx + y)$ for each i, j. The $\widetilde{a}_{ij}$ are uniformly elliptic, with the same λ, Λ as the a_{ij}, so applying the result above gives

$$\left\|u^+\right\|^2_{L^\infty(B_{\frac{1}{2}(1-\theta)r}(y))} = \left\|v^+\right\|^2_{L^\infty(B_{\frac{1}{2}})} \leq C \cdot \left\|v^+\right\|^2_{L^2(B_1)} \leq C((1-\theta))r^{-m} \left\|u^+\right\|^2_{L^2(B_r)}.$$

Finally, $B_{\theta r}$ is contained in a finite union of balls $B_{\frac{1}{2}(1-\theta)r}(y_i)$ with $y_i \in B_{\theta r}$ for each i, so the result follows. □

We also have the following result, which can be obtained via direct calculation [HL11, theorem 4.10].

Theorem 1.5. *If $u \in H^1(B_2)$ is a bounded weak solution of $-\partial_j(a_{ij}\partial_i u) = 0$, where the $a_{ij} \in L^\infty(B_2)$ are bounded and uniformly elliptic, then*

$$\operatorname*{osc}_{B_{\frac{1}{2}}} u \leq \gamma \operatorname*{osc}_{B_1} u$$

for some $\gamma = \gamma\left(m, \frac{\lambda}{\Lambda}\right)$, where $\operatorname{osc}_\Omega u := \operatorname{ess\,sup}_\Omega u - \operatorname{ess\,inf}_\Omega u$.

By dilation, theorem 1.5 holds on any B_{2r}, B_r, $B_{\frac{r}{2}}$, with the same γ.

Combining theorems 1.4 and 1.5 yields the following ([HL11, theorem 4.11] - given without proof in [HL11], so we give our own proof here).

Theorem 1.6. *If $u \in H^1(B_1)$ is a weak solution of $-\partial_j(a_{ij}\partial_i u) = 0$, where the $a_{ij} \in L^\infty(B_1)$ are bounded and uniformly elliptic, then $u \in C^{0,\alpha}(B_{\frac{1}{2}})$, with*

$$\|u\|_{C^{0,\alpha}(B_{\frac{1}{2}})} \leq C \|u\|_{L^2(B_1)}$$

for some $\alpha = \alpha\left(m, \frac{\lambda}{\Lambda}\right)$, $C = C\left(m, \frac{\lambda}{\Lambda}\right)$.

Proof. We begin with our key estimate. Take any $z \in B_{\frac{1}{2}}$, $k \geq 0$. Now $B_{\frac{1}{4}}(z) \subseteq B_{\frac{3}{4}}$, and u is bounded on $B_{\frac{3}{4}}$ by theorem 1.4 (applied to $\pm u$), so theorem 1.5 gives

$$\operatorname*{osc}_{B_{2^{-2-k}}(z)} u \leq \gamma \operatorname*{osc}_{B_{2^{-1-k}}(z)} u \leq \cdots \leq \gamma^k \operatorname*{osc}_{B_{\frac{1}{4}}(z)} u \leq \gamma^k \operatorname*{osc}_{B_{\frac{3}{4}}} u \leq 2\gamma^k \sup_{B_{\frac{3}{4}}} u.$$

First, we show that $u \in C^0(B_{\frac{1}{2}})$. Set $a_k(z) = \operatorname{ess\,inf}_{B_{2^{-2+k}}(z)} u$, $b_k(z) = \operatorname{ess\,sup}_{B_{2^{-2+k}}(z)} u$ for each $z \in B_{\frac{1}{2}}$, so $a_k(z)$ is increasing, $b_k(z)$ is decreasing. Additionally,

$$0 \leq b_k(z) - a_k(z) = \operatorname*{osc}_{B_{2^{-2-k}}(z)} u \leq C \|u\|_{L^2(B_1)} \gamma^k \to 0,$$

so $a_k, b_k \to \overline{u}$ uniformly for some $\overline{u} : B_{\frac{1}{2}} \to \mathbb{R}$. Further, if $c_k(z) = \frac{1}{|B_{2^{-2+k}}(z)|} \int_{B_{2^{-2+k}}(z)} u$, then $a_k \leq c_k \leq b_k$, so $c_k \to \overline{u}$ uniformly. The c_k are all continuous functions, so $\overline{u}$ is continuous. Finally, $c_k(u) \to u$ *a.e.* by the Lebesgue differentiation theorem

Finally, we show that $\overline{u} \in C^{0,\alpha}(B_{\frac{1}{2}})$. Given $x, y \in B_{\frac{1}{2}}$ with $|x-y| < \frac{1}{2}$, take $z = \frac{x+y}{2}$, and $k \in \mathbb{N}$ such that $2^{-3-k} \leq \frac{|x-y|}{2} < 2^{-2-k}$. Then

$$|\overline{u}(x) - \overline{u}(y)| \leq \operatorname*{osc}_{B_{2^{-2-k}}(z)} \overline{u} = \operatorname*{osc}_{B_{2^{-2-k}}(z)} u \leq 2\gamma^k \operatorname*{ess\,sup}_{B_{\frac{3}{4}}} u,$$

and $\gamma^k \leq \gamma^{-2}|x-y|^{-\frac{\log\gamma}{\log 2}}$. Now $\gamma < 1$ so $-\frac{\log\gamma}{\log 2} > 0$, so we get $\alpha = \alpha(\gamma) \in (0,1)$ such that

$$\frac{|\overline{u}(x) - \overline{u}(y)|}{|x-y|^\alpha} \leq C\left(m, \tfrac{\lambda}{\Lambda}\right) \operatorname*{ess\,sup}_{B_{\frac{3}{4}}} u.$$

If $x, y \in B_{\frac{1}{2}}$, $|x-y| > \frac{1}{2}$, then

$$\frac{|\overline{u}(x) - \overline{u}(y)|}{|x-y|^\alpha} \leq 2^{1-\alpha} \sup_{B_{\frac{3}{4}}} \overline{u} = 2^{1-\alpha} \operatorname*{ess\,sup}_{B_{\frac{3}{4}}} u.$$

Finally, theorem 1.4 gives

$$\sup_{x \in B_{\frac{1}{2}}} |\overline{u}(x)| + \sup_{x,y \in B_{\frac{1}{2}}} \frac{|\overline{u}(x) - \overline{u}(y)|}{|x-y|^\alpha} \leq C\left(m, \tfrac{\lambda}{\Lambda}\right) \operatorname*{ess\,sup}_{B_{\frac{3}{4}}} u \leq C\left(m, \tfrac{\lambda}{\Lambda}\right) \|u\|_{L^2(B_1)}.$$

$\square$

The following result is an easy consequence of this (see [HL11, lemma 4.12]).

Lemma 1.7. *If $w \in H^1(B_r)$ is a weak solution of $-\partial_j(a_{ij}\partial_i w) = 0$, where the $a_{ij} \in L^\infty(B_r)$ are bounded and uniformly elliptic, then there exist $\alpha = \alpha\left(m, \frac{\lambda}{\Lambda}\right) \in (0,1)$, $C = C\left(m, \frac{\lambda}{\Lambda}\right)$ such that*

$$\|Dw\|^2_{L^2(B_\rho)} \leq C\left(\frac{\rho}{r}\right)^{m-2+2\alpha} \|Dw\|^2_{L^2(B_r)}$$

for any $0 < \rho < r$.

This has the same conclusion as lemma 1.1, but does not require constant coefficients. This means that we can use the method outlined in §1.2, but without freezing the coefficients, so we do not need continuity of the a_{ij}. This gives the following result [HL11, theorem 4.13].

Theorem 1.8. *Suppose u satisfies (1.2), where the $a_{ij} \in L^\infty(\Omega)$ are bounded and uniformly elliptic, and $f \in L^q(\Omega)$ for some $q \in (\frac{m}{2}, m)$.*

Then $u \in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha = \alpha\left(m, \frac{\lambda}{\Lambda}, q\right) \in (0,1)$, and for each $\widetilde{\Omega} \Subset \Omega$ there exists $C = C(\lambda, \Lambda, m, q, \widetilde{\Omega})$ such that

$$\|u\|_{C^{0,\alpha}(\widetilde{\Omega})} \leq C\left(\|u\|_{H^1(\Omega)} + \|f\|_{L^q(\Omega)}\right).$$

Unfortunately, theorem 1.4 does not hold for systems (see §3.1), so we cannot get an analogue of lemma 1.7, even when u is bounded, and the method does not carry over to the systems case. However, this idea can be applied in other contexts, so we will see it again in §4.5.

2 Elliptic systems

In §1.2, we briefly examined the method of freezing coefficients. We will now explore this idea in more detail in the context of elliptic systems, following the structure of the proof in [HL11, §3]. In [HL11], the proof is given in the scalar case for $B_1 \subseteq \mathbb{R}^m$, so we have adapted it to work for systems and any domain $\Omega \subseteq \mathbb{R}^m$, using ideas from [Giu03] and [Gia83] to assist this goal. We have also extended it to allow the case $g \neq 0$ (see (2.1)). We have restricted to equations with only leading-order coefficients to streamline the discussion. In the rest of this section, we discuss an extension of the method to give boundary regularity.

2.1 The problem

Suppose $u : \Omega \to \mathbb{R}^n$ for $\Omega \subseteq \mathbb{R}^m$. We write $u(x) = (u^1(x), \dots u^m(x))$, and consider the problem

$$\begin{aligned} -\partial_j(A^{ij}_{\alpha\beta}\partial_i u^\alpha) &= f_\beta - \partial_j g^j_\beta \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where we require this to hold for each β. We use the convention that repeated indices in upper and lower pairs are summed over, and take $i, j = 1, \dots, m$, and $\alpha, \beta = 1, \dots, n$.

We assume that the coefficients $A^{ij}_{\alpha\beta} \in L^\infty(\Omega)$ of the system are symmetric in i, j and in α, β, and that they are bounded and uniformly elliptic, that is there exist λ, Λ such that

$$\lambda|\xi|^2 \leq A^{ij}_{\alpha\beta}(x)\xi^\alpha_i \xi^\beta_j \leq \Lambda|\xi|^2 \qquad \forall \xi \in \mathbb{R}^{mn}$$

for almost all $x \in \Omega$.

The weak formulation of (2.1) is

$$\int_\Omega A^{ij}_{\alpha\beta}(x)\partial_i u^\alpha \partial_j \varphi^\beta = \int_\Omega f_\beta \varphi^\beta + g^j_\beta \partial_j \varphi^\beta \qquad \forall \varphi \in H^1_0(\Omega, \mathbb{R}^n), \tag{2.2}$$

with $u \in H^1_0(\Omega, \mathbb{R}^n)$. The notation $(\cdot, \mathbb{R}^n)$ indicates that functions in the relevant function space have codomain $\mathbb{R}^n$, although we will suppress this notation where convenient.

2.2 Campanato spaces

The method of freezing coefficients will give us integral estimates for the solution u, but we wish to conclude that u is locally α-Hölder continuous. This means we need an integral characterisation of $C^{0,\alpha}$ functions, so we give the characterisation from [Giu03, §2.3], beginning with the following definition.

Definition 2.1. Let $\Omega \subseteq \mathbb{R}^m$, $1 \leq p < \infty$, and $\lambda \geq 0$. Then the *Campanato space* $\mathcal{L}^{p,\lambda}(\Omega, \mathbb{R}^n)$ is the space of functions $u \in L^p(\Omega, \mathbb{R}^n)$ such that

$$[u]^p_{p,\lambda} := \sup_{\substack{x_0 \in \Omega \\ r > 0}} r^{-\lambda} \left\| u - \overline{u}_{x_0,r} \right\|^p_{L^p(\Omega_r(x_0))} < \infty,$$

where $\Omega_r(x_0) := \Omega \cap B_r(x_0)$, and

$$\overline{u}_{x_0,r} := \frac{1}{|\Omega_r(x_0)|} \int_{\Omega_r(x_0)} u.$$

The associated norm is

$$\|u\|_{\mathcal{L}^{p,\lambda}} := [u]_{p,\lambda} + \|u\|_{L^p}.$$

Remark 2.1. In [Giu03], this definition is given using cubes instead of balls, although it is noted that this gives an isomorphic space.

Remark 2.2. If $r > \varepsilon$, then

$$r^{-\lambda} \left\| u - \overline{u}_{x_0,r} \right\|_{L^p(\Omega_r(x_0))}^p \leq \varepsilon^{-\lambda} \left\| u - \overline{u}_{x_0,r} \right\|_{L^p(\Omega)}^p ,$$

so $[u]_{p,\lambda}$ only depends on the behaviour for small values of r.

We will also need the following assumption on Ω.

Definition 2.2. We say $\Omega \subseteq \mathbb{R}^m$ has *no external cusps* if there exists $A \geq 0$ such that

$$|\Omega_r(x_0)| \geq A |B_r(x_0)|$$

for each $x_0 \in \overline{\Omega}$ and $0 < r \leq 1$, where we recall from definition 2.1 that $\Omega_r(x_0) := \Omega \cap B_r(x_0)$.

Now we have the following theorem [Tah15, theorem 18.12].[5]

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^m$ be a bounded open set with no external cusps, and suppose $m < \lambda \leq m + p$, so that $\alpha := \frac{\lambda - m}{p} \in (0, 1]$. Then the spaces $\mathcal{L}^{p,\lambda}(\Omega, \mathbb{R}^n)$ and $C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^n)$ are isomorphic. In particular, there is $C = C(p, A, \lambda, m)$ such that*

$$\|u\|_{C^{0,\alpha}} \leq C \|u\|_{\mathcal{L}^{p,\lambda}}$$

for each $u \in \mathcal{L}^{p,\lambda}(\Omega, \mathbb{R})$.

In [Tah15], the dependence of the constant is not mentioned explicitly. However, we will later need the fact that it is independent of λ, so we sketch the proof here to show where the constant comes from.

Proof (sketch). First, we have that if $x \in \Omega$, $0 < \rho < r \leq \min\{1, \operatorname{diam}(\Omega)\}$, then there is $C = C(p, A, \lambda, m)$ such that

$$|\overline{u}_{x,r} - \overline{u}_{x,\rho}|^p \leq C r^{\frac{\lambda - m}{p}} [u]_{p,\lambda} \tag{2.3}$$

(see [Tah15, proof of lemmas 18.7, 18.8, and case 1 in subsequent calculations]).

Then given $0 < R \leq 1$, (2.3) can be used to show that the sequence of averages $(\overline{u}_{x,2^{-i}R})$ converges uniformly to a limit $\overline{u}$ which is independent of the choice of R. The $\overline{u}_{x,2^{-i}R}$ are all continuous, so $\overline{u}$ is continuous, and $\overline{u} = u$ *a.e.* by the Lebesgue differentiation theorem.

Hence it is enough to show that $\overline{u}$ is Hölder continuous. If $x, y \in \Omega$, $r := |x - y| < \frac{1}{2}$, then

$$|\overline{u}(x) - \overline{u}(y)| \leq |\overline{u}(x) - \overline{u}_{x,2r}| + |\overline{u}_{x,2r} - \overline{u}_{y,2r}| + |\overline{u}_{y,2r} - \overline{u}(y)|.$$

Then (2.3) immediately gives a bound on the first and third terms. Integrating

$$|\overline{u}_{x,2r} - \overline{u}_{y,2r}| \leq |\overline{u}_{x,2r} - \overline{u}(z)| + |\overline{u}(z) - \overline{u}_{y,2r}|$$

over $z \in \Omega_{2r}(x) \cap \Omega_{2r}(y)$, and using (2.3) and the fact that Ω has no external cusps, gives a bound for the second term. Hence we have that

$$|\overline{u}(x) - \overline{u}(y)| \leq C r^{\frac{\lambda - m}{p}} [u]_{p,\lambda} .$$

□

[5]A similar result is also in [Giu03, §2.3], but it requires Ω to have no internal cusps instead. This cannot be correct, as the example we give in §3.4 is for a domain with no internal cusps.

2.3 Additional preliminary results

We now collect the other preliminary results we need to prove our main theorem.

First, we need the following analogue of lemma 1.1 [Gia83, §III, theorem 2.1].

Lemma 2.2. *Suppose $w \in H^1(B_r(x_0), \mathbb{R}^n)$ is a weak solution of*

$$\widetilde{A}^{ij}_{\alpha\beta}\partial_i\partial_j w^\alpha = 0 \qquad \text{in } B_r(x_0) \text{ for each } \beta,$$

where the $\widetilde{A}^{ij}_{\alpha\beta}$ are constants, symmetric in i, j and α, β, such that

$$\lambda|\xi|^2 \le \widetilde{A}^{ij}_{\alpha\beta}\xi^\alpha_i\xi^\beta_j \le \Lambda|\xi|^2 \qquad \forall \xi \in \mathbb{R}^{mn}$$

for some $0 < \lambda \le \Lambda$. Then there exists $C = C(\lambda, \Lambda)$ such that

$$\|w\|^2_{L^2(B_\rho(x_0))} \le C\left(\frac{\rho}{r}\right)^m \|w\|^2_{L^2(B_r(x_0))},$$

$$\|w - \overline{w}_{x_0,\rho}\|^2_{L^2(B_\rho(x_0))} \le C\left(\frac{\rho}{r}\right)^{m+2} \|w - \overline{w}_{x_0,r}\|^2_{L^2(B_r(x_0))},$$

for each $0 < \rho \le r$.

Finally, we will need the following technical lemma (special case of [HL11, Lemma 3.4]).[6]

Lemma 2.3. *Given constants $A, B, \alpha, \beta \ge 0$ with $\beta < \alpha$, there exist constants $\varepsilon > 0$, $C \ge 0$ depending only on A, α, β, such that if $\varphi : [0, R] \to [0, \infty)$ is an increasing function, and*

$$\varphi(\rho) \le A\left(\left(\frac{\rho}{r}\right)^\alpha + \varepsilon\right)\varphi(r) + Br^\beta$$

for each $0 < \rho \le r \le R$, then

$$\varphi(r) \le C\left(\left(\frac{r}{R}\right)^\beta \varphi(R) + Br^\beta\right)$$

for each $0 < r \le R$.

Proof. Note that for $\tau \in (0, 1)$, $r \le R$, we have that

$$\varphi(\tau r) \le A\left(\tau^\alpha + \varepsilon\right)\varphi(r) + Br^\beta$$

Without loss of generality, assume that $2A > 1$, so we can take $\tau \in (0, 1)$ such that $\tau^{-(\alpha-\gamma)} = 2A$, where $\gamma := \frac{\alpha+\beta}{2}$. Assume $\varepsilon \le \tau^\alpha$.

Then

$$\varphi(\tau r) \le 2A\tau^\alpha\varphi(r) + Br^\beta = \tau^\gamma\varphi(r) + Br^\beta.$$

For $k \in \mathbb{N}$, taking $r = \tau^k R$ gives

$$\varphi(\tau^{k+1}R) \le \tau^\gamma\varphi(\tau^k R) + B\tau^{k\beta}R^\beta,$$

[6] Note that [HL11, Lemma 3.4] requires the exponent γ of $\frac{r}{R}$ in the conclusion to be different to the exponent β of r, with $\gamma > \beta$. In the subsequent proof of the analogue to our theorem 2.4, the text requires $\gamma = \beta$, so we have amended the proof of this lemma to allow this.

so that

$$\begin{aligned}
\varphi(\tau^k R) &\le \tau^{k\gamma}\varphi(R) + BR^\beta \sum_{j=0}^{k-1} \tau^{j\gamma+(k-1-j)\beta} \\
&= \tau^{k\gamma}\varphi(R) + B(\tau^{k-1}R)^\beta \sum_{j=0}^{k-1} \tau^{j(\gamma-\beta)} \\
&\le \tau^{k\gamma}\varphi(R) + \frac{1}{1-\tau^{\gamma-\beta}} \cdot B(\tau^{k-1}R)^\beta \\
&= \tau^{k\gamma}\varphi(R) + \frac{2A}{2A-1} \cdot B(\tau^{k-1}R)^\beta,
\end{aligned}$$

where we used $\gamma - \beta = \alpha - \gamma$ and $\tau^{-(\alpha-\gamma)} = 2A$ for the final step.

