How ‘Big’ is the Moduli Space of Riemann Surfaces?

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Contents

1 Introduction

We begin with the following rough definition of the moduli space.

Definition 1.1. The *moduli space* $\mathcal{M}(S)$ of a compact orientable topological surface S is the set of isomorphism classes of Riemann surfaces homeomorphic to S.

We are interested in compact Riemann surfaces without boundary, which are orientable and hence fully characterised by their genus g. This means that we will be investigating $\mathcal{M}(S_g)$, where S_g is the closed genus g surface.

Our main focus of this essay is to look at the 'size' of the moduli space, which we do in two ways. In chapters 2 and 3, we find local coordinates for the moduli space, use this to see how 'close' $\mathcal{M}(S_g)$ is to being compact, and explore a natural compactification of the space. Then in chapters 4 and 5, we aim to numerically quantify the size: first, we examine a metric on the moduli space, then we use what we've learned to investigate Mirzakhani's calculation of the volume of $\mathcal{M}(S_g)$.

1.1 The three different cases

The following theorem allows us to separate the spaces $\mathcal{M}(S_g)$ into three different cases, depending on the genus g.

Theorem 1.1 (Uniformization theorem for Riemann surfaces)**.** *Any simply connected Riemann surface is isomorphic to one of:*

- *the Riemann sphere* $\mathbb{C} \cup \{\infty\}$*,*
- *the complex plane* $\mathbb{C}$*,*
- *the unit disc* $\mathbb{D}$*.*

These surfaces can be given constant curvature metrics (compatible with their complex structures) of curvature $+1$, 0, and -1 respectively. So as noted in [Til07], any Riemann surface X has universal cover isomorphic to one of these possibilities, and can hence be given a constant curvature metric. By Gauss–Bonnet, the curvature of the metric has the same sign as $\chi(X)$, so we get the cases $g = 0$, $g = 1$, and $g \geq 2$.

If $g = 0$, then the moduli space is a single point, since by uniformization all compact simply connected Riemann surfaces are isomorphic to the Riemann sphere. Hence the simplest nontrivial case is $g = 1$. The space $\mathcal{M}(T^2)$ is much easier to work with directly than moduli spaces for higher genus, and many of its features are analogous to those of $\mathcal{M}(S_g)$ for $g \geq 2$, so in chapters 2 and 3 we will use $\mathcal{M}(T^2)$ to guide our exploration of these more complicated spaces. In chapters 4 and 5, our constructions are more reliant on working with hyperbolic surfaces, which is only applicable for $g \geq 2$. Nevertheless, it is still possible to apply similar ideas to the moduli space of the torus, so we return to this case at the end of each chapter.

1.2 Independent calculations and proofs

We wish to draw attention to the following calculations and proofs, which are our own independent work.

- Proposition 3.3, where we calculate the lengths of simple closed geodesics in a unit-area flat torus.
- Theorems 3.5 and 3.6, where we prove that $\widehat{\mathcal{M}}(S_g)$ is sequentially compact.
- The calculations in §4.2 justifying the choices of area element and matrix inner product, and the calculation in §4.3 justifying the formula for Φ.
- The (incomplete) argument in §4.6 establishing that the Poincaré metric on $\mathbb{H}$ is the analogue to the Weil–Petersson metric.
- The short calculation in §5.5 showing that $V_{g