Now given any $0 < r \le R$, choose k such that $\tau^{k+1} \le \frac{r}{R} < \tau^k$. Then the above gives

$$\begin{aligned}
\varphi(r) &\le \frac{1}{\tau^\gamma}\left(\frac{r}{R}\right)^\gamma \varphi(R) + \frac{1}{\tau^{2\beta}}\frac{2A}{2A-1} \cdot Br^\beta \\
&\le \frac{1}{\tau^{2\beta}}\left(\left(\frac{r}{R}\right)^\beta \varphi(R) + Br^\beta\right),
\end{aligned}$$

so the result follows with $\varepsilon = \tau^\alpha$, $C = \tau^{-2\beta}$.

□

2.4 The main theorem

We now give our adaptation of [HL11, theorem 3.8], the main theorem of this section. We begin by assuming that the coefficients $A^{ij}_{\alpha\beta}$ are uniformly continuous, so we can find a "modulus of continuity"[7] $\tau : [0,\infty) \to [0,\infty)$ such that for each i, j, α, β,

$$|A^{ij}_{\alpha\beta}(x) - A^{ij}_{\alpha\beta}(y)| \le \tau(\delta) \quad \text{whenever} \quad |x-y| < \delta. \tag{2.4}$$

Theorem 2.4. *Suppose $\Omega \subseteq \mathbb{R}^m$ is bounded, and u satisfies (2.2), where the $A^{ij}_{\alpha\beta} \in C^0(\Omega, \mathbb{R}^n)$ are bounded, uniformly elliptic, and uniformly continuous, $f \in L^q(\Omega, \mathbb{R}^n)$ for some $q \in (\frac{m}{2}, m)$, and $g \in L^{q'}(\Omega, \mathbb{R}^{mn})$ for some $q' > m$.*

Then $u \in C^{0,\alpha}_{loc}(\Omega)$ for $\alpha = \min\left\{2 - \frac{m}{q}, 1 - \frac{m}{q}\right\}$, and for each $\widetilde{\Omega} \Subset \Omega$ there exists C depending only on λ, Λ, m, τ, q, q', $\operatorname{dist}(\widetilde{\Omega}, \partial\Omega)$ such that[8]

$$\|u\|_{C^{0,\alpha}(\widetilde{\Omega})} \le C\left(\|u\|_{H^1(\Omega)} + \|f\|_{L^q(\Omega)}\|g\|_{L^{q'}(\Omega)}\right),$$

where τ is as defined in (2.4).

Proof. Fix x_0, r such that $B_r(x_0) \subseteq \Omega$, and write $\widetilde{A}^{ij}_{\alpha\beta} := A^{ij}_{\alpha\beta}(x_0) \in \mathbb{R}$ for each i, j, α, β. Then there is a unique solution $w \in H^1(B_r(x_0), \mathbb{R}^n)$ to the "frozen" equation

$$\int_{B_r(x_0)} \widetilde{A}^{ij}_{\alpha\beta}\partial_i w^\alpha \partial_j \varphi^\beta = 0 \qquad \forall \varphi \in H^1_0(B_r(x_0), \mathbb{R}^n) \tag{2.5}$$

[7] This definition is slightly different to that taken in [HL11], but ensures that τ is increasing, which we will need later.

[8] In [HL11], the dependence on m, q is not mentioned, however it does follow from the given proof.

with $u - w \in H_0^1(B_r(x_0), \mathbb{R}^n)$ (see [Gia83, §I, thoerem 3.1]).

Note that if w solves (2.5), then so does Dw, so writing $v = u - w$ and using lemma 2.2 gives that

$$\begin{aligned}
\|Du\|^2_{L^2(B_\rho(x_0))} &\leq 2\|Dw\|^2_{L^2(B_\rho(x_0))} + 2\|Dv\|^2_{L^2(B_\rho(x_0))} \\
&\leq C\left(\frac{\rho}{r}\right)^m \|Dw\|^2_{B_r(x_0)} + 2\|Dv\|^2_{L^2(B_\rho(x_0))} \\
&\leq C\left(\frac{\rho}{r}\right)^m \left(2\|Du\|^2_{B_r(x_0)} + 2\|Dv\|^2_{B_r(x_0)}\right) + 2\|Dv\|^2_{L^2(B_\rho(x_0))} \\
&\leq C\left(\left(\frac{\rho}{r}\right)^m \|Du\|^2_{L^2(B_r(x_0))} + \|Dv\|^2_{L^2(B_r(x_0))}\right), \qquad (2.6)
\end{aligned}$$

where C depends on λ, Λ. So we want to estimate $\|Dv\|^2_{L^2(B_r(x_0))}$.

Note that given $\varphi \in H_0^1(B_r(x_0), \mathbb{R}^n)$, we have from (2.2) and (2.5) that

$$\begin{aligned}
\int_{B_r(x_0)} \widetilde{A}^{ij}_{\alpha\beta}\partial_i v^\alpha \partial_j \varphi^\beta &= \int_{B_r(x_0)} \widetilde{A}^{ij}_{\alpha\beta}\partial_i u^\alpha \partial_j \varphi^\beta \\
&= \int_{B_r(x_0)} \left(\left(A^{ij}_{\alpha\beta} - \widetilde{A}^{ij}_{\alpha\beta}\right)\partial_i u^\alpha \partial_j \varphi^\beta + f_\beta \varphi^\beta + g^j_\beta \partial_j \varphi^\beta\right).
\end{aligned}$$

Now $v \in H_0^1(B_r(x_0), \mathbb{R}^n)$, so we can set $\varphi = v$ and use the fact that τ is increasing to give

$$\begin{aligned}
\int_{B_r(x_0)} \widetilde{A}^{ij}_{\alpha\beta}\partial_i v^\alpha \partial_j v^\beta &= \int_{B_r(x_0)} \left(\left(A^{ij}_{\alpha\beta} - \widetilde{A}^{ij}_{\alpha\beta}\right)\partial_i u^\alpha \partial_j v^\beta + f_\beta v^\beta + g^j_\beta \partial_j v^\beta\right) \\
&\leq \int_{B_r(x_0)} \left(\tau(r)\partial_i u^\alpha \partial_j v^\beta + f_\beta v^\beta + g^j_\beta \partial_j v^\beta\right) \\
&\leq C(m)\Big(\tau(r)\|Du\|_{L^2(B_r(x_0))}\|Dv\|_{L^2(B_r(x_0))} \\
&\qquad + \|f\|_{L^\gamma(B_r(x_0))}\|v\|_{L^{2^*}(B_r(x_0))} + \|g\|_{L^2(B_r(x_0))}\|Dv\|_{L^2(B_r(x_0))}\Big), \qquad (2.7)
\end{aligned}$$

where $\gamma = \frac{2m}{m+2}$, $2^* = \frac{2m}{m-2}$. We chose these exponents for the second term because 2^* is the largest possible exponent in the Sobolev inequality $\|v\|_{L^{2^*}} \leq C\|Dv\|_{L^2}$, which gives the smallest possible exponent γ for f, and hence the largest possible α (once g is taken into account), which we can see from the following calculation.

$$\begin{aligned}
\|f\|^2_{L^\gamma(B_r(x_0))} &= \left(\int_{B_r(x_0)} \mathbf{1}\cdot|f|^\gamma\right)^{\frac{2}{\gamma}} \\
&\leq \left(\left(\int_{B_r(x_0)} \mathbf{1}^{\frac{q}{q-\gamma}}\right)^{1-\frac{\gamma}{q}} \cdot \left(\int_{B_r(x_0)} (|f|^\gamma)^{\frac{q}{\gamma}}\right)^{\frac{\gamma}{q}}\right)^{\frac{2}{\gamma}} \\
&\leq C(m)\left(r^m\right)^{2\left(\frac{1}{\gamma}-\frac{1}{q}\right)}\left(\int_\Omega |f|^q\right)^{\frac{2}{q}} \\
&= C(m)r^{m-2+2\left(2-\frac{m}{q}\right)}\|f\|^2_{L^q(\Omega)}. \qquad (2.8)
\end{aligned}$$

Similarly,

$$\|g\|^2_{L^\gamma(B_r(x_0))} \leq C(m)r^{m-2+2\left(1-\frac{m}{q'}\right)}\|g\|^2_{L^{q'}(\Omega)}. \qquad (2.9)$$

So returning to (2.7), using the Sobolev inequality mentioned above and uniform ellipticity, and then substituting (2.8), (2.9) with $\alpha = \min\left\{2 - \frac{m}{q}, 1 - \frac{m}{q}\right\}$ gives

$$\begin{aligned}
&\int_{B_r(x_0)} \lambda |Dv|^2 \leq C\left(\tau(r)\left\|Du\right\|_{L^2(B_r(x_0))} + \|f\|_{L^\gamma(B_r(x_0))} + \|g\|_{L^2(B_r(x_0))}\right)\|Dv\|_{L^2(B_r(x_0))}\\
&\|Dv\|_{L^2(B_r(x_0))} \leq C\left(\tau(r)\left\|Du\right\|_{L^2(B_r(x_0))} + \|f\|_{L^\gamma(B_r(x_0))} + \|g\|_{L^2(B_r(x_0))}\right)\\
&\|Dv\|^2_{L^2(B_r(x_0))} \leq C\left(\tau(r)\left\|Du\right\|^2_{L^2(B_r(x_0))} + r^{m-2+2\alpha}\left(\|f\|^2_{L^q(\Omega)} + \|g\|^2_{L^{q'}(\Omega)}\right)\right).
\end{aligned}$$

Now we can substitute this into (2.6) to get

$$\|Du\|^2_{L^2(B_\rho(x_0))} \leq C(\lambda, \Lambda, m)\left(\left(\left(\frac{\rho}{r}\right)^m + \tau(r)\right)\|Du\|^2_{L^2(B_r(x_0))} + r^{m-2+2\alpha}F^2\right), \tag{2.10}$$

where $F := \|f\|^2_{L^q(\Omega)} + \|g\|^2_{L^{q'}(\Omega)}$

Finally, we apply lemma 2.3 with $\varphi(t) = \|Du\|^2_{L^2(B_t(x_0))}$. Note that if $x_0 \in \widetilde{\Omega} \Subset \Omega$ then $B_t(x_0) \subseteq \Omega$ for each $t \in [0, R]$ provided $R \leq \operatorname{dist}(\widetilde{\Omega}, \partial\Omega)$. Then we get C, ε depending on[9] $C(\lambda, \Lambda)$, m, $m - 2 + 2\alpha$ (that is depending on λ, Λ, m, q), such that if $\tau(R) < \varepsilon$ then

$$\|Du\|^2_{L^2(B_r(x_0))} \leq Cr^{m-2+2\alpha}\left(\frac{1}{R^{m-2+2\alpha}}\|Du\|^2_{L^2(\Omega)} + F^2\right). \tag{2.11}$$

Hence if we take R_0 such that $R_0 \leq \operatorname{dist}(\widetilde{\Omega}, \partial\Omega)$, $\tau(R_0) < \varepsilon$, then for each $r < R_0$ we get

$$\|Du\|^2_{L^2(B_r(x_0))} \leq C(\lambda, \Lambda, m, \tau, q, \operatorname{dist}(\widetilde{\Omega}, \partial\Omega))r^{m-2+2\alpha}\left(\|Du\|^2_{L^2(\Omega)} + F^2\right).$$

Now for $x_0 \in \widetilde{\Omega}$, take $r_0 > 0$ such that $B_{r_0}(x_0) \subseteq \widetilde{\Omega}$. Then for each $r < r_0$, we have $\widetilde{\Omega}_r(x_0) = B_r(x_0)$, so we may use the Poincaré-Wirtinger inequality

$$\|u - \overline{u}\|^2_{L^2(B_r(x_0))} \leq C(m)r^2\|Du\|^2_{L^2(B_r(x_0))}$$

(see [Eva10, §5.8.1]) to give

$$\begin{aligned}
r^{-(m+2\alpha)}\|u - \overline{u}_{x_0,r}\|^2_{L^2(\widetilde{\Omega}_r(x_0))} &\leq Cr^{-(m+2\alpha)} \cdot r^2\|Du\|^2_{L^2(B_r(x_0))}\\
&= Cr^{-(m-2+2\alpha)}\|Du\|^2_{L^2(B(x_0,r))}\\
&\leq C(\lambda, \Lambda, m, \tau, q, \operatorname{dist}(\widetilde{\Omega}, \partial\Omega))\left(\|Du\|^2_{L^2(\Omega)} + F^2\right).
\end{aligned}$$

So $u \in \mathcal{L}^{2,m+2\alpha}(\widetilde{\Omega})$, with

$$\|u\|_{\mathcal{L}^{2,m+2\alpha}(\widetilde{\Omega})} = [u]_{2,m+2\alpha} + \|u\|_{L^2(\widetilde{\Omega})} \leq C(\lambda, \Lambda, m, \tau, q, \operatorname{dist}(\widetilde{\Omega}, \partial\Omega))\left(\|u\|_{H^1(\Omega)} + F\right). \tag{2.12}$$

However, we need $\widetilde{\Omega}$ to have no external cusps to apply theorem 2.1.

We add our own note about how to handle this. Write $\delta = \frac{1}{2}\operatorname{dist}(\widetilde{\Omega}, \partial\Omega)$, and take

$$\widehat{\Omega} = B_\delta(\widetilde{\Omega}) := \{x \in \mathbb{R}^m : \operatorname{dist}(x, \widetilde{\Omega}) < \delta\}.$$

[9] Note that this is the only point in this proof which gives dependence on q.

Then $\operatorname{dist}(\widehat{\Omega}, \partial\Omega) = \frac{1}{2}\operatorname{dist}(\widetilde{\Omega}, \partial\Omega)$, and if $x \in \widehat{\Omega}$ then there is $\widetilde{x} \in \widetilde{\Omega}$ such that $x \in B_\delta(\widetilde{x}) \subseteq \widehat{\Omega}$. Now

$$|\widehat{\Omega}_r(x)| = |\widehat{\Omega} \cap B_r(x)| \geq |B_\delta(\widetilde{x}) \cap B_r(x)| \geq 2^{-m}|B_r(x)|$$

for $0 < r \leq 2\delta$, and

$$|\widehat{\Omega}_r(x)| \geq |B_\delta(\widetilde{x}) \cap B_r(x)| \geq |B_\delta(\widetilde{x})| \geq \delta^m r^{-m}|B_r(x)|$$

for $2\delta < r \leq 1$, so taking $A = \min\{2^{-m}, \delta^m\}$ gives that $\widehat{\Omega}$ has no external cusps.

Now $\mathcal{L}^{2,m+2\alpha}(\widetilde{\Omega}) \subseteq \mathcal{L}^{2,m+2\alpha}(\widehat{\Omega})$, so using theorem 2.1, and applying (2.12) to $\widehat{\Omega}$, gives that $u \in C^{0,\alpha}(\widetilde{\Omega})$, with

$$\begin{aligned}
\|u\|_{C^{0,\alpha}(\widetilde{\Omega})} &\leq C(\delta, \alpha, m)\,\|u\|_{\mathcal{L}^{2,m+2\alpha}(\widetilde{\Omega})} \\
&= C(\operatorname{dist}(\widetilde{\Omega}, \partial\Omega), m, q)\,\|u\|_{\mathcal{L}^{2,m+2\alpha}(\widehat{\Omega})} \\
&\leq C(\lambda, \Lambda, m, \tau, q, \operatorname{dist}(\widetilde{\Omega}, \partial\Omega))\left(\|u\|_{H^1(\Omega)} + F\right).
\end{aligned}$$

Recalling that $F = \|f\|^2_{L^q(\Omega)} + \|g\|^2_{L^{q'}(\Omega)}$ gives the result. □

This theorem does not quite match theorem 1.2, as we are requiring the coefficients $A^{ij}_{\alpha\beta}$ to be uniformly continuous. However, we add our own note that if the $A^{ij}_{\alpha\beta}$ are continuous on Ω and we fix some $\widetilde{\Omega} \Subset \Omega$, then we can find $\widehat{\Omega}$ such that $\widetilde{\Omega} \Subset \widehat{\Omega} \Subset \Omega$. Then the $A^{ij}_{\alpha\beta}$ are uniformly continuous in $\widehat{\Omega}$, so we can apply theorem 2.4 in $\widehat{\Omega}$ to see that

$$\|u\|_{C^{0,\alpha}(\widetilde{\Omega})} \leq C\left(\|u\|_{H^1(\Omega)} + \|f\|_{L^q(\Omega)} + \|g\|^2_{L^{q'}(\Omega)}\right).$$

This gives the following result.

Theorem 2.5. *Suppose the conditions of theorem 2.4 hold with the $A^{ij}_{\alpha\beta} \in C^0(\Omega, \mathbb{R}^n)$ not necessarily uniformly continuous. Then the conclusion of theorem 2.4 also holds, with the constant C now depending on the $A^{ij}_{\alpha\beta}$, m, q, and* $\operatorname{dist}(\widetilde{\Omega}, \partial\Omega)$.

We may also obtain regularity results for the derivatives of u in the same way as in §1.2. Note that in theorem 1.3 we require $a_{ij} \in C^{0,\alpha}(\overline{\Omega})$. The argument used to obtain theorem 2.5 can be used here to replace this with the condition $a_{ij} \in C^{0,\alpha}_{loc}(\Omega)$.

2.5 Boundary regularity

Theorem 2.4 only gives interior regularity, so we give our own exploration of how to adapt this proof to get boundary regularity.

In the proof, we only restrict x_0 to lie in $\widetilde{\Omega} \Subset \Omega$ because we need $R > 0$ such that $B_R(x_0) \subseteq \Omega$ for each x_0. In the context of boundary regularity, it is natural to replace $B_r(x_0)$ with $\Omega_r(x_0)$ to avoid this issue. This leads us to ask whether (with suitable assumptions on Ω) we can get an analogous result to lemma 2.2 with $B_\rho(x_0)$, $B_r(x_0)$ replaced with $\Omega_\rho(x_0)$, $\Omega_r(x_0)$.

Also, when using this lemma in our proof, we take $w = u$ on $\partial\Omega$, where $u \in H^1_0(\Omega)$, so we can restrict to the case when $w = 0$ on $\partial\Omega$. Hence we need the following analogue of lemma 2.2.

Remark 2.3. Let $\Omega \subseteq \mathbb{R}^m$ be a Lipschitz domain, and $\widetilde{A}^{ij}_{\alpha\beta}$ be constants, symmetric in i, j and α, β, such that

$$\lambda|\xi|^2 \leq \widetilde{A}^{ij}_{\alpha\beta}\xi^\alpha_i\xi^\beta_j \leq \Lambda|\xi|^2 \qquad \forall \xi \in \mathbb{R}^{mn}$$

for some $0 < \lambda \leq \Lambda$. Let $w \in H^1(\Omega_r(x_0), \mathbb{R}^n)$ be a weak solution of

$$\widetilde{A}^{ij}_{\alpha\beta}\partial_{ij}w^\alpha = 0 \qquad \text{for each } \beta \tag{2.13}$$

in $\Omega_r(x_0)$, with $u = 0$ on $\partial\Omega$ in the sense of trace. Then it is reasonable to expect[10] that there exists $C = C(\lambda, \Lambda, \Omega)$ such that

$$\|Dw\|^2_{L^2(\Omega_\rho(x_0))} \leq C\left(\frac{\rho}{r}\right)^m \|Dw\|^2_{L^2(\Omega_r(x_0))}$$

for each $0 < \rho \leq r \leq \operatorname{diam}(\Omega)$.

Unfortunately, we did not have time to obtain a proof of this.

It may be the case that this is true for more general domains (using a suitable notion of "$w = 0$ on $\partial\Omega$"). In that case, we would probably need to impose a cusp condition. However, when applying this to boundary regularity, we wish to apply theorem 2.1 to obtain Hölder continuity, which already has a cusp condition which we need to impose.

Theorem 2.6. *Let $\Omega \subseteq \mathbb{R}^m$ be such that the conclusion of remark 2.3 holds. Suppose u satisfies* (2.2)*, where the $A^{ij}_{\alpha\beta} \in C^0(\Omega, \mathbb{R}^n)$ are bounded and uniformly continuous, τ is as defined in* (2.4)*, and $f \in L^q(\Omega, \mathbb{R}^n)$ for some $q \in (\frac{m}{2}, m)$.*

Then $u \in \mathcal{L}^{2,m+2\alpha}(\widetilde{\Omega})$, where $\alpha = 2 - \frac{m}{q}$, and there exists $C(\lambda, \Lambda, \Omega, m, \tau, q)$ such that

$$\|u\|_{\mathcal{L}^{2,m+2\alpha}(\widetilde{\Omega})} \leq C\left(\|u\|_{H^1(\Omega)} + \|f\|_{L^q(\Omega)}\right).$$

Proof. This time, we fix $x_0 \in \Omega$ and some $r > 0$. If we set $\widetilde{A}^{ij}_{\alpha\beta} := A^{ij}_{\alpha\beta}(x_0)$, then as before we have a unique solution $w \in H^1(\Omega_r(x_0), \mathbb{R}^n)$ to

$$\int_{\Omega_r(x_0)} \widetilde{A}^{ij}_{\alpha\beta}\partial_i w^\alpha \partial_j \varphi^\beta = 0 \qquad \forall \varphi \in H^1_0(\Omega_r(x_0), \mathbb{R}^n)$$

with $u - w \in H^1_0(\Omega_r(x_0), \mathbb{R}^n)$.

Further, since $u \in H^1_0(\Omega)$, $u - w \in H^1_0(\Omega_r(x_0))$, both u and $u - w$ are in the H^1 closure of the set

$$\{u \in C^\infty(\Omega_r(x_0)) : \operatorname{dist}(\operatorname{supp} \mathrm{u}, \partial\Omega) > 0\},$$

so w is also in this set, and we can use remark 2.3.

From here, we continue as in the proof of theorem 2.4 with $B_\rho(x_0), B_r(x_0)$ replaced with $\Omega_\rho(x_0), \Omega_r(x_0)$, until we get an analogue of (2.10), that is

$$\|Du\|^2_{L^2(\Omega_\rho(x_0))} \leq C(\lambda, \Lambda, \Omega, n)\left(\left(\tau(r) + \left(\frac{\rho}{r}\right)^m\right)\|Du\|^2_{L^2(\Omega_r(x_0))} + r^{m-2+2\alpha}\|f\|^2_{L^q(\Omega)}\right).$$

[10] [Gia83, §VIII.2] mentions that if $\partial\Omega$ is smooth, then the method used to prove theorem 2.4 "can be straightforwardly extended up to the boundary", citing [Cam65]. This paper is in Italian so we were unable to read it, but it makes it seem plausible that this holds in this case. Certainly the result can be obtained when $\Omega_r(x_0)$ is a half-ball by taking an odd extension. From here we could proceed by "flattening the boundary", as described in [Eva10, §C.1].

Again, we apply lemma 2.3, this time setting $\varphi(t) = \|Du\|^2_{L^2(\Omega_t(x_0))}$. Again, this gives C, ε depending on $C(\lambda, \Lambda)$, m, $m-2+2\alpha$ (that is, on λ, Λ, m, q), such that if $\tau(R) < \varepsilon$ then

$$\|Du\|^2_{L^2(B_r(x_0))} \leq Cr^{m-2+2\alpha} \left(\frac{1}{R^{m-2+2\alpha}} \|Du\|^2_{L^2(\Omega)} + \|f\|^2_{L^q(\Omega)} \right).$$

So we can take R_0 such that $\tau(R_0) < \varepsilon_0$Then for $r < R_0$ we get

$$\|Du\|^2_{L^2(B_r(x_0))} \leq C(\lambda, \Lambda, \Omega, m, \tau, q) r^{m-2+2\alpha} \left(\|Du\|^2_{L^2(\Omega)} + \|f\|^2_{L^q(\Omega)} \right).$$

Finally, since $u \in H^1_0(\Omega_r(x_0))$, its extension by 0 is in $H^1_0(B_r(x_0))$. Hence we can apply the Poincaré-Wirtinger inequality from [Eva10, §5.8.1] as before to conclude that $u \in \mathcal{L}^{2,m+2\alpha}(\widetilde{\Omega})$, with

$$\|u\|_{\mathcal{L}^{2,m+2\alpha}(\widetilde{\Omega})} \leq C(\lambda, \Lambda, \Omega, m, \tau, q) \left(\|u\|_{H^1(\Omega)} + \|f\|_{L^q(\Omega)} \right).$$

□

If $\Omega \subseteq \mathbb{R}^m$ is Lipschitz and bounded, then it has no external cusps. Hence if the conditions of theorem 2.6 hold, then theorem 2.1 gives that $u \in C^{0,\alpha}(\Omega)$ for $\alpha = 2 - \frac{m}{q}$, and

$$\|u\|_{C^{0,\alpha}(\Omega)} \leq C \left(\|u\|_{H^1(\Omega)} + \|f\|_{L^q(\Omega)} \right),$$

so we recover an analogue of theorem 2.4.

3 Counterexamples

We devote this section to examples which show that certain assumptions cannot be dropped from the theorems we have seen in §1 and 2.

In all calculations, sums will be shown explicitly and lower indices will be used to avoid confusion.

3.1 A system with bounded coefficients

Theorem 1.4 gives that elliptic equations with bounded coefficients have locally bounded solutions, but this is not true for systems, as can be seen from an example due to De Giorgi [DG68]. We found this example in [Gia83, §2.3], where only the system and associated solution are given, so we present our own calculations to explain the reasoning behind it.

We aim to find a system which has

$$u : B_1 \to \mathbb{R}^m, \qquad u(x) = \frac{x}{|x|^\gamma}$$

as a solution, where $\gamma \geq 1$ is to be determined and $B_1 \subseteq \mathbb{R}^m$. Note that u is discontinuous, and is unbounded when $\gamma > 1$.

We will need the following lemma ([Gia83, §II, lemma 3.1], special case).

Lemma 3.1. *Suppose $\Omega \subseteq \mathbb{R}^m$ is bounded, $m \geq 2$, and $u \in H^1(\Omega) \cap C^2(\Omega \backslash \{x_0\}, \mathbb{R}^n)$, where $x_0 \in \Omega$. Then if u is a classical solution of*

$$\sum_{i,j,\beta} \partial_i (A^{ij}_{\alpha\beta}(x, u(x))\, \partial_j u_\beta) = 0 \qquad \forall \alpha, \tag{3.1}$$

in $\Omega \backslash \{x_0\}$, and

$$A^{ij}_{\alpha\beta}(x, u(x)) \in L^\infty(\Omega) \cap C^1(\Omega \backslash \{x_0\})$$

for each i, j, α, β, then u is a weak solution of (3.1) *in Ω.*

This means we need $u \in H^1(B_1, \mathbb{R}^m)$, so we must take $\gamma < \frac{m}{2}$. Already this means that in order to have $\gamma \geq 1$ we must have $m \geq 3$.

We look for a system with coefficients $A^{ij}_{\alpha\beta}$ depending only on x. In that case, a classical solution to (3.1) must satisfy

$$\sum_{i,j,\beta} A^{ij}_{\alpha\beta}(x)\, \partial_{ij} u_\beta + \sum_{i,j,\beta} \partial_i A^{ij}_{\alpha\beta}(x)\, \partial_j u_\beta = 0 \qquad \forall \alpha.$$

We will need the following derivatives.

$$\partial_k \left(\frac{1}{|x|^c} \right) = -c \frac{x_k}{|x|^{c+2}}$$

$$\partial_j u_\beta = \partial_j \left(\frac{x_\beta}{|x|^\gamma} \right) = -\gamma \frac{x_j x_\beta}{|x|^{\gamma+2}} + \frac{1}{|x|^\gamma} \delta_{j\beta}$$

$$\partial_{ij} u_\beta = \gamma(\gamma+2) \frac{x_i x_j x_\beta}{|x|^{\gamma+4}} - \gamma \frac{x_\beta}{|x|^{\gamma+2}} \delta_{ij} - \gamma \frac{x_j}{|x|^{\gamma+2}} \delta_{i\beta} - \gamma \frac{x_i}{|x|^{\gamma+2}} \delta_{j\beta}$$

First, we note that if we take the usual Laplacian, that is if $A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta}$, then

$$\begin{aligned}\sum_{i,j,\beta}\partial_i(\delta_{ij}\delta_{\alpha\beta}\partial_j u_\beta) &= \sum_i \partial_i^2 u_\alpha \\ &= \sum_i\left(\gamma(\gamma+2)\frac{x_i^2 x_\alpha}{|x|^{\gamma+4}} - \gamma\frac{x_\alpha}{|x|^{\gamma+2}} - \gamma\frac{x_i}{|x|^{\gamma+2}}\delta_{i\alpha} - \gamma\frac{x_i}{|x|^{\gamma+2}}\delta_{i\alpha}\right) \\ &= \gamma(\gamma+2)\frac{|x|^2 x_\alpha}{|x|^{\gamma+4}} - m\gamma\frac{x_\alpha}{|x|^{\gamma+2}} - 2\gamma\frac{x_\alpha}{|x|^{\gamma+2}} \\ &= -\gamma(m-\gamma)\frac{x_\alpha}{|x|^{\gamma+2}}.\end{aligned}$$

Next, we wish to find an additional term which will cancel this out. The resulting operator must be elliptic and bounded, and a simple way to do this is to take $A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta} + a^i_\alpha a^j_\beta$ for some choice of bounded coefficients $a^i_\alpha(x)$, as this gives

$$\sum_{i,j,\alpha,\beta} A^{ij}_{\alpha\beta}\xi^i_\alpha\xi^j_\beta = \sum_{i,\alpha}\left(\xi^i_\alpha\right)^2 + \left(\sum_{i,\alpha} a^i_\alpha\xi^i_\alpha\right)^2 \geq \left(1 + \sup_{i,\alpha}\left\|a^i_\alpha\right\|_{L^\infty}\right)|\xi|^2.$$

A first guess for a^i_α is $\frac{x_\alpha}{|x|^c}$ for some $c > 0$, since

$$\partial_i\left(\frac{x_\alpha x_\beta}{|x|^c}\right) = -c\frac{x_i x_\alpha x_\beta}{|x|^{c+2}} + \frac{x_\beta}{|x|^c}\delta_{i\alpha} + \frac{x_\alpha}{|x|^c}\delta_{i\beta}.$$

However, multiplying this with $\partial_j u_\beta$ gives terms which are multiplied by a single copy of x_i, and will not cancel out easily.

So it is better to try $a^i_\alpha = \frac{x_i x_\alpha}{|x|^c}$. We need $c \leq 2$ for this to be bounded, and if $c < 2$ then it is continuous, so u cannot be a solution to the associated system by theorem 2.4. Hence we take $c = 2$, and calculate for $x \neq 0$

$$\begin{aligned}&\sum_{i,j,\beta}\frac{x_i x_j x_\alpha x_\beta}{|x|^4}\partial_{ij}u_\beta + \sum_{i,j,\beta}\partial_i\frac{x_i x_j x_\alpha x_\beta}{|x|^4}\partial_j u_\beta \\ &= \sum_{i,j,\beta}\frac{x_i x_j x_\alpha x_\beta}{|x|^4}\left(\gamma(\gamma+2)\frac{x_i x_j x_\beta}{|x|^{\gamma+4}} - \gamma\frac{x_\beta}{|x|^{\gamma+2}}\delta_{ij} - \gamma\frac{x_j}{|x|^{\gamma+2}}\delta_{i\beta} - \gamma\frac{x_i}{|x|^{\gamma+2}}\delta_{j\beta}\right) \\ &\quad + \sum_{i,j,\beta}\left(-4\frac{x_i^2 x_j x_\alpha x_\beta}{|x|^6} + \frac{x_j x_\alpha x_\beta}{|x|^4} + \frac{x_i x_\alpha x_\beta}{|x|^4}\delta_{ij} + \frac{x_i x_j x_\beta}{|x|^4}\delta_{i\alpha} + \frac{x_i x_j x_\alpha}{|x|^4}\delta_{i\beta}\right)\left(-\gamma\frac{x_j x_\beta}{|x|^{\gamma+2}} + \frac{1}{|x|^\gamma}\delta_{j\beta}\right) \\ &= \sum_{i,j,\beta}\frac{x_\alpha}{|x|^4}\left(\gamma(\gamma+2)\frac{x_i^2 x_j^2 x_\beta^2}{|x|^{\gamma+4}} - \gamma\frac{x_i^2 x_\beta^2}{|x|^{\gamma+2}}\delta_{ij} - \gamma\frac{x_i^2 x_j^2}{|x|^{\gamma+2}}\delta_{i\beta} - \gamma\frac{x_i^2 x_j^2}{|x|^{\gamma+2}}\delta_{j\beta}\right) \\ &\quad + \sum_{j,\beta}\frac{x_\alpha}{|x|^4}\left(-4\frac{|x|^2 x_j x_\beta}{|x|^2} + m x_j x_\beta + x_j x_\beta + x_\alpha x_j x_\beta + x_j x_\beta\right)\left(-\gamma\frac{x_j x_\beta}{|x|^{\gamma+2}} + \frac{1}{|x|^\gamma}\delta_{j\beta}\right) \\ &= \frac{x_\alpha}{|x|^{\gamma+4}}\left(\gamma(\gamma+2)\frac{|x|^6}{|x|^4} - \gamma\frac{|x|^4}{|x|^2} - \gamma\frac{|x|^4}{|x|^2} - \gamma\frac{|x|^4}{|x|^2}\right) \\ &\quad + \sum_{j,\beta}\frac{x_\alpha}{|x|^{\gamma+4}}\left(-4\frac{|x|^2 x_j x_\beta}{|x|^2} + (m+3)x_j x_\beta\right)\left(-\gamma\frac{x_j x_\beta}{|x|^2} + \delta_{j\beta}\right) \\ &= \frac{x_\alpha}{|x|^{\gamma+2}}(\gamma(\gamma-1) + (-4+m+3)(-\gamma+1)) \\ &= -(\gamma(m-\gamma) - (m-1))\frac{x_\alpha}{|x|^{\gamma+2}}.\end{aligned}$$

This suggests we take

$$A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta} - \frac{\gamma(m-\gamma)}{\gamma(m-\gamma)-(m-1)}\frac{x_i x_j x_\alpha x_\beta}{|x|^4}.$$

Unfortunately, when $1 \le \gamma < \frac{m}{2}$, we have

$$\frac{\gamma(\gamma-m)}{\gamma(\gamma-m)+m-1} \ge \frac{\frac{m}{2}\left(m-\frac{m}{2}\right)}{\frac{m}{2}\left(m-\frac{m}{2}\right)-m+1} = \left(\frac{m}{m-2}\right)^2 > 1.$$

So if

$$x = \left(\frac{1}{2}, 0, \dots, 0\right) \in B_1, \quad \xi^i_\alpha = \begin{cases} 1 & i=\alpha=1 \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\sum_{i,j,\alpha,\beta} A^{ij}_{\alpha\beta}\xi^i_\alpha\xi^j_\beta = A^{11}_{11} = 1 - \frac{\gamma(m-\gamma)}{\gamma(m-\gamma)-(m-1)}\cdot 1 < 0,$$

so the system is not elliptic.

However, we can continue to adjust the coefficients a^i_α chosen above. Note that

$$\partial_i\left(\frac{x_i x_\alpha}{|x|^2}\right) = -2\frac{x_i^2 x_\alpha}{|x|^4} + \frac{x_\alpha}{|x|^2} + \frac{x_i}{|x|^2}\delta_{i\alpha},$$

and that $\partial_j u_\beta \delta_{j\beta}$ collapses nicely once summed, that is

$$\sum_{j,\beta}\partial_j u_\beta \delta_{j\beta} = \sum_{j,\beta}\left(-\gamma\frac{x_j x_\beta}{|x|^{\gamma+2}} + \frac{1}{|x|^\gamma}\delta_{j\beta}\right)\delta_{j\beta} = (1-\gamma)\frac{1}{|x|^\gamma}.$$

This suggests that $\frac{x_i x_\alpha}{|x|^2}\delta_{j\beta}$ would also work well as a term of $A^{ij}_{\alpha\beta}$, so we try

$$A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta} + \left(C_1\delta_{i\alpha} + C_2\frac{x_i x_\alpha}{|x|^2}\right)\left(C_1\delta_{j\beta} + C_2\frac{x_j x_\beta}{|x|^2}\right).$$

Next we calculate

$$\begin{aligned}
&\sum_{i,j,\beta}\frac{x_i x_\alpha}{|x|^2}\delta_{j\beta}\partial_{ij}u_\beta + \sum_{i,j,\beta}\partial_i\frac{x_i x_\alpha}{|x|^2}\delta_{j\beta}\partial_j u_\beta \\
&= \sum_{i,j,\beta}\frac{x_i x_\alpha}{|x|^2}\delta_{j\beta}\left(\gamma(\gamma+2)\frac{x_i x_j x_\beta}{|x|^{\gamma+4}} - \gamma\frac{x_\beta}{|x|^{\gamma+2}}\delta_{ij} - \gamma\frac{x_j}{|x|^{\gamma+2}}\delta_{i\beta} - \gamma\frac{x_i}{|x|^{\gamma+2}}\delta_{j\beta}\right) \\
&\quad + \sum_{i,j,\beta}\left(-2\frac{x_i^2 x_\alpha}{|x|^4} + \frac{x_\alpha}{|x|^2} + \frac{x_i}{|x|^2}\delta_{i\alpha}\right)\delta_{j\beta}\left(-\gamma\frac{x_j x_\beta}{|x|^{\gamma+2}} + \frac{1}{|x|^\gamma}\delta_{j\beta}\right) \\
&= \sum_{i,\beta}\frac{x_\alpha}{|x|^2}\left(\gamma(\gamma+2)\frac{x_i^2 x_\beta^2}{|x|^{\gamma+4}} - \gamma\frac{x_i x_\beta}{|x|^{\gamma+2}}\delta_{i\beta} - \gamma\frac{x_i x_\beta}{|x|^{\gamma+2}}\delta_{i\beta} - \gamma\frac{x_i^2}{|x|^{\gamma+2}}\right) \\
&\quad + \sum_i\left(-2\frac{x_i^2 x_\alpha}{|x|^4} + \frac{x_\alpha}{|x|^2} + \frac{x_i}{|x|^2}\delta_{i\alpha}\right)\sum_{j,\beta}\left(-\gamma\frac{x_j x_\beta}{|x|^{\gamma+2}}\delta_{j\beta} + \frac{1}{|x|^\gamma}\delta_{j\beta}\right) \\
&= \sum_i\frac{x_\alpha}{|x|^2}\left(\gamma(\gamma+2)\frac{x_\beta^2}{|x|^{\gamma+2}} - 2\gamma\frac{x_\beta^2}{|x|^{\gamma+2}} - \gamma\frac{1}{|x|^\gamma}\right) \\
&\quad + \left(-2\frac{x_\alpha}{|x|^2} + m\frac{x_\alpha}{|x|^2} + \frac{x_\alpha}{|x|^2}\right)\sum_\beta\left(-\gamma\frac{x_\beta^2}{|x|^{\gamma+2}} + \frac{1}{|x|^\gamma}\right) \\
&= (-\gamma(m-\gamma) + (m-1)(m-\gamma))\frac{x_\alpha}{|x|^{\gamma+2}},
\end{aligned}$$

$$\begin{aligned}
&\sum_{i,j,\beta} \frac{x_j x_\beta}{|x|^2} \delta_{i\alpha} \partial_{ij} u_\beta + \sum_{i,j,\beta} \partial_i \frac{x_j x_\beta}{|x|^2} \delta_{i\alpha} \partial_j u_\beta \\
&= \sum_{i,j,\beta} \frac{x_j x_\beta}{|x|^2} \delta_{i\alpha} \left(\gamma(\gamma+2) \frac{x_i x_j x_\beta}{|x|^{\gamma+4}} - \gamma \frac{x_\beta}{|x|^{\gamma+2}} \delta_{ij} - \gamma \frac{x_j}{|x|^{\gamma+2}} \delta_{i\beta} - \gamma \frac{x_i}{|x|^{\gamma+2}} \delta_{j\beta} \right) \\
&\quad + \sum_{i,j,\beta} \left(-2 \frac{x_i x_j x_\beta}{|x|^4} + \frac{x_\beta}{|x|^2} \delta_{ij} + \frac{x_j}{|x|^2} \delta_{i\beta} \right) \delta_{i\alpha} \left(-\gamma \frac{x_j x_\beta}{|x|^{\gamma+2}} + \frac{1}{|x|^\gamma} \delta_{j\beta} \right) \\
&= \sum_{j,\beta} \frac{1}{|x|^2} \left(\gamma(\gamma+2) \frac{x_\alpha x_j^2 x_\beta^2}{|x|^{\gamma+4}} - \gamma \frac{x_j x_\beta^2}{|x|^{\gamma+2}} \delta_{j\alpha} - \gamma \frac{x_j^2 x_\beta}{|x|^{\gamma+2}} \delta_{\alpha\beta} - \gamma \frac{x_\alpha x_j x_\beta}{|x|^{\gamma+2}} \delta_{j\beta} \right) \\
&\quad + \sum_{j,\beta} \frac{1}{|x|^2} \left(-2 \frac{x_\alpha x_j x_\beta}{|x|^2} + \frac{x_\beta}{|x|^2} \delta_{j\alpha} + \frac{x_j}{|x|^2} \delta_{\alpha\beta} \right) \left(-\gamma \frac{x_j x_\beta}{|x|^{\gamma+2}} + \frac{1}{|x|^\gamma} \delta_{j\beta} \right) \\
&= \sum_{\beta} \frac{1}{|x|^2} \left(\gamma(\gamma+2) \frac{x_\alpha x_\beta^2}{|x|^{\gamma+2}} - \gamma \frac{x_\alpha x_\beta^2}{|x|^{\gamma+2}} - \gamma \frac{x_\beta}{|x|^\gamma} \delta_{\alpha\beta} - \gamma \frac{x_\alpha x_\beta^2}{|x|^{\gamma+2}} \right) \\
&\quad + \sum_{\beta} \frac{1}{|x|^2} \left(2\gamma \frac{x_\alpha x_\beta^2}{|x|^{\gamma+2}} - \gamma \frac{x_\alpha x_\beta^2}{|x|^{\gamma+4}} - \gamma \frac{x_\beta}{|x|^{\gamma+2}} \delta_{\alpha\beta} - 2 \frac{x_\alpha x_\beta^2}{|x|^{\gamma+2}} + \frac{x_\beta}{|x|^{\gamma+2}} \delta_{\alpha\beta} + \frac{x_\beta}{|x|^{\gamma+2}} \delta_{\alpha\beta} \right) \\
&= (\gamma(\gamma+2) - \gamma - \gamma - \gamma) \frac{x_\alpha}{|x|^{\gamma+2}} + (2\gamma - \gamma - \gamma - 2 + 1 + 1) \frac{x_\alpha}{|x|^{\gamma+2}} \\
&= \gamma(\gamma-1) \frac{x_\alpha}{|x|^{\gamma+2}},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i,j,\beta} \delta_{i\alpha} \delta_{j\beta} \partial_{ij} u_\beta + \sum_{i,j,\beta} \partial_i \delta_{i\alpha} \delta_{j\beta} \partial_j u_\beta \\
&= \sum_{i,j,\beta} \delta_{i\alpha} \delta_{j\beta} \left(\gamma(\gamma+2) \frac{x_i x_j x_\beta}{|x|^{\gamma+4}} - \gamma \frac{x_\beta}{|x|^{\gamma+2}} \delta_{ij} - \gamma \frac{x_j}{|x|^{\gamma+2}} \delta_{i\beta} - \gamma \frac{x_i}{|x|^{\gamma+2}} \delta_{j\beta} \right) + 0 \\
&= \sum_{j} \left(\gamma(\gamma+2) \frac{x_\alpha x_j^2}{|x|^{\gamma+4}} - \gamma \frac{x_j}{|x|^{\gamma+2}} \delta_{j\alpha} - \gamma \frac{x_j}{|x|^{\gamma+2}} \delta_{j\alpha} - \gamma \frac{x_\alpha}{|x|^{\gamma+2}} \right) \\
&= (\gamma(\gamma+2) - \gamma - \gamma - m\gamma) \frac{x_\alpha}{|x|^{\gamma+2}} \\
&= -\gamma(m-\gamma) \frac{x_\alpha}{|x|^{\gamma+2}}.
\end{aligned}$$

So if $A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta} + \left(C_1 \delta_{i\alpha} + C_2 \frac{x_i x_\alpha}{|x|^2}\right)\left(C_1 \delta_{j\beta} + C_2 \frac{x_j x_\beta}{|x|^2}\right)$, then

$$\begin{aligned}
\sum_{i,j,\beta} \partial_i (A^{ij}_{\alpha\beta}(x) \partial_j u_\beta) = \Big(&-\gamma(m-\gamma) - C_1^2 \gamma(m-\gamma) + C_1 C_2 (-\gamma(m-\gamma) + (m-1)(m-\gamma)) \\
&+ C_1 C_2 \gamma(\gamma-1) - C_2^2(\gamma(m-\gamma) - (m-1)) \Big) \frac{x_\alpha}{|x|^{\gamma+2}}.
\end{aligned}$$

Hence, if the term in the brackets is zero, then lemma 3.1 gives that $u(x) = \frac{x}{|x|^\gamma}$ is a weak solution to our system on B_1. Provided $C_1, C_2 > 0$,the system will be elliptic.

Rewriting this condition gives

$$\begin{aligned}0 &= -\gamma(m-\gamma)(1+C_1^2+C_1C_2+C_2^2)+C_1C_2(m^2-m-m\gamma+\gamma+\gamma^2-\gamma)+C_2^2(m-1)\\ &= -\gamma(m-\gamma)(1+C_1^2+C_1C_2+C_2^2)+C_1C_2(m(m-1)-\gamma(m-\gamma))+C_2^2(m-1)\\ &= -\gamma(m-\gamma)(1+(C_1+C_2)^2)+C_2(m-1)(C_1m+C_2).\end{aligned}$$

We write $\gamma = \frac{m}{2}(1-d)$ for $d > 0$, giving

$$\begin{aligned}\frac{m^2}{4}(1-d)(1+d)(1+(C_1+C_2)^2) &= C_2(m-1)(C_1m+C_2)\\ m^2(1-d^2)(1+(C_1+C_2)^2) &= 4C_2(m-1)(C_1m+C_2)\\ m^2d^2(1+(C_1+C_2)^2) &= m^2(1+(C_1+C_2)^2)-4C_2(m-1)(C_1m+C_2)\\ d^2(1+(C_1+C_2)^2) &= 1+m^{-2}(C_1m-C_2(m-2))^2,\end{aligned}$$

so

$$d = \sqrt{\frac{1+(C_1-\frac{m-2}{m}C_2)^2}{1+(C_1+C_2)^2}}.$$

By varying C_1, C_2, we can obtain any value of $d \in (0,1)$, and hence any $\gamma \in (0, \frac{m}{2})$. Therefore, any function $u \in H^1(B_1, \mathbb{R}^m)$ of the form $u(x) = x|x|^{-\gamma}, \gamma \in (1, \frac{m}{2})$, is a solution to an elliptic system with bounded coefficients. In particular, taking $C_1 = m-2$, $C_2 = m$ gives De Giorgi's example, that is the elliptic system

$$\sum_{i,j,\alpha,\beta}\int_{B_1} A^{ij}_{\alpha\beta}(x)\partial_j u_\beta \partial_i \varphi_\alpha = 0,$$

$$A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta} + \left((m-2)\delta_{i\alpha} + m\frac{x_i x_\alpha}{|x|^2}\right)\left((m-2)\delta_{j\beta} + m\frac{x_j x_\beta}{|x|^2}\right),$$

which has the discontinuous and unbounded solution $u \in H^1(B_1, \mathbb{R}^m)$ given by

$$u(x) = \frac{x}{|x|^\gamma}, \qquad \gamma = \frac{m}{2}\left(1 - \sqrt{\frac{1}{1+(2m-2)^2}}\right).$$

3.2 A system with smooth coefficients depending on u

Theorem 2.4 gives that elliptic systems with continuous coefficients depending only on x have Hölder continuous solutions. In §4, we will look at harmonic maps, and the systems which these solve have coefficients depending on $u(x)$ also. An example due to Giusti and Miranda [GM68] shows that theorem 2.4 does not hold in this case. Again, we found this in [Gia83, §2.3], where only the system and associated solution are given, so we present it alongside our own calculations.

This time, we seek a system of the form (3.1) with smooth coefficients $A^{ij}_{\alpha\beta}$ depending only on u, with solution

$$u : B_1 \to \mathbb{R}^m \qquad u(x) = \frac{x}{|x|}.$$

Again, we need $m \geq 3$ to have $u \in H^1(B_1, \mathbb{R}^m)$.

As before, we begin with the usual Laplacian, and setting $\gamma = 1$ in our previous computation gives

$$\sum_{i,j,\beta}\partial_i(\delta_{ij}\delta_{\alpha\beta}\partial_j u_\beta) = -1(m-1)\frac{x_\alpha}{|x|^{1+2}} = -(m-1)\frac{x_\alpha}{|x|^3}.$$

Again we want coefficients $A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta} + a^i_\alpha a^j_\beta$, now for smooth $a^i_\alpha(u)$. As before, a first guess is $a^i_\alpha = u_\alpha = \frac{x_\alpha}{|x|}$, but

$$\partial_i\left(\frac{x_\alpha x_\beta}{|x|^2}\right) = -2\frac{x_i x_\alpha x_\beta}{|x|^4} + \frac{x_\beta}{|x|^2}\delta_{i\alpha} + \frac{x_\alpha}{|x|^2}\delta_{i\beta}$$

has terms which do not sum nicely.

A better guess is $a^i_\alpha = u_i u_\alpha = \frac{x_i x_\alpha}{|x|^2}$, so that $a^i_\alpha a^j_\beta = u_i u_j u_\alpha u_\beta = \frac{x_i x_j x_\alpha x_\beta}{|x|^4}$. From previous computations, we have

$$\sum_{i,j,\beta} \partial_i\left(\frac{x_i x_j x_\alpha x_\beta}{|x|^4}\partial_j u_\beta\right) = -(1(m-1)-(m-1))\frac{x_\alpha}{|x|^{1+2}} = 0.$$

We appear to have found an appropriate system, but taking $A^{ij}_{\alpha\beta} = u_i u_j u_\alpha u_\beta$ does not satisfy the ellipticity condition

$$A^{ij}_{\alpha\beta}\xi^i_\alpha\xi^j_\beta \geq \lambda|\xi|^2.$$

We try the same trick as before, and take

$$A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta} + \left(C_1\delta_{i\alpha} + C_2 u_i u_\alpha\right)\left(C_1\delta_{j\beta} + C_2 u_j u_\beta\right).$$

Then, again making use of previous calculations, we have

$$\sum_{i,j,\beta} \partial_i\left(\frac{u_i u_\alpha \delta_{j\beta}}{|x|^4}\partial_j u_\beta\right) = (-1(m-1)+(m-1)(m-1))\frac{x_\alpha}{|x|^{1+2}} = (m-2)(m-1)\frac{x_\alpha}{|x|^3},$$

$$\sum_{i,j,\beta} \partial_i\left(\frac{u_j u_\beta}{|x|^4}\delta_{i\alpha}\partial_j u_\beta\right) = -1(1-1)\frac{x_\alpha}{|x|^{1+2}} = 0,$$

$$\sum_{i,j,\beta} \partial_i\left(\delta_{i\alpha}\delta_{j\beta}\partial_j u_\beta\right) = -1(m-1)\frac{x_\alpha}{|x|^{1+2}} = -(m-1)\frac{x_\alpha}{|x|^3},$$

so

$$\partial_i\left(A^{ij}_{\alpha\beta}\partial_j u_\beta\right) = (-(m-1) - C_1^2(m-1) + C_1C_2(m-1)(m-2))\frac{x_\alpha}{|x|^3}.$$

Setting this equal to 0 and dividing by $m-1$ gives

$$1 = C_1(C_2(m-2) - C_1),$$

so taking $C_1 = 1$, $C_2 = \frac{2}{m-2}$ gives the required system.

However, while $a^i_\alpha = u_i u_\alpha$ is smooth, it is not unbounded as u varies. But by noting that $|u| = 1$ when $u = \frac{x}{|x|}$, we can adjust the coefficients a^i_α without re-doing the calculations above. In particular,

$$a^i_\alpha = \frac{u_i u_\alpha}{1+|u|^2}$$

gives smooth, bounded coefficients.

This gives Giusti and Miranda's example, which is the elliptic system

$$\sum_{i,j,\alpha,\beta}\int_{B_1} A^{ij}_{\alpha\beta}(x)\partial_j u_\beta \partial_i\varphi_\alpha = 0,$$

$$A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta} + \left(\delta_{i\alpha} + \frac{4}{m-2}\cdot\frac{u_i u_\alpha}{1+|u|^2}\right)\left(\delta_{j\beta} + \frac{4}{m-2}\cdot\frac{u_j u_\beta}{1+|u|^2}\right).$$

Applying lemma 3.1 shows it has the discontinuous solution $u \in H^1(B_1, \mathbb{R}^m)$ given by

$$u(x) = \frac{x}{|x|}.$$

3.3 Extending the counterexamples

The example in §3.1 only applies when $n = m \geq 3$, so we give our own extension of this example to any $m, n \geq 3$.

In all cases, we take a system of the form

$$\sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} \int_{B_1} A^{ij}_{\alpha\beta}(x) \partial_j u_\beta \partial_i \varphi_\alpha = 0 \qquad \forall \varphi \in H^1_0(B_1, \mathbb{R}^n),$$

or equivalently

$$\sum_{i,j=1}^{m} \sum_{\beta=1}^{n} \int_{B_1} A^{ij}_{\alpha\beta}(x) \partial_j u_\beta \partial_i \varphi = 0 \qquad \forall \varphi \in H^1_0(B_1),\ \alpha = 1, \ldots, n.$$

If $n > m$, then set

$$A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta} + \left((m-2)\delta_{i\alpha} + m\frac{x_i x_\alpha}{|x|^2} \right) \left((m-2)\delta_{j\beta} + m\frac{x_j x_\beta}{|x|^2} \right),$$

$$u_\alpha = \frac{x_\alpha}{|x|^\gamma} \quad \text{where} \quad \gamma = \frac{m}{2}\left(1 - \sqrt{\frac{1}{1 + (2m-2)^2}} \right),$$

with the convention that $x_k = 0$ for $k > m$.

Note that the system is still elliptic, and that $\partial_j u_\beta = 0$ when $\beta > m$.

Then for each $\varphi \in H^1_0(B_1)$, $\alpha \leq m$, we have

$$\sum_{i,j=1}^{m} \sum_{\beta=1}^{n} \int_{B_1} A^{ij}_{\alpha\beta} \partial_j u_\beta \partial_i \varphi = \sum_{i,j,\beta=1}^{m} \int_{B_1} A^{ij}_{\alpha\beta} \partial_j u_\beta \partial_i \varphi = 0,$$

from §3.1. When $\alpha > m$, we have

$$A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta} + \left((m-2)\cdot 0 + m\frac{x_i \cdot 0}{|x|^2} \right) \left((m-2)\delta_{j\beta} + m\frac{x_j x_\beta}{|x|^2} \right) = \delta_{ij}\delta_{\alpha\beta},$$

so

$$\sum_{i,j=1}^{m} \sum_{\beta=1}^{n} \int_{B_1} A^{ij}_{\alpha\beta} \partial_j u_\beta \partial_i \varphi = \sum_{i=1}^{m} \int_{B_1} \partial_i u_\alpha \partial_i \varphi = \sum_{i=1}^{m} \int_{B_1} 0 \cdot \partial_i \varphi = 0.$$

Hence u is an unbounded discontinuous solution to the system given above.

If $m > n$, then given $x \in \mathbb{R}^m$, write $\widetilde{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n$. We take

$$A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta} + \left((n-2)\delta_{i\alpha} + n\frac{\widetilde{x}_i \widetilde{x}_\alpha}{|\widetilde{x}|^2} \right) \left((n-2)\delta_{j\beta} + n\frac{\widetilde{x}_j \widetilde{x}_\beta}{|\widetilde{x}|^2} \right),$$

$$u_\alpha = \frac{\widetilde{x}_\alpha}{|\widetilde{x}|^\gamma} \quad \text{where} \quad \gamma = \frac{n}{2}\left(1 - \sqrt{\frac{1}{1 + (2n-2)^2}} \right),$$

again with the convention that $\widetilde{x}_k = 0$ for $k > n$. Note that $\partial_j u_\beta = 0$ for $j > n$.

Now if $i > n$, then

$$A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta} + \left((n-2)\cdot 0 + n\frac{0 \cdot \widetilde{x}_\alpha}{|\widetilde{x}|^2} \right) \left((n-2)\delta_{j\beta} + n\frac{\widetilde{x}_j \widetilde{x}_\beta}{|\widetilde{x}|^2} \right) = \delta_{ij}\delta_{\alpha\beta},$$

so

$$
\begin{aligned}
\sum_{i,j=1}^{m}\sum_{\beta=1}^{n}\int_{B_1} A^{ij}_{\alpha\beta}\partial_j u_\beta \partial_i\varphi &= \sum_{i=1}^{m}\sum_{j,\beta=1}^{n}\int_{B_1} A^{ij}_{\alpha\beta}\partial_j u_\beta \partial_i\varphi \\
&= \sum_{i,j,\beta=1}^{n}\int_{B_1} A^{ij}_{\alpha\beta}\partial_j u_\beta \partial_i\varphi + \sum_{i=n+1}^{m}\sum_{j,\beta=1}^{n}\int_{B_1} \delta_{ij}\delta_{\alpha\beta}\partial_j u_\beta \partial_i\varphi \\
&= \sum_{i,j,\beta=1}^{n}\int_{B_1} A^{ij}_{\alpha\beta}\partial_j u_\beta \partial_i\varphi + \sum_{i=n+1}^{m}\sum_{j,\beta=1}^{n}\int_{B_1} 0\cdot\delta_{\alpha\beta}\partial_j u_\beta \partial_i\varphi \\
&= \sum_{i,j,\beta=1}^{n}\int_{B_1} A^{ij}_{\alpha\beta}\partial_j u_\beta \partial_i\varphi \\
&= 0
\end{aligned}
$$

from §3.1, so u is an unbounded discontinuous solution to our system.

Note that when $n > m$, u is only discontinuous at a point, but when $m > n$, the singular set is $B_1 \cap \{x_1, \dots, x_N = 0\}$, which has Hausdorff dimension[11] $m - n$. We can generate further examples with larger singular sets as follows.

Given $x \in \mathbb{R}^m$, write $\widetilde{x} := (x_1, x_2, x_3)$, and set

$$
A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta} + \left(\delta_{i\alpha} + 3\frac{\widetilde{x}_i\widetilde{x}_\alpha}{|\widetilde{x}|^2}\right)\left(\delta_{j\beta} + 3\frac{\widetilde{x}_j\widetilde{x}_\beta}{|\widetilde{x}|^2}\right),
$$

$$
u_\alpha = \frac{\widetilde{x}_\alpha}{|\widetilde{x}|^\gamma} \quad \text{where} \quad \gamma = \frac{3}{2}\left(1 - \frac{1}{\sqrt{17}}\right).
$$

Then $\partial_j u_\beta = 0$ whenever $j > 3$ or $\beta > 3$, and if $i > 3$ or $\alpha > 3$ then $A^{ij}_{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta}$. So u is a solution of the associated system, by similar calculations to those above. Hence we have an elliptic system with bounded coefficients, whose solution has a singular set of Hausdorff dimension $m - 3$.

This procedure can also be carried out with the example in §3.2.

From this, it is reasonable to guess that we can establish *partial regularity* results - that is, solutions to elliptic systems with "nice" coefficients should themselves be "nice" outside a singular set of Hausdorff dimension some constant amount less than that of the domain. This indeed turns out to be true in many cases (see for example [Giu03, §9]).

3.4 The Campanato space for a domain with a cusp

Theorem 2.1 gives that if Ω has no external cusps, then the spaces $C^{0,\alpha}(\Omega)$ and $\mathcal{L}^{p,m+p\alpha}$ are isomorphic. To see that the inclusion $C^{0,\alpha}(\Omega) \subseteq \mathcal{L}^{p,m+p\alpha}(\Omega)$ holds for any domain Ω, we note that for fixed $y = (y_1, \dots, y_m) \in \Omega$, $1 \le p < \infty$, $r > 0$, we have

$$
\int_{\Omega_r(y)} |u(x) - \overline{u}_{y,r}|^p \,\mathrm{d}x \le C(p)\int_{\Omega_r(y)} |u(x) - \xi|^p \,\mathrm{d}x \quad \forall \xi \in \mathbb{R}^n \tag{3.2}
$$

[11] A definition of Hausdorff dimension can be found in ***, although the precise definition is not important here

(see [Giu03, remark 2.2]). Hence if $u \in C^{0,\alpha}(\Omega)$, then

$$
\begin{aligned}
r^{-m-p\alpha} \int_{\Omega_r(y)} |u(x) - \overline{u}_{y,r}|^p \, \mathrm{d}x &\leq C r^{-m-p\alpha} \int_{\Omega_r(y)} |u(x) - u(y)|^p \, \mathrm{d}x \\
&\leq C r^{-p\alpha} \sup_{x \in \Omega_r(y)} |u(x) - u(y)|^p \\
&\leq C \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{p\alpha}} \\
&= C \, \|u\|^p_{C^{0,\alpha}(\Omega)} .
\end{aligned}
$$

We give our own example to show that the assumption on Ω is required for the reverse inclusion.

Fix any $m \geq 2$, $p \in [1, \infty)$ and $\alpha \in (0,1)$. Then for $\beta \geq 1$, $\gamma \in (0, \alpha)$ to be chosen later, define $\Omega \subseteq \mathbb{R}^m$, $u : \Omega \to \mathbb{R}$ by

$$
\Omega = \{(x_1, \widetilde{x}) \in \mathbb{R} \times \mathbb{R}^{m-1} : 0 < x_1 < 1, |\widetilde{x}| < x_1^\beta\}, \qquad u(x_1, \widetilde{x}) = x_1^\gamma .
$$

Now $u \notin C^{0,\alpha}(\Omega)$, so our aim is to find β such that $u \in \mathcal{L}^{p,m+p\alpha}(\Omega)$. We will bound $(*) := r^{-m-p\alpha} \int_{\Omega_r(y)} |u(x) - \overline{u}_{y,r}|^p \, \mathrm{d}x$ for fixed $y \in \Omega$ by separating into cases (see figure 3.1). By remark 2.2, we may consider only $r \leq 1$.

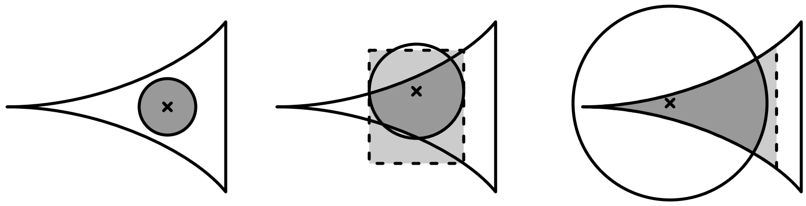

Figure 3.1: The different cases for our calculation.

Case 1 ($0 < r \leq y_1$). Using (3.2) with $\xi = y_1^\gamma$ gives

$$
\begin{aligned}
(*) &\leq C r^{-m-p\alpha} \int_{\Omega_r(y)} |x_1^\gamma - y_1^\gamma|^p \, \mathrm{d}x \\
&\leq C r^{-m-p\alpha} |\Omega_r(y)| \sup_{x \in \Omega_r(y)} |x_1^\gamma - y_1^\gamma|^p \\
&= C r^{-m-p\alpha} |\Omega_r(y)| \left(\sup_{|t|<r} |(y_1 + t)^\gamma - y_1^\gamma| \right)^p \\
&= C r^{-m-p\alpha} |\Omega_r(y)| (y_1^\gamma - (y_1 - r)^\gamma)^p .
\end{aligned}
$$

Now $\frac{r}{y_1} \in (0,1)$ so $1 - \frac{r}{y_1} \leq (1 - \frac{r}{y_1})^\gamma$, giving that $y_1^\gamma - (y_1 - r)^\gamma \leq y_1^{\gamma-1} r$. Hence

$$
(*) \leq C r^{-m+p(1-\alpha)} \, y_1^{-p(1-\gamma)} |\Omega_r(y)| .
$$

If $r \leq y_1^\beta$, then $\Omega_r(y) \subseteq B_r(y)$ and $r^{1-\alpha} \leq y_1^{\beta(1-\alpha)}$, so

$$
(*) \leq C r^{p(1-\alpha)} \, y_1^{-p(1-\gamma)} \leq C \left(y_1^{\beta(1-\alpha)-(1-\gamma)} \right)^p .
$$

And if $r \geq y_1^\beta$, then

$$\Omega_r(y) \subseteq \{(x_1, \widetilde{x}) : |x_1 - y_1| < r, |\widetilde{x}| < (y_1 + r)^\beta\},$$

which has volume $Cr(y_1 + r)^{\beta(m-1)} \leq Cry_1^{\beta(m-1)}$, so

$$(*) \leq Cr^{-(m-1)+p(1-\alpha)}\, y_1^{\beta(m-1)-p(1-\gamma)}.$$

Depending on the sign of $-(m-1)+p(1-\alpha)$, this either achieves its maximum at $r = y_1^\beta$ or $r = y_1$. Hence

$$(*) \leq C \max\left\{ \left(y_1^{\beta(1-\alpha)-(1-\gamma)} \right)^p,\ y_1^{(\beta-1)(m-1)-p(\alpha-\gamma)} \right\}.$$

Case 2 $(y_1 < r \leq 1)$. Now

$$\Omega_r(y) \subseteq \{(x_1, \widetilde{x}) \in \mathbb{R} \times \mathbb{R}^{m-1} : 0 < x_1 < 2r, |\widetilde{x}| < x_1^\beta\}.$$

Using this, and (3.2) with $\xi = 0$, gives

$$\begin{aligned}(*) &\leq Cr^{-m-p\alpha} \int_0^{2r} t^{\beta(m-1)} \cdot t^{p\gamma}\, \mathrm{d}t \\ &= Cr^{(\beta-1)(m-1)-p(\alpha-\gamma)}.\end{aligned}$$

Depending on the sign of $-(m-1)+p(1-\alpha)$, this either achieves its maximum at $r = y_1$ or $r = 1$, giving

$$(*) \leq C \max\left\{ y_1^{(\beta-1)(m-1)-p(\alpha-\gamma)}, 1 \right\}.$$

Combining both cases gives that for fixed $y \in \Omega$,

$$\sup_{r>0} r^{-m-p\alpha} \int_{\Omega_r(y)} |u(x) - \overline{u}_{y,r}|^p\, \mathrm{d}x \leq C \max\left\{ \left(y_1^{\beta(1-\alpha)-(1-\gamma)} \right)^p,\ y_1^{(\beta-1)(m-1)-p(\alpha-\gamma)}, 1 \right\},$$

so

$$[u]_{p,\lambda}^p \leq C \sup_{y_1 \in (0,1)} \max\left\{ \left(y_1^{\beta(1-\alpha)-(1-\gamma)} \right)^p,\ y_1^{(\beta-1)(m-1)-p(\alpha-\gamma)}, 1 \right\}.$$

This is finite if $\beta(1-\alpha) - (1-\gamma)$, $(\beta-1)(m-1) - p(\alpha-\gamma)$ are both non-negative, that is if

$$\beta \geq 1 + \frac{\alpha - \gamma}{1 - \alpha}, \qquad \beta - 1 \geq \frac{p(\alpha-\gamma)}{m-1}.$$

So if

$$\beta \geq 1 + \max\left\{ \frac{1}{1-\alpha}, \frac{p}{m-1} \right\} (\alpha - \gamma),$$

then $u \in \mathcal{L}^{p,m+p\alpha}(\Omega)$.

We can also show that if $u \in \mathcal{L}^{p,m+p\alpha}(\Omega)$, then β must satisfy the inequality above. To do this, we use remark 2.1 to equivalently take $\Omega_r(y) = \Omega \cap Q_r(y)$, where $Q_r(y)$ is the cube of side length $2r$ centred at y and aligned with the axes. Given $u \in \mathcal{L}^{p,m+p\alpha}(\Omega)$, for some fixed $m \geq 2$, $p \in [1, \infty)$, $0 < \gamma \leq \alpha, < 1$, $\beta \geq 1$, we carry out two calculations.

Calculation 1. Take $r \in (0,1)$. Then $\Omega_r(0) = \{x \in \Omega : 0 < x_1 < r\}$, and

$$\overline{u}_{0,r} = \frac{\int_0^r t^{\beta(m-1)} t^\gamma \, \mathrm{d}t}{\int_0^r t^{\beta(m-1)} \, \mathrm{d}t} = \frac{\beta(m-1)+1}{\beta(m-1)+1+\gamma} \, r^\gamma.$$

Now write $\widetilde{C} = \left(\frac{\beta(m-1)+1}{\beta(m-1)+1+\gamma}\right)^{1/\gamma} \in (0,1)$, so

$$\begin{aligned} r^{-m-p\alpha} \int_{\Omega_r(0)} |u(x) - \overline{u}_{0,r}|^p \, \mathrm{d}x &= r^{-m-p\alpha} \int_0^r t^{\beta(m-1)} |(\widetilde{C}r)^\gamma - t^\gamma|^p \, \mathrm{d}t \\ &\geq r^{-m-p\alpha} \int_0^{\frac{1}{2}\widetilde{C}r} t^{\beta(m-1)} ((\widetilde{C}r)^\gamma - t^\gamma)^p \, \mathrm{d}t \\ &\geq r^{-m-p\alpha} ((\widetilde{C}r)^\gamma - (\tfrac{1}{2}\widetilde{C}r)^\gamma)^p \int_0^{\frac{1}{2}\widetilde{C}r} t^{\beta(m-1)} \, \mathrm{d}t \\ &= C r^{(\beta-1)(m-1)-p(\alpha-\gamma)}, \end{aligned}$$

where $C > 0$ is independent of r. Now if $r > 0$, then the function $y \mapsto r^{-\lambda} \left\| u - \overline{u}_{y,r} \right\|_{L^p(\Omega_r(y))}$ is continuous, so

$$\begin{aligned} [u]_{p,\lambda}^p \geq \sup_{x_0 \in \overline{\Omega}} r^{-\lambda} \left\| u - \overline{u}_{y,r} \right\|_{L^p(\Omega_r(y))}^p [u]_{p,\lambda}^p \\ = \sup_{x_0 \in \Omega} r^{-\lambda} \left\| u - \overline{u}_{y,r} \right\|_{L^p(\Omega_r(y))}^p \\ \geq C r^{(\beta-1)(m-1)-p(\alpha-\gamma)} \end{aligned}$$

for each $r > 0$. Hence $(\beta-1)(m-1) - p(\alpha-\gamma) \geq 0$, that is

$$\beta \geq 1 + \frac{p}{m-1}(\alpha - \gamma).$$

Calculation 2. Take $y_1 \in (0, \frac{1}{2})$, and set $y = (y_1, 0, 0, \ldots, 0)$, $r = \left(\frac{1}{2} y_1\right)^\beta$. Then $(y_1 - r)^\beta \geq (2r^{1/\beta} - r^{1/\beta})^\beta = r$, so $\Omega_r(y) = Q_r(y)$. Further,

$$\overline{u}_{y,r} = (2r)^{-m} \int_{y_1-r}^{y_1+r} (2r)^{m-1} t^\gamma \, \mathrm{d}t = \frac{1}{2(\gamma+1)r} \left((y_1+r)^{\gamma+1} - (y_1-r)^{\gamma+1} \right).$$

Now for $\gamma \in (0,1)$, we have

$$\frac{d}{ds} \left((y_1+s)^{\gamma+1} - (y_1-s)^{\gamma+1} \right) = 2(\gamma+1) \cdot \frac{1}{2}((y_1+s)^\gamma + (y_1-s)^\gamma) \leq 2(\gamma+1) y_1^\gamma$$

by concavity of $s \mapsto s^\gamma$. Integrating this inequality, noting that $(y_1+0)^{\gamma+1} - (y_1-0)^{\gamma+1} = 0$, dividing by $2(\gamma+1)s$ and taking $s = r$ then gives

$$\overline{u}_{y,r} \leq y_1^\gamma.$$

Hence

$$\begin{aligned} r^{-m-p\alpha} \int_{\Omega_r(y)} |u(x) - \overline{u}_{y,r}|^p \, \mathrm{d}x &= r^{-m-p\alpha} \int_{y_1-r}^{y_1+r} (2r)^{m-1} |t^\gamma - \overline{u}_{y,r}|^p \, \mathrm{d}t \\ &\geq r^{-1-p\alpha} \int_{y_1+\frac{r}{2}}^{y_1+r} (t^\gamma - \overline{u}_{y,r})^p \, \mathrm{d}t \\ &\geq r^{-1-p\alpha} \int_{y_1+\frac{r}{2}}^{y_1+r} (t^\gamma - y_1^\gamma)^p \, \mathrm{d}t \\ &\geq \frac{1}{2} r^{-p\alpha} \left(\left(y_1 + \frac{r}{2}\right)^\gamma - y_1^\gamma \right)^p. \end{aligned}$$

Now for $s \le y_1$, $\gamma < 1$, we have

$$\frac{d}{ds}\left((y_1+s)^\gamma - y_1^\gamma\right) = \gamma\,(y_1+s)^{\gamma-1} \ge 2^{\gamma-1}\gamma\, y_1^{\gamma-1}.$$

Integrating this inequality, noting that $(y_1+0)^\gamma - y_1^\gamma = 0$, and taking $s = \frac{r}{2} \le y_1$ gives

$$\left(y_1 + \frac{r}{2}\right)^\gamma - y_1^\gamma \ge C(\gamma)\, y_1^{\gamma-1} r,$$

so

$$\begin{aligned}
[u]_{p,\lambda}^p &\ge r^{-m-p\alpha} \int_{\Omega_r(y)} |u(x) - \overline{u}_{y,r}|^p \,\mathrm{d}x \\
&\ge C r^{-p\alpha}\left(y_1^{\gamma-1} r\right)^p \\
&= C y_1^{p\,(\beta(1-\alpha)-(1-\gamma))}
\end{aligned}$$

for each $y_1 > 0$, where $C > 0$ is independent of y_1. Hence $\beta(1-\alpha)-(1-\gamma) \ge 0$, that is

$$\beta \ge 1 + \frac{1}{1-\alpha}(\alpha - \gamma).$$

These calculations show that given $m \ge 2$, $p \in [1,\infty)$, $0 < \gamma \le \alpha, < 1$, $\beta \ge 1$, if we define $\Omega \subseteq \mathbb{R}^m$, $u : \Omega \to \mathbb{R}$ by

$$\Omega = \{(x_1, \widetilde{x}) \in \mathbb{R} \times \mathbb{R}^{m-1} : 0 < x_1 < 1, |\widetilde{x}| < x_1^\beta\}, \qquad u(x_1, \widetilde{x}) = x_1^\gamma$$

then

$$u \in \mathcal{L}^{p,m+p\alpha}(\Omega) \quad \Leftrightarrow \quad \beta \ge 1 + \max\left\{\frac{1}{1-\alpha}, \frac{p}{m-1}\right\}(\alpha - \gamma).$$

Hence the "pointiness" of the cusps of Ω regulates the regularity of $u \in \mathcal{L}^{p,m+p\alpha}(\Omega)$.

It seems reasonable to expect that something similar holds in general. In particular, we suggest the following.

> If Ω has *at worst $C^{0,\beta}$ cusps*, that is there exists $A \ge 0$ such that for each $x_0 \in \overline{\Omega}$ and $0 < r \le \min\{1, \operatorname{diam}(\Omega)\}$, we have $|\Omega_r(x_0)| \ge A r^{\beta(m-1)+1}$, then
>
> $$\mathcal{L}^{p,m+p\alpha}(\Omega) \subseteq C^{0,\gamma}(\Omega), \qquad \text{where} \qquad \gamma = \alpha - \min\left\{1-\alpha, \frac{m-1}{p}\right\}(\beta - 1).$$

Returning to §2.5, this would mean that we could relax the condition that Ω has no external cusps (provided we don't run into the cusp condition mentioned after remark 2.3), and still be able to use theorem 2.6 to conclude that solutions of elliptic systems with uniformly continuous coefficients are Hölder continuous. Unfortunately, we did not have time to explore this any further, but [GL23] would be our starting point for further exploration in this direction.

3.5 Boundary regularity

The original aim of finding the example in §3.4 was to construct an example to show that the cusp condition is needed to apply theorem 2.6 to Hölder continuity. While we can find equations of the desired form which have u (as defined in §3.4) as a solution (for example by noting that for any q, we can have $\Delta u \in L^q(\Omega)$ by taking β sufficiently large), we do not have that u is equal to zero on the boundary.

If we adapt u to get appropriate boundary conditions, for example by taking

$$u(x_1, \widetilde{x}) = x_1^{\gamma}(1 - x_1)(1 - x_1^{-2\beta}|\widetilde{x}|^2),$$

then

$$\begin{aligned}\Delta u = &\left(\gamma(\gamma-1)x_1^{\gamma-2} - (\gamma-2\beta)(\gamma-2\beta-1)x_1^{\gamma-2\beta-2}|\widetilde{x}|^2\right) \\ &- \left((\gamma+1)\gamma x_1^{\gamma-1} - (\gamma-2\beta+1)(\gamma-2\beta)x_1^{\gamma-2\beta-1}|\widetilde{x}|^2\right) \\ &+ 2(m-1)\left(x_1^{\gamma-2\beta} - x_1^{\gamma-2\beta+1}\right).\end{aligned}$$

Now if $x \in \Omega$ (with Ω as in §3.4), then $x_1^{-2\beta}|\widetilde{x}|^2 \leq 1$, so the first term behaves like $x_1^{\gamma-2}$ and the second term behaves like $x_1^{\gamma-1}$. Now

$$\int_{\Omega} x_1^{q(\gamma-2)}\,\mathrm{d}x = \int_{t=0}^{1} t^{\beta} \cdot t^{q(\gamma-2)}\,\mathrm{d}t,$$

so by making β large we can ensure the first two terms are in $L^q(\Omega)$ for our desired q. However, the third term (which comes from derivatives with respect to $x_2, \ldots, x_m$) is only in $L^q(\Omega)$ when β is small. This prevents us from adjusting β to get the first terms in $L^q(\Omega)$.

This suggests adding an "adjustment" term to our equation to cancel the third term, but we were unable to find an appropriate adjustment which is uniformly continuous.

4 Harmonic Maps

In §1, we studied elliptic equations, where the solution u is a map from $\Omega \subseteq \mathbb{R}^m$ to $\mathbb{R}$, and in §2, we studied elliptic systems, where the solution is a map from $\Omega \subseteq \mathbb{R}^m$ to $\mathbb{R}^n$. We now consider the situation where the solution is a map between Riemannian manifolds, but we will restrict our attention to those which satisfy an analogue of the equation $\Delta u = 0$. These are known as *harmonic maps*.

Examples of harmonic maps include geodesics and minimal surfaces (see §4.2), but they also include other situations, for example the behaviour of nematic liquid crystals [Hél02, introduction]. These can be thought of as thin rod-shaped molecules which want to be parallel to each other. Considering their direction (a unit-length vector) at each point in space gives a map to the unit sphere which minimises some variational energy, and this turns out to be very close to the energy given in definition 4.1.

4.1 Defining harmonic maps

Suppose we have Riemannian manifolds $\mathcal{M}, \mathcal{N}$ of dimensions m, n, with C^1 Riemannian metrics g, h respectively.

We begin by defining harmonic maps $\mathcal{M} \to \mathbb{R}^n$, following [Hél02, §1.1]. Given local coordinates $(x^1, \ldots, x^m)$ on $U \subseteq \mathcal{M}$, and a function $u : U \to \mathbb{R}$, we define the *Laplacian* by

$$\Delta_{\mathcal{M}} u = \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g}\, g^{ij}(x) \partial_j u \right),$$

where $g_{ij}(x) = g(x)\left(\partial_{x^i}, \partial_{x^j}\right)$, and $g^{ij}(x)$ is the $(i, j)^{\text{th}}$ element of the inverse matrix of (g_{ij}). It is clear from a routine calculation that this is coordinate-independent.

Note that if $\mathcal{M} = \mathbb{R}^m$ with the Euclidean metric, then $g_{ij} = \delta_{ij}$, and $\Delta_{\mathcal{M}} u = \frac{1}{\sqrt{1}} \partial_i \left(\sqrt{1} \delta_{ij} \partial_j u \right) = \partial_i^2 u$, giving the usual Laplacian.

Then we say that $u : \mathcal{M} \to \mathbb{R}^n$ is *harmonic* if each component $u^\alpha : \mathcal{M} \to \mathbb{R}$ satisfies $\Delta_{\mathcal{M}} u^\alpha = 0$.

To help move on to the general case, note that if $\Omega \subseteq \mathbb{R}^m$, then harmonic maps $\Omega \to \mathbb{R}^n$ are critical points of the energy functional $u \mapsto \int_\Omega |Du|^2 \, \mathrm{d}x = \sum_{i,\alpha} \int_\Omega (\partial_i u^\alpha)^2 \, \mathrm{d}x$. Trying an analogous expression for functions $u : \mathcal{M} \to \mathbb{R}^n$, we see that if

$$e(u) := \sum_\alpha g^{ij}(x) \partial_i u^\alpha \partial_j u^\alpha, \qquad E(u) := \int_{\mathcal{M}} e(u) \, \mathrm{dvol}_g,$$

where $\mathrm{dvol}_g = \sqrt{\det g(x)}\, dx^1 \ldots dx^m$, then $e(u)$ is coordinate-independent, and harmonic maps are critical points of E.

This suggests the following definition of harmonic maps $\mathcal{M} \to \mathcal{N}$ [SY97, §IX.1]. Note we are retaining the convention that repeated indices are summed over, with $i, j = 1, \ldots, m$ and $\alpha, \beta = 1, \ldots, n$.

Definition 4.1. Given a C^1 function $u : (\mathcal{M}, g) \to (\mathcal{N}, h)$, and local coordinates $x^1, \ldots, x^m$ on $\mathcal{M}$, $y^1, \ldots, y^n$ on $\mathcal{N}$, the *energy density* is

$$e(u) := g^{ij}(x) h_{\alpha\beta}(u(x)) \partial_i u^\alpha \partial_j u^\beta,$$

where we write $u = (u^1, \ldots, u^n)$, $g_{ij} = g\left(\partial_{x^i}, \partial_{x^j}\right)$, $h_{\alpha\beta} = h\left(\partial_{y^\alpha}, \partial_{y^\beta}\right)$. Then u is a *harmonic map* if it is a critical point of the energy functional

$$E(u) := \int_{\mathcal{M}} e(u) \, \mathrm{dvol}_g. \tag{4.1}$$

Note that $e(u) = \mathrm{tr}_g(u^*h)$, so this is coordinate-independent.

Further, if ∇ is the Levi-Civita connection on $\mathcal{N}$ with Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ in the coordinates $y_1, \ldots, y_n$, then the Euler-Lagrange equations for (4.1) are

$$\Delta_{\mathcal{M}} u^\alpha + g^{ij}(x)\Gamma^\alpha_{\beta\gamma}(u(x))\partial_i u^\beta \partial_j u^\gamma = 0 \qquad \forall \alpha, \tag{4.2}$$

so we can also take this as the definition of harmonic maps (see [Hél02, §1.2]).

If $\mathcal{N} = \mathbb{R}^n$, then all the Christoffel symbols are 0, giving our previous definition of harmonic maps into $\mathbb{R}^n$.

Additionally, if $\mathcal{N}$ is isometrically embedded in $\mathbb{R}^N$, we have an alternative formulation of (4.2), namely

$$\Delta_{\mathcal{M}} u + g^{ij} B(u(x))\,(\partial_i u, \partial_j u) = 0, \tag{4.3}$$

where B is the second fundamental form of $\mathcal{N}$ (see [Hél02, (1.17), lemma 1.2.4]).

4.2 Examples

In order to motivate our discussion, we give a few examples of harmonic maps.

Example 4.1 (Geodesics [Hél02, example 1.2.6])**.** If $\mathcal{M} = [0, L] \subseteq \mathbb{R}$ with coordinate t, then (4.2) becomes

$$\ddot{u}^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\,\dot{u}^\beta\,\dot{u}^\gamma = 0 \qquad \forall \alpha.$$

These are the geodesic equations for curves in $(\mathcal{N}, h)$.

Example 4.2 (Minimal submanifolds [HW08, §2.2, example 7])**.** If $u : \mathcal{M} \to \mathcal{N}$ is an isometric immersion, then it can be shown that u is harmonic if and only if $\mathcal{M}$ is a minimal submanifold of $\mathcal{N}$.

Example 4.3 (Holomorphic maps (adapted from [HW08, §2.2, example 10]))**.** If $\mathcal{M} = \mathcal{N} = \mathbb{C}$, with complex coordinate $z = x + iy$ on $\mathcal{M}$, then $g_{ij} = \delta_{ij}$ and all the Christoffel symbols for h are 0. Hence (4.2) becomes

$$0 = \left(\partial_x^2 + \partial_y^2\right) u = \partial_{\bar{z}}\partial_z u,$$

So holomorphic and antiholomorphic maps are harmonic.

Example 4.4 (Maps into the sphere (adapted from [HW08, §3.1, example]))**.** Suppose $\mathcal{N} = S^n \subseteq \mathbb{R}^{n+1}$ with the round metric. The second fundamental form of $S^n \subseteq \mathbb{R}^{n+1}$ is

$$B(p)(X, Y) = \langle X, Y \rangle p,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^{n+1}$. So (4.3) becomes

$$\Delta_{\mathcal{M}} u = -e(u)u, \tag{4.4}$$

where $e(u) := \sum_\alpha g^{ij}\partial_i u^\alpha \partial_j u^\alpha$.

Now if $\mathcal{M} = B^{n+1} \subseteq \mathbb{R}^{n+1}$ is the unit ball, then (4.4) becomes

$$\Delta u = -|Du|^2 u,$$

which has weak solution $x \mapsto \frac{x}{|x|}$. This suggests that it would be useful to find a weak formulation of definition 4.1, and also that maps satisfying such a definition may not be continuous.

4.3 Different notions of weakly harmonic maps

We now seek a notion of weakly harmonic maps. As mentioned in [SY97, §IX.1], it makes sense to first restrict our attention to maps with finite energy. Recalling that

$$E(u) = \int_{\mathcal{M}} g^{ij}(x) h_{\alpha\beta}(u(x)) \partial_i u^\alpha \partial_j u^\beta \, \mathrm{dvol}_g,$$

we see that our maps should be bounded (to control $h_{\alpha\beta}(u)$) with derivatives in L^2 (to control $\partial_i u^\alpha \cdot \partial_j u^\beta$), so they should be in the space

$$L^\infty(\mathcal{M}, \mathcal{N}) \cap H^1(\mathcal{M}, \mathcal{N}).$$

Unfortunately, we don't have a good notion of what this space means, since we want to work in local coordinates, but this requires continuity (to assume that the image of an open set in $\mathcal{M}$ is contained in a coordinate chart of $\mathcal{N}$), which we want to avoid imposing. We can get around this by isometrically embedding $\mathcal{N}$ in $\mathbb{R}^N$ (although a smooth embedding can be used instead [SY97, §IX.1]), and considering maps $u : \mathcal{M} \to \mathbb{R}^N$ taking values in $\mathcal{N}$ almost everywhere, giving the space

$$\mathcal{D} := \left\{ u \in L^\infty(\mathcal{M}, \mathbb{R}^N) \cap H^1(\mathcal{M}, \mathbb{R}^N) : u(x) \in \mathcal{N} \ a.e. \right\}.$$

We could now give a weak formulation of (4.2), but again this requires local coordinates on $\mathcal{N}$, and hence continuity. However, the weak formulation of (4.3) makes sense without assuming continuity, and if u is continuous then the two weak formulations agree. This suggests the following definition [SY97, §IX.1].

Definition 4.2. A map $u \in \mathcal{D}$ is *weakly harmonic* if

$$\int_{\mathcal{M}} \sum_\alpha \left[g^{ij} \partial_i u^\alpha \partial_j \varphi^\alpha + g^{ij} \varphi^\alpha B^\alpha(u(x)) \left(\partial_i u, \partial_j u \right) \right] \mathrm{dvol}_g = 0 \tag{4.5}$$

for each coordinate patch $U \subseteq \mathcal{M}$, and any $\varphi \in H^1_0(U, \mathbb{R}^N) \cap L^\infty(U, \mathbb{R}^N)$.[12] (Recall that B is the second fundamental form of the embedding of $\mathcal{N}$ in $\mathbb{R}^N$.)

There are other weak formulations of definition 4.1, so we briefly outline two such formulations from [HW08, §3.1]. A map $u \in \mathcal{D}$ is called *minimising* if any $v \in \mathcal{D}$ such that $v = u$ outside a compact subset of $\mathcal{M}$, satisfies $E(v) \geq E(u)$. And a weakly harmonic map is called *stationary* if it is stationary with respect to a larger class of deformations than those used to find the Euler-Lagrange equations for (4.1). We have the inclusions

$$\{\text{minimising maps}\} \subseteq \{\text{stationary maps}\} \subseteq \{\text{weakly harmonic maps}\},$$

and these are strict in general.

[12] In [SY97], φ is restricted to $C^\infty_c(U, \mathbb{R}^N)$, but we have opted for the more general class of functions used in [Hél02, definition 1.4.9], as this will be useful to us later on. Note that this gives an equivalent definition.

4.4 Regularity of continuous weakly harmonic maps, via freezing coefficients

We begin with the following surprising theorem [Hél02, theorem 1.5.1].

Theorem 4.1. *Suppose $\mathcal{N}$ is compact without boundary, and $u \in H^1(\mathcal{M},\mathcal{N})$ is a continuous harmonic map.*[13] *Then u is smooth, that is if the metric on $\mathcal{M}$ is $C^{k,\alpha}$, and the metric on $\mathcal{N}$ is $C^{l,\alpha}$, then u is $C^{\min\{k,l\}+1,\alpha}$.*

The difficult step in the proof of this theorem is showing that u is Lipschitz [HW08, §4.1]. We will prove the weaker statement that Hölder continuous weakly harmonic maps are Lipschitz continuous, because it shows another freezing coefficients argument. The idea is that the continuity of u means we can assume that the image of a small set in $\mathcal{M}$ lies in a single coordinate patch in $\mathcal{N}$, and therefore work in local coordinates. We cannot use theorem 2.4 here, since our coefficients depend on u as well as x (see §3.2), but we can choose coordinates in such a way that the frozen equation has a nice form.

Proposition 4.2. *If a weakly harmonic map $u : \mathcal{M} \to \mathcal{N}$ is α-Hölder continuous for some $\alpha \in (0,1)$, then u is locally Lipschitz.*

The following proof is from [SY97, §IX.7], with some parts reordered and additional explanation added for clarity.

Proof. First, we obtain an estimate which will be useful later. Take normal coordinates $(x^1,\dots,x^m)$ in a normal neighbourhood $B_{r_0}(x_0)$ of $\mathcal{M}$ and the usual coordinates $(y^1,\dots,y^N)$ on $\mathbb{R}^N$, and write $u = (u^1,\dots,u^N)$. Since u is continuous and weakly harmonic, it satisfies the weak form of (4.2), so

$$\int_{B_{r_0}(x_0)} \sum_\beta \Big[-g^{ij}\partial_i u^\beta \partial_j \varphi^\beta + \varphi^\beta g^{ij}\Gamma^\beta_{\gamma\delta}(u)\partial_i u^\gamma \partial_j u^\delta\Big]\,\mathrm{d}x = 0 \tag{4.6}$$

for each $\varphi \in H^1_0(B_{r_0}(x_0),\mathbb{R}^N)$.

Now if g_0 is the Euclidean metric on $\mathbb{R}^N$, then $\sum_\beta \partial_i u^\beta \partial_j\varphi^\beta = g_0\left(\partial_i u^\beta \partial_{y^\beta}, \partial_j \varphi^\gamma \partial_{y^\gamma}\right)$ for fixed i,j. However, u takes values in $\mathcal{N}$, so $\partial_i u^\beta \partial_{y^\beta}$ is tangent to $\mathcal{N}$, and we can replace $\partial_j\varphi^\gamma\partial_{y^\gamma}$ with its projection $h_{\gamma\delta}\partial_j\varphi^\delta\partial_{y^\delta}$ onto the tangent space to $\mathcal{N}$. Hence

$$\sum_\beta \partial_i u^\beta \partial_j \varphi^\beta = g_0\left(\partial_i u^\beta \partial_{y^\beta}, \partial_j\varphi^\gamma\partial_{y^\gamma}\right) = h_{\beta\gamma}\partial_i u^\beta \partial_j\varphi^\gamma.$$

So if we write $\langle \nabla u, \nabla\varphi\rangle := g^{ij}h_{\beta\gamma}\partial_i u^\beta\partial_j\varphi^\gamma$, then for each $r \in (0,r_0)$ we have

$$\begin{aligned}
\left|\int_{B_r(x_0)} \langle\nabla u,\nabla\varphi\rangle\,\mathrm{dvol}_g\right| &= \left|\int_{B_r(x_0)} \sum_\beta g^{ij}\partial_i u^\beta \partial_j\varphi^\beta\,\mathrm{dvol}_g\right| \\
&= \left|\int_{B_r(x_0)} \sum_\beta \varphi^\beta\Gamma^\beta_{\gamma\delta}(u) g^{ij}\partial_i u^\gamma\partial_j u^\delta\,\mathrm{dvol}_g\right| \\
&\le C_1 \sup_{B_r(x_0)}|\varphi| \int_{B_r(x_0)} g^{ij}h_{\gamma\delta}\partial_i u^\gamma \partial_j u^\delta\,\mathrm{dvol}_g \\
&= C_1 \sup_{B_r(x_0)}|\varphi| \int_{B_r(x_0)} e(u)\,\mathrm{dvol}_g,
\end{aligned} \tag{4.7}$$

[13]Note that when $\mathcal{N}$ is compact, the condition $u \in L^\infty(\mathcal{M},\mathcal{N})$ is automatically satisfied.

where C_1 depends only on the behaviour of h and embedding of $\mathcal{N}$ into $\mathbb{R}^N$ near $u(x_0)$.

This is a coordinate-independent statement on $\mathcal{N}$, so we are free to change coordinates. Hence we assume (by decreasing r_0 if necessary) that $u(B_{r_0}(x_0))$ lies in a normal neighbourhood of $u(x_0)$ with normal coordinates $(y^1, \ldots, y^n)$, and that $r < 1$. We now write $u = (u^1, \ldots, u^n)$.

If $|x| \leq r$, then the normality of the coordinates and Hölder continuity of u gives

$$
\begin{aligned}
|g_{ij}(x) - \delta_{ij}| &\leq C_2 r^2, \\
|h_{\beta\gamma}(u(x)) - \delta_{\beta\gamma}| &\leq C_3 r^{2\alpha},
\end{aligned}
\tag{4.8}
$$

where C_2 depends only on the behaviour of g near x_0, and C_3 depends only on $\|u\|_{C^{0,\alpha}}$ and the behaviour of h near $u(x_0)$.

Further, $g_{ij} = \delta_{ij}$ at x_0, and the $\Gamma^\beta_{\gamma\delta}$ are all 0 at $u(x_0)$. We want to freeze coefficients as we did in §2.4, but this time our equation has coefficients depending on both x and $u(x)$, so we need to freeze coefficients in both the domain and codomain. Plugging $g_{ij} = \delta_{ij}$ and $\Gamma^\beta_{\gamma\delta} = 0$ into (4.6) gives

$$\sum_i \partial_i^2 v = 0.$$

Take v to be the unique weak solution to this on $B_r(x_0)$ with $v = u$ on $\partial B_r(x_0)$.

In §2.4, we set $\varphi = u - v$ in the equation for u. Doing this with (4.7) gives

$$\left| \int_{B_r(x_0)} \langle \nabla u, \nabla(u - v) \rangle \,\mathrm{dvol}_g \right| \leq C_1 \sup_{B_r(x_0)} |u - v| \int_{B_r(x_0)} e(u) \,\mathrm{dvol}_g. \tag{4.9}$$

Further, the maximum principle gives

$$\max_{\overline{B}_r(x_0)} v = \max_{\partial B_r(x_0)} v = \max_{\partial B_r(x_0)} u \leq \max_{\overline{B}_r(x_0)} u,$$

$$\min_{\overline{B}_r(x_0)} v = \min_{\partial B_r(x_0)} v = \min_{\partial B_r(x_0)} u \geq \min_{\overline{B}_r(x_0)} u,$$

so

$$\operatorname*{osc}_{\overline{B}_r(x_0)} v \leq \operatorname*{osc}_{\overline{B}_r(x_0)} u \leq C r^\alpha,$$

where $C = 2^\alpha \|u\|_{C^{0,\alpha}}$. Hence if $\widetilde{x} \in \partial B_r(x_0)$, then

$$
\begin{aligned}
\sup_{x \in B_r(x_0)} |u(x) - v(x)| &\leq \sup_{x \in B_r(x_0)} |u(x) - u(\widetilde{x})| + |u(\widetilde{x}) - v(\widetilde{x})| + \sup_{x \in B_r(x_0)} |v(\widetilde{x}) - v(x)| \\
&\leq C r^\alpha + 0 + C r^\alpha.
\end{aligned}
$$

Combining this with (4.9), we have

$$\left| \int_{B_r(x_0)} \langle \nabla u, \nabla(u - v) \rangle \,\mathrm{dvol}_g \right| \leq C(C_1, \|u\|_{C^{0,\alpha}}) r^\alpha \int_{B_r(x_0)} e(u) \,\mathrm{dvol}_g. \tag{4.10}$$

We can also take $\varphi = u - v$ in the equation for v. If we write $\partial v \cdot \partial \varphi := \sum_{i,\beta} \partial_i v^\beta \partial_i \varphi^\beta$, then this equation is

$$\int_{B_r(x_0)} \partial v \cdot \partial \varphi \,\mathrm{d}x = 0 \quad \forall \varphi \in H^1_0(B_r(x_0)),$$

and setting $\varphi = u - v$ gives

$$\int_{B_r(x_0)} |\partial(u-v)|^2 \,\mathrm{d}x = \int_{B_r(x_0)} \partial u \cdot \partial(u-v)\,\mathrm{d}x. \tag{4.11}$$

Now if $|x| \leq r$, then $|\sqrt{\det g(x)} - 1| \leq Cr$ by (4.8), so

$$|g^{ij} h_{\beta\gamma}(u)\sqrt{\det g} - \delta_{ij}\delta_{\beta\gamma}| \leq C(C_2, C_3) r^{\min\{1,2\alpha\}}.$$

Noting that if $\alpha \in (0,1)$ then $\alpha \leq \min(2\alpha, 1)$, we have

$$\begin{aligned}
&\left| \int_{B_r(x_0)} \partial u \cdot \partial(u-v)\,\mathrm{d}x - \int_{B_r(x_0)} \langle \nabla u, \nabla(u-v)\rangle \,\mathrm{dvol}_g \right| \\
&\qquad = \left| \sum_{i,j} \int_{B_r(x_0)} \left(\delta_{ij}\delta_{\beta\gamma} - g^{ij} h_{\beta\gamma}(u)\sqrt{\det g}\right) \partial_i u^\beta \,\partial_j (u-v)^\gamma \,\mathrm{d}x \right| \\
&\qquad \leq C(C_2, C_3) r^\alpha \int_{B_r(x_0)} |\partial u||\partial(u-v)|\,\mathrm{d}x \\
&\qquad \leq C r^\alpha \int_{B_r(x_0)} \left(|\partial u|^2 + |\partial v|^2\right)\,\mathrm{d}x \\
&\qquad \leq C r^\alpha \int_{B_r(x_0)} |\partial u|^2 \,\mathrm{d}x,
\end{aligned}$$

where for the final inequality we used the fact that v is energy minimising.

Combining this with (4.10) and (4.11), we get

$$\begin{aligned}
\int_{B_r(x_0)} |\partial(u-v)|^2\,\mathrm{d}x &= \int_{B_r(x_0)} \partial(u-v)\cdot\partial u\,\mathrm{d}x \\
&\leq \int_{B_r(x_0)} \langle \nabla(u-v), \nabla u\rangle \,\mathrm{dvol}_g + C(C_2,C_3)\,r^\alpha \int_{B_r(x_0)} |\partial u|^2\,\mathrm{d}x \\
&\leq C(C_1, \|u\|_{C^{0,\alpha}})\,r^\alpha \int_{B_r(x_0)} e(u)\,\mathrm{dvol}_g + C(C_2,C_3)\,r^\alpha \int_{B_r(x_0)} |\partial u|^2\,\mathrm{d}x \\
&\leq C(C_1, C_2, C_3, \|u\|_{C^{0,\alpha}})\,r^\alpha \int_{B_r(x_0)} |\partial u|^2\,\mathrm{d}x,
\end{aligned} \tag{4.12}$$

where we used (4.8) for the final inequality.

This inequality is better for smaller values of r, which suggests that we iterate.[14] We aim to compare $A_{\frac{r}{2}}(u)$ to $A_r(u)$, where we write $A_\rho(\cdot) = \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} |\partial \cdot|^2\,\mathrm{d}x$ for convenience.

Note that for any w, we have

$$A_{\frac{r}{2}}(w) = \frac{2^n}{|B_r(x_0)|}\int_{B_{\frac{r}{2}}(x_0)} |\partial w|^2\,\mathrm{d}x \leq \frac{2^n}{|B_r(x_0)|}\int_{B_r(x_0)} |\partial w|^2\,\mathrm{d}x = 2^n A_r(w).$$

[14] If we don't have Hölder continuity, then the analogue of (4.12) we obtain is missing the factor of r^α, which prevents us from iterating.

If $w = v$, we can improve this. Note that

$$\begin{aligned}
\Delta|\partial v|^2 &= \sum_j \partial_j^2 |\partial v|^2 \\
&= \sum_{i,j,\beta} 2\left(\left(\partial_i \partial_j v^\beta\right)^2 + \partial_i v^\beta \partial_i \partial_j^2 v^\beta \right) \\
&= 2\sum_\beta \left(|D^2 v^\beta|^2 + \sum_i \partial_i v^\beta \partial_i (\Delta v^\beta) \right) \\
&\geq 0,
\end{aligned}$$

so $|\partial v|^2$ is subharmonic, giving us the mean value inequality

$$\frac{d}{d\rho}\left(\frac{1}{B_\rho(x_0)} \int_{B_\rho(x_0)} |\partial v|^2 \,\mathrm{d}x \right) \geq 0 \quad \text{for } 0 \leq \rho \leq r.$$

In particular, $A_{\frac{r}{2}}(v) \leq A_r(v)$.

Finally, the energy minimising property of v gives that for any ρ, w,

$$A_\rho(v) \leq A_\rho(w).$$

Now

$$|\partial u|^2 = \partial u \cdot \partial(u-v) + \partial(u-v)\cdot \partial v + |\partial v|^2 \leq |\partial(u-v)|(|\partial u| + |\partial v|) + |\partial v|^2.$$

Integrating this over $B_{\frac{r}{2}}(x_0)$ and using Cauchy-Schwarz gives

$$\begin{aligned}
A_{\frac{r}{2}}(u) &\leq \sqrt{A_{\frac{r}{2}}(u-v)}\left(\sqrt{A_{\frac{r}{2}}(u)} + \sqrt{A_{\frac{r}{2}}(v)}\right) + A_{\frac{r}{2}}(v) \\
&\leq C\sqrt{A_r(u-v)}\left(\sqrt{A_{\frac{r}{2}}(u)} + \sqrt{A_r(v)}\right) + A_r(v) \\
&\leq C\sqrt{A_r(u-v)}\left(\sqrt{A_{\frac{r}{2}}(u)} + \sqrt{A_r(u)}\right) + A_r(u).
\end{aligned}$$

Then (4.12) gives that

$$\begin{aligned}
A_{\frac{r}{2}}(u) &\leq Cr^{\frac{\alpha}{2}}\sqrt{A_r(u)}\left(\sqrt{A_{\frac{r}{2}}(u)} + \sqrt{A_r(u)}\right) + A_r(u) \\
&\leq Cr^{\frac{\alpha}{2}}\left(\frac{C}{2}A_r(u) + \frac{1}{2C}A_{\frac{r}{2}}(u)\right) + \left(1 + Cr^{\frac{\alpha}{2}}\right)A_r(u) \\
&\leq Cr^{\frac{\alpha}{2}}\left(\frac{C}{4}A_r(u) + \frac{1}{C}A_{\frac{r}{2}}(u)\right) + \left(1 + Cr^{\frac{\alpha}{2}}\right)A_r(u) \\
&\leq \left(1 - r^{\frac{\alpha}{2}}\right)^{-1}\left(1 + Cr^{\frac{\alpha}{2}}\right)A_r(u) \\
&\leq \left(1 + Cr^{\frac{\alpha}{2}}\right)A_r(u),
\end{aligned}$$

where C depends only on $C_1, C_2, C_3, \|u\|_{C^{0,\alpha}}$.

If we set $r_i = 2^{-i}r_0$, then

$$\frac{1}{|B_{r_i}(x_0)|}\int_{B_{r_i}(x_0)} |\partial u|^2 \,\mathrm{d}x \leq \prod_{j=0}^{\infty}\left(1 + Cr_0^{\frac{\alpha}{2}}2^{\frac{i\alpha}{2}}\right)\frac{1}{|B_{r_0}(x_0)|}\int_{B_{r_0}(x_0)} |\partial u|^2 \,\mathrm{d}x,$$

by iteration, so for each $0 < r < r_0$ we have

$$\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |\partial u|^2 \,\mathrm{d}x \le C.$$

Applying (4.8) for the final time gives that

$$\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} e(u)\,\mathrm{dvol}_g \le C\,\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |\partial u|^2 \,\mathrm{d}x \le C$$

for each $0 < r < r_0$.

Now x_0 was arbitrary and C does not depend on x_0 (only on u and the local behaviour of g, h). This means that $e(u) = \langle \nabla u, \nabla u\rangle$ is locally bounded, so ∇u is a L^∞_{loc} function, and u is therefore locally Lipschitz.

□

Once we have that u is Lipschitz, the conclusion of theorem 4.1 follows from a bootstrap argument, which we will briefly outline, first in the case where g, h are C^∞, following part of [HW08, §4.2]. We have that $\nabla u \in L^\infty_{loc}$, so working in local coordinates, as u is a weak solution of (4.2), this means that $\Delta_{\mathcal{M}} u \in L^\infty_{loc}$. We can then apply regularity results for systems similar to those we have seen so far (see [Mor08, theorem 6.2.5]) to get that $u \in W^{2,p}_{loc}$ for each $1 \le p < \infty$. Then given any $1 \le p' < \infty$, we see that $g^{ij}(x)\Gamma^\alpha_{\beta\gamma}(u(x))\partial_i u^\beta \partial_j u^\gamma \in W^{1,p'}_{loc}$ because $Du \in W^{1,2p'}_{loc}$. Hence (4.2) gives that $\Delta_{\mathcal{M}} u \in W^{1,2p'}_{loc}$, so again applying regularity results gives that $u \in W^{3,p}_{loc}$ for each $1 \le p < \infty$. Repeating this process, we see that $u \in W^{k,p}_{loc}$ for any $1 \le p < \infty$, $k \in \mathbb{N}$, so u is smooth. In the case where g, h are not C^∞, this process stops after some number of iterations, giving that u is locally $W^{k,p}$ for some $k \in \mathbb{N}$. In order to recover theorem 4.1, we must carry out this process with results which give local $C^{k,\alpha}$ regularity of u, similar to the results obtained at the end of §1.2.

This means that the regularity problem for harmonic maps can be reduced to the question of continuity. We briefly outline the situation regarding this, following ideas from [Hél02, §1.5].

In the case m $(= \dim \mathcal{M}) = 1$, note that any map $u \in \mathcal{D}$ is in $H^1(\mathcal{M}, B_R)$ for $R = \|u\|_{L^\infty((M))}$, $B_R \subseteq \mathbb{R}^N$. Then $u \in C^{0,\frac{1}{2}}(\mathcal{M})$ by Morrey's inequality, so any harmonic map $u : \mathcal{M} \to \mathcal{N}$ is continuous, hence smooth.

If $m = 2$, we cannot use the argument above. However, provided $\mathcal{N}$ is compact and without boundary, it is still true that all weakly harmonic maps are continuous [Hél02, §4.1].

When $m \ge 3$, the picture is much more complicated. The possibility of a regularity result for all weakly harmonic maps has been ruled out by the construction by Rivière of (finite energy) weakly harmonic maps with value in S^2 [Riv95]. However, the picture for minimizing and stationary maps is more favourable - minimising maps must be continuous outside a singular set of Hausdorff dimension at most $m - 3$, and stationary maps must be continuous outside a singular set of Hausdorff dimension at most $m - 2$.

It turns out that the example $u : B^3 \to S^2$, $B_1 \subseteq \mathbb{R}^3$, $x \mapsto \frac{x}{|x|}$ is a minimising map [Hél02, §1.5]. This shows that the result for minimising maps is optimal.

4.5 Scalar methods applied to the energy

To finish this section, we follow [SY97, §IX.4], to see how ideas similar to those in §1.3 can be applied to the energy of harmonic maps, and how this gives us interesting results about harmonic maps themselves.

We begin with the Bochner formula for harmonic maps ([SY97, §IX.4],[EL78, pg 12]),

$$\frac{1}{2}\Delta_{\mathcal{M}}e(u) = |\nabla(du)|^2 - \sum_{i,j} h(\mathrm{Riem}^{\mathcal{N}}(u_*e_i, u_*e_j)u_*e_i, u_*e_j) + \sum_{\alpha} \mathrm{Ricci}^{\mathcal{M}}(u^*\theta_\alpha, u^*\theta_\alpha).$$

Here ∇ is the pullback of the connection on $\mathcal{N}$, $e_1, \ldots, e_m$ is an orthonormal basis for TM, and $\theta_1, \ldots, \theta_n$ is an orthonormal basis for T^*N. Suppose $\mathcal{M}$ is compact, so we can bound the third term from below by $-C|du|^2 = -Ce(u)$. Hence if $\mathcal{N}$ has nonpositive sectional curvature, then discarding the first two terms gives

$$\Delta_{\mathcal{M}}e(u) \geq -C(\mathcal{M})e(u). \tag{4.13}$$

Alternatively, the second term is also bounded from below, this time by $-C(\mathcal{N})|du|^4 = -Ce(u)^2$, so if $e(u) \leq 1$, then we can discard the first term and bound $e(u)^2$ by $e(u)$ to get

$$\Delta_{\mathcal{M}}e(u) \geq -C_0(\mathcal{M},\mathcal{N})e(u). \tag{4.14}$$

This lets us use the following generalisation of theorem 1.4 [Mor08, theorem 5.3.1].

Theorem 4.3. *Suppose $w \in H^1_{loc}(\Omega) \cap L^2(\Omega)$ satisfies $w \geq 1$ in $\Omega \subseteq \mathbb{R}^m$. Suppose also that*

$$-\partial_i(a_{ij}\partial_j w + b_i w) + c_i\partial_i w + dw \leq 0$$

in the weak sense in Ω, where the coefficients $a_{ij} \in L^\infty(\Omega)$, $b_i, c_i \in L^m(\Omega)$, $d \in L^{\frac{n}{2}}(\Omega)$ are such that

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \forall\xi \in \mathbb{R}^m$$

for almost all $x \in \Omega$, and

$$\|b_i\|^2_{L^m(B_r(x))} + \|c_i\|^2_{L^m(B_r(x))} + \|d\|_{L^{\frac{m}{2}}(B_r(x))} \leq C_1 r^\mu$$

for each $B_r(x) \subseteq \Omega$.

Then $w \in L^\infty_{loc}(\Omega)$, and there exists $C = C(m, \lambda, \Lambda, C_1, \mu)$ such that if $0 < \rho \leq R$ and $B_{R+\rho}(x_0) \subseteq \Omega$, then

$$\|w\|^2_{L^\infty(B_R(x_0))} \leq C\rho^{-m}\|w\|^2_{L^2(B_{R+\rho}(x_0))}$$

for each $x \in B_R(x_0)$.

We give our own explanation for how this applies in our case.

Suppose $u \in C^2(\overline{\Omega}, \mathcal{N})$ is a harmonic map with respect to the Euclidean metric g_0 on a compact set $\overline{\Omega} \subseteq \mathbb{R}^m$, and that $e(u) \leq 1$, so that (4.14) gives $-\Delta e(u) - C_0 e(u) \leq 0$. Now $e(u)$ is defined using only the first derivatives of u, so $e(u) \in C^1(\Omega)$.

Given $\varepsilon \in (0,1)$, write $v_\varepsilon = \frac{1}{\varepsilon}(e(u) + \varepsilon) \in C^1(\Omega)$, so that

$$-\Delta v_\varepsilon - C_0 v_\varepsilon = \frac{1}{\varepsilon}(-\Delta_{\mathcal{M}}e(u) - C_0 e(u) - C_0\varepsilon) \leq 0$$

in the weak sense, and $1 \leq v_\varepsilon \leq M_\varepsilon$, $\partial_i v_\varepsilon \leq M_\varepsilon$ for each i, where $M_\varepsilon = \frac{1}{\varepsilon}\|e(u)\|_{C^1(\Omega)}$.

Next, take $w_\varepsilon \geq 0$ such that $w_\varepsilon^2 = v_\varepsilon$. Then $w_\varepsilon \in C^1(\Omega)$ because v_ε stays away from 0, so by compactness of $\overline{\Omega}$ we have that $w_\varepsilon \in H^1_{loc}(\Omega) \cap L^2(\Omega)$. Further, $1 \leq w_\varepsilon \leq \sqrt{M_\varepsilon}$, $\partial_i v_\varepsilon \leq \sqrt{M_\varepsilon}$ for each i, so

$$\begin{aligned} 0 &\geq -\Delta_{\mathcal{M}}(w_\varepsilon^2) - C_0 w_\varepsilon^2 \\ &= -2w_\varepsilon \Delta_{\mathcal{M}} w_\varepsilon - 2|\nabla w_\varepsilon|^2 - C_0 w_\varepsilon^2 \\ &\geq \sqrt{M_\varepsilon}\Big(-2\Delta w_\varepsilon - 2\sum_i \partial_i w_\varepsilon - C_0 w_\varepsilon\Big), \end{aligned}$$

again in the weak sense. Dividing by $\sqrt{M_\varepsilon}$ gives that

$$-2\Delta w_\varepsilon - 2\sum_i \partial_i w_\varepsilon - C_0 w_\varepsilon \leq 0,$$

so we can apply theorem 4.3 to get

$$|w_\varepsilon(x)|^2 \leq C(m, C_0)\rho^{-m} \int_{B_{R+\rho}(x_0)} w_\varepsilon^2$$

whenever $0 < \rho \leq R$, $B_{R+\rho}(x_0) \subseteq \Omega$, $x \in B_R(x_0)$.

Finally, substituting $w_\varepsilon^2 = v_\varepsilon = \frac{1}{\varepsilon}(e(u) + \varepsilon)$ and multiplying by $\frac{1}{\varepsilon}$ gives

$$e(u)(x) + \varepsilon \leq C\rho^{-m} \int_{B_{R+\rho}(x_0)} (e(u) + \varepsilon) \leq C\rho^{-m} \int_{B_{R+\rho}(x_0)} e(u) + C\rho^{-m}(R+\rho)^m \varepsilon.$$

Sending $\varepsilon \to 0$ and rewriting ρ, R, we have shown that

$$\sup_{B_\rho(x_0)} e(u) \leq C(R-\rho)^{-m} \int_{B_R(x_0)} e(u)$$

whenever $0 < R - \rho \leq \frac{R}{2}$, $B_R(x_0) \subseteq \Omega$. We see that if $\frac{R}{2} \leq R - \rho < R$, then $\left(\frac{R}{2}\right)^{-m} \leq (R-\rho)^{-m}$ and $\emptyset \neq B_{2\rho}(x_0) \subseteq B_R(x_0)$, so

$$\sup_{B_\rho(x_0)} e(u) \leq C(\rho)^{-m} \int_{B_{2\rho}(x_0)} e(u) \leq C(R-\rho)^{-m} \int_{B_R(x_0)} e(u).$$

Finally, we that $e(u)(x_0) \leq C(R-\rho)^{-m} \int_{B_R(x_0)} e(u)$ for each $\rho > 0$, so sending $\rho \to 0$ gives the same result for $\rho = 0$. Noting that C_0 depends on $\mathcal{M} = \overline{\Omega}$, $\mathcal{N}$, we have shown the following.

Lemma 4.4. *Let $u \in C^2(\overline{\Omega}, \mathcal{N})$ be a harmonic map with respect to the Euclidean metric g_0 on a compact set $\overline{\Omega} \subseteq \mathbb{R}^m$, such that $e(u) \leq 1$. Then there exists $C = C(m, \overline{\Omega}, \mathcal{N})$ such that*

$$\sup_{B_\rho(x_0)} e(u) \leq C(R-\rho)^{-m} \int_{B_R(x_0)} e(u)$$

whenever $0 \leq \rho < R$, $B_R(x_0) \subseteq \Omega$.

Note the same argument works when $\mathcal{N}$ is nonpositively curved, and in that case C does not depend on $\mathcal{N}$.

We will now use this to prove the main theorem of this subsection, following a proof from [SY97, §IX.4]. We have rearranged it to show where the ideas come from, and added additional explanation for clarity.

Theorem 4.5. *Suppose $u \in C^2(\overline{B}_{r_0}, \mathcal{N})$ is harmonic with respect to the Euclidean metric g_0 on $B_{r_0} \subseteq \mathbb{R}^m$, $m \geq 3$. Then there exist $\varepsilon > 0$, $C \geq 0$ depending only on $m, \mathcal{N}$, such that if*

$$r_0^{2-m} \int_{B_{r_0}} e(u) \leq \varepsilon, \tag{4.15}$$

then

$$\sup_{B_{\frac{r_0}{2}}} e(u) \leq C r_0^{-m} \int_{B_{r_0}} e(u). \tag{4.16}$$

Proof. We begin by stating the following lemma (consequence of [SY97, §IX, lemma 1.2]).

Lemma 4.6. *If u is a stationary map from a domain $\Omega \subseteq \mathbb{R}^m$, $m \geq 3$, to a manifold $\mathcal{N}$, then for each $x \in \Omega$ and $0 < \rho < r < \operatorname{dist}(x, \partial\Omega)$ we have*

$$r^{2-m} \int_{B_r(x)} e(u) - \rho^{2-m} \int_{B_\rho(x)} e(u) \geq 0.$$

Hence

$$\rho^{2-m} \int_{B_\rho(x)} e(u) \leq r^{2-m} \int_{B_r(x)} e(u)$$

whenever $x \in B_{r_0}$ and $0 < \rho < r < r_0 - |x|$.

Note that given $r_1 < r_0$, $x \in B_{r_1}$, $0 < \rho \leq r_1 - |x|$, by applying lemma 4.6 and increasing the area of integration (see figure 4.1) we get

$$\rho^{2-m} \int_{B_\rho(x)} e(u) \leq \left(\frac{r_0 + r_1}{2} - |x|\right)^{2-m} \int_{B_{\frac{1}{2}(r_0+r_1)-|x|}(x)} e(u) \leq C \left(\frac{r_0}{r_0 - r_1}\right)^{m-2} r_0^{2-m} \int_{B_{r_0}} e(u),$$

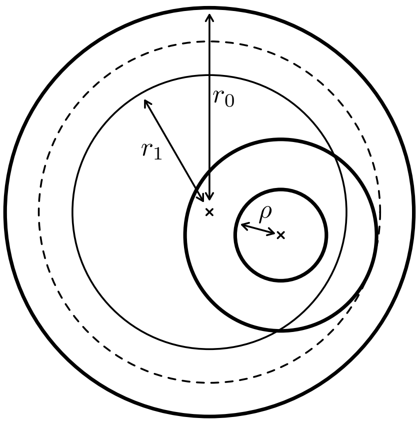

Figure 4.1

and setting $r_1 = \frac{3}{4} r_0$ gives[15]

$$\rho^{2-m} \int_{B_\rho(x)} e(u) \leq 4^{m-2} C r_0^{2-m} \int_{B_{r_0}} e(u) \tag{4.17}$$

[15] Note that [SY97] has $\left(\frac{3}{4}\right)^{m-2}$ in the expression below, and the original paper [Sch84] containing this proof has $\left(\frac{4}{3}\right)^{m-2}$. We have corrected this here.

for any $x \in B_{r_1}$ and $0 < \rho \leq r_1 - |x|$.

Now take some $B_{\rho_0}(x_0) \subseteq B_{r_1}$ to be fixed later. We aim to use lemma 4.4, so we must rescale u. Given $a > 0$, write $\rho_1 = a\rho_0$, and consider the map $v : (B_{\rho_1}, g_0) \to \mathcal{N}$ given by $v(y) = u\left(\frac{1}{a}y + x_0\right)$, which is also harmonic. Then for each $y \in B_{\rho_1}$, if $x = f(y)$ then

$$\begin{aligned} e(v)(y) &= \sum_{i,j} \delta_{ij} h_{\alpha\beta}(v(y)) \partial_{y^i} v^\alpha(y) \partial_{y^j} v^\beta(y) \\ &= \sum_{i,j} \delta_{ij} h_{\alpha\beta}(u(x)) \cdot \frac{1}{a}\partial_{x^i} u^\alpha(x) \cdot \frac{1}{a}\partial_{x^j} u^\beta(x) \\ &= \frac{1}{a^2} e(u)(x). \end{aligned}$$

Note that if $a = \sqrt{e_0}$, where $e_0 = \sup_{B_{\rho_0}(x_0)} e(u)$, then $e(v) \leq 1$. Assume for now that $\rho_1 \leq 1$, that is that $\sqrt{e_0}\,\rho_0 \leq 1$. Then we can instead take $a = (\rho_0)^{-1} \geq \sqrt{e_0}$ to get a harmonic map $v : B_1 \to \mathcal{N}$ with energy at most 1 (and rescaling to B_1 avoids the dependence of C on $\overline{\Omega}$ in lemma 4.4). Then using lemma 4.4 with $R = \rho_1$, $\rho = 0$ gives

$$\begin{aligned} \rho_0^2\, e(u)(x_0) &= e(v)(0) \\ &\leq C(m, \mathcal{N}) \int_{B_1} e(v) \\ &\leq C\rho_0^{-m} \int_{B_{\rho_0}(x)} \rho_0^2\, e(u) \\ &\leq 4^{m-2} C r_0^{2-m} \int_{B_{r_0}} e(u), \end{aligned}$$

where for the last line we used (4.17). So if $e_0 \leq \rho_0^{-2}$, then

$$\rho_0^2\, e(u)(x_0) \leq C(m, \mathcal{N})\, r_0^{2-m} \int_{B_{r_0}} e(u).$$

Now we choose x_0. Given $r_2 < r_1$, note that the function $x \mapsto e(u)(x)$ is continuous on $\overline{B}_{r_2}$, so we take x_0 to be the point where it attains its maximum (possibly on ∂B_{r_2}).

We could set $r_2 = \frac{1}{2}r_0$, $\rho_0 = \frac{1}{4}r_0$ to get (4.16), but the condition $\rho_1 \leq 1$ becomes $\sup_{B_{\rho_0}(x_0)} e(u) \leq \rho_0^{-2}$. This still contains a supremum, so is not what we want. Instead, we wait to fix r_2. We could take $\rho_0 = r_1 - r_2$, but it will be helpful to instead set $\rho_0 = \frac{1}{2}(r_1 - r_2)$, so that $r_1 - (r_2 + \rho_0) = \rho_0$ (see figure 4.2).

This gives

$$\frac{1}{4}(r_1 - r_2)^2 \sup_{B_{r_2}} e(u) \leq C r_0^{2-m} \int_{B_{r_0}} e(u).$$

The function $r \mapsto (r_1 - r)^2 \sup_{B_r} e(u)$ is continuous on $[0, r_1]$, so attains its maximum, and is 0 at r_1. We take $r_2 < r_1$ to be the point where this maximum is attained, so that

$$\sup_{B_r} e(u) \leq C \left(\frac{r_1 - r}{r_0}\right)^{-2} \cdot r_0^{-m} \int_{B_{r_0}} e(u) \tag{4.18}$$

for each $r \leq r_1$. Setting $r_2 = \frac{r_0}{2}$ and recalling that $r_1 = \frac{3r_0}{4}$ gives (4.16), provided $e_0 = \sup_{B_{\rho_0}(x_0)} e(u) \leq \rho_0^{-2}$.

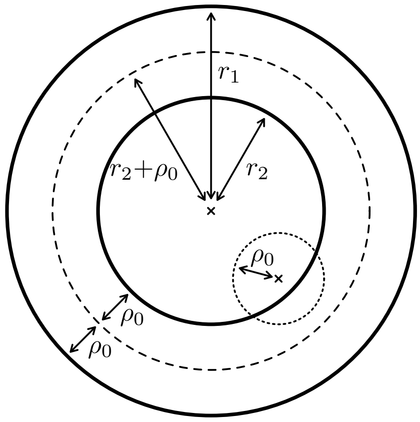

Figure 4.2

Now we return to this condition. Suppose $e_0 \geq \rho_0^{-2}$, so that taking $a = \sqrt{e_0}$ and rescaling as above gives $\rho_1 > 1$. Then $v|_{B_1}$ is a harmonic map with energy at most 1, so applying lemma 4.4 on B_1 and using (4.17) gives

$$\begin{aligned}
e_0^{-1} e(u)(x_0) &= e(v)(0) \\
&\leq C(m, \mathcal{N}) \int_{B_1} e(v) \\
&= C\left(\sqrt{e_0}\right)^m \int_{B_{e_0^{-\frac{1}{2}}}(x_0)} e_0^{-1} e(u) \\
&= C e_0^{-\frac{1}{2}(2-m)} \int_{B_{\sqrt{e_0}}(x_0)} e(u) \\
&\leq 4^{m-2} C r_0^{2-m} \int_{B_{r_0}} e(u).
\end{aligned}$$

But we also have

$$\begin{aligned}
e_0 &= \sup_{B_{\rho_0}(x_0)} e(u) \\
&\leq \sup_{B_{r_2+\rho_0}} e(u) \\
&= \frac{1}{\rho_0^2}(r_1 - (r_2 + \rho_0))^2 \sup_{B_{r_2+\rho_0}} e(u) \\
&\leq \frac{1}{\rho_0^2}(r_1 - r_2)^2 \sup_{B_{r_2}} e(u) \\
&= \frac{1}{\rho_0^2}(2\rho_0)^2 e(u)(x_0) \\
&= 4\, e(u)(x_0),
\end{aligned}$$

so combining these inequalities gives

$$1 \leq C r_0^{2-m} \int_{B_{r_0}} e(u). \tag{4.19}$$

If we require that (4.15) holds for a suitable ε, then we exclude this possibility. $\square$

We add our own note that in the proof above, if we take $r_1 = \beta r_0$ in (4.17) for $\beta \in (0,1)$, avoid absorbing the resulting factor of β^{-m} into the constant, and write $r = \alpha r_0$ for $\alpha \in (0, \beta)$, then (4.18) becomes

$$\begin{aligned}\sup_{B_{\alpha r_0}} e(u) &\leq C \left(\frac{1}{1-\beta}\right)^{m-2} \left(\frac{r_1 - \alpha r_0}{r_0}\right)^{-2} r_0^{-m} \int_{B_{r_0}} e(u) \\ &= \frac{1}{(1-\beta)^{m-2}(\beta-\alpha)^2} \cdot C r_0^{-m} \int_{B_{r_0}} e(u),\end{aligned}$$

and (4.19) becomes

$$1 \leq \left(\frac{1}{1-\beta}\right)^{m-2} \cdot C r_0^{2-m} \int_{B_{r_0}} e(u),$$

where C is independent of α, β. For fixed α, we have $\inf_{\beta \in (\alpha,1)} (1-\beta)^{2-m}(\beta-\alpha)^{-2} = C(m)(1-\alpha)^{-m}$, and if $\beta \in (0,1)$ then $(1-\beta)^{2-m} r_0^{2-m} \int_{B_{r_0}} e(u) > r_0^{2-m} \int_{B_{r_0}} e(u)$, so we have shown the following.

Theorem 4.7. *Suppose u satisfies the conditions of theorem 4.5. Then there exist $\varepsilon > 0$, $C \geq 0$ depending only on $m, \mathcal{N}$, such that if*

$$r_0^{2-m} \int_{B_{r_0}} e(u) \leq \varepsilon,$$

then

$$\sup_{B_{\alpha r_0}} e(u) \leq C \frac{1}{(1-\alpha)^m} r_0^{-m} \int_{B_{r_0}} e(u).$$

Hence we have an analogue of theorem 1.4 for the energy of harmonic maps. While we have given the result for the Euclidean metric on subsets of $\mathbb{R}^m$, it can be obtained in a more general case (for example, in [SY97, §IX.4], it is stated for the case when $\lambda \delta_{ij} \leq g_{ij} \leq \Lambda \delta_{ij}$). We also note that if $\mathcal{N}$ is negatively curved, then (4.13) holds with no condition on $e(u)$, so the small energy condition can be dropped from theorem 4.5 in this case.

We can use theorem 4.5 to prove the following [SY97, §IX, corollary 4.4].

Corollary 4.8. *Suppose $\mathcal{M}$, $\mathcal{N}$ are compact. Define the set*

$$\mathcal{F}_\Lambda := \left\{ u \in C^\infty(\mathcal{M}, \mathcal{N} : u \text{ is harmonic}, E(u) = \int_{\mathcal{M}} e(u) \,\mathrm{dvol}_g \leq \Lambda \right\}.$$

Then any map u in the weak H^1 closure of $\mathcal{F}_\Lambda$ is smooth and harmonic outside some set S which is relatively closed in the interior of $\mathcal{M}$, and has Hausdorff dimension at most $m-2$.

Estimates like theorem 4.5 are very useful for proving other results, and hold in a wide variety of cases. For example, it is remarked in [Lin99, proposition 1.4] that theorem 4.5 is true for stationary harmonic maps, and so corollary 4.8 also holds in this case.

Conclusion

In this essay, we have examined two methods for establishing the regularity of elliptic equations (or systems if the coefficients are continuous). The results we initially obtained (theorems 1.4 and 2.4) couldn't be immediately generalised to harmonic maps (as we saw in §3.1 and §3.2), but the methods were still able to be adapted to give partial regularity results, for example theorems 4.1 and 4.8.

We also looked at how the method of freezing coefficients can be adapted to give boundary regularity, provided the conclusion of remark 2.3 is true. Unfortunately, we were not able to prove the result in this remark. With more time, it would have been interesting to explore techniques which we could have used to prove this. The question of boundary regularity also prompted an exploration into Campanato spaces with the intention of generating an example showing the necessity of a cusp assumption on Ω. Unfortunately, we did not obtain such an example, but the exploration did suggest a way to extend the result we had obtained (see §3.4).

If we had more time, we would also have investigated applications of Campanato spaces to harmonic maps, to better tie this topic in to our essay. In particular, Campanato spaces can be defined on compact manifolds and behave as expected, at least in the scalar-valued case [Gei88], so we hoped to try and adapt the proof of theorem 2.4, with the same choice of coordinates as in the proof of theorem 4.1, to show that continuous harmonic maps between compact manifolds are Hölder continuous. This would have first required more work to define Campanato spaces for maps between manifolds, which prevented us from doing this.

We would also have enjoyed further exploring other topics, for example the partial regularity results mentioned at the end of §4.4, or the applications of theorem 4.5 at the end of §4.5. However, we hope that what we were able to explore was enough to showcase the versatility of the methods we covered.

References

[Cam65] Sergio Campanato, *Equazioni ellittiche del* II deg *ordine espazi* $\mathfrak{L}^{(2,\lambda)}$, Ann. Mat. Pura Appl. (4) **69** (1965), 321–381, DOI 10.1007/BF02414377 (Italian). MR192168

[DG68] Ennio De Giorgi, *Un esempio di estremali discontinue per un problema variazionale di tipo ellittico*, Boll. Un. Mat. Ital. (4) **1** (1968), 135–137 (Italian). MR0227827

[EL78] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978), no. 1, 1–68, DOI 10.1112/blms/10.1.1. MR495450

[Eva10] Lawrence C. Evans, *Partial differential equations*, 2nd ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010. MR2597943

[Gei88] M. Geisler, *Morrey-Campanato spaces on manifolds*, Comment. Math. Univ. Carolin. **29** (1988), no. 2, 309–318. MR957401

[Gia83] Mariano Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Mathematics Studies, vol. 105, Princeton University Press, Princeton, NJ, 1983. MR717034

[GT01] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR1814364

[Giu03] Enrico Giusti, *Direct methods in the calculus of variations*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003. MR1962933

[GM68] Enrico Giusti and Mario Miranda, *Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni*, Boll. Un. Mat. Ital. (4) **1** (1968), 219–226 (Italian). MR0232265

[GL23] Damiano Greco and Pier Domenico Lamberti, *Extension and embedding theorems for Campanato spaces on $C^{0,\gamma}$ domains* (2023). `arXiv:2302.11990`.

[HL11] Qing Han and Fanghua Lin, *Elliptic partial differential equations*, 2nd ed., Courant Lecture Notes in Mathematics, vol. 1, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2011. MR2777537

[Hél02] Frédéric Hélein, *Harmonic maps, conservation laws and moving frames*, 2nd ed., Cambridge Tracts in Mathematics, vol. 150, Cambridge University Press, Cambridge, 2002. Translated from the 1996 French original; With a foreword by James Eells. MR1913803

[HW08] Frédéric Hélein and John C. Wood, *Harmonic maps*, Handbook of global analysis, Elsevier Sci. B. V., Amsterdam, 2008, pp. 417–491, 1213, DOI 10.1016/B978-044452833-9.50009-7. MR2389639

[Lin99] Fang-Hua Lin, *Gradient estimates and blow-up analysis for stationary harmonic maps*, Ann. of Math. (2) **149** (1999), no. 3, 785–829, DOI 10.2307/121073. MR1709303

[Mor08] Charles B. Morrey Jr., *Multiple integrals in the calculus of variations*, Classics in Mathematics, Springer-Verlag, Berlin, 2008. Reprint of the 1966 edition [MR0202511]. MR2492985

[Riv95] Tristan Rivière, *Everywhere discontinuous harmonic maps into spheres*, Acta Math. **175** (1995), no. 2, 197–226, DOI 10.1007/BF02393305. MR1368247

[Sch84] Richard M. Schoen, *Analytic aspects of the harmonic map problem*, Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983), Math. Sci. Res. Inst. Publ., vol. 2, Springer, New York, 1984, pp. 321–358, DOI 10.1007/978-1-4612-1110-5_17. MR765241

[SY97] R. Schoen and S. T. Yau, *Lectures on harmonic maps*, Conference Proceedings and Lecture Notes in Geometry and Topology, II, International Press, Cambridge, MA, 1997. MR1474501

[Tah15] Ali Taheri, *Function spaces and partial differential equations. Vol. 2*, Oxford Lecture Series in Mathematics and its Applications, vol. 41, Oxford University Press, Oxford, 2015. Contemporary analysis. MR3443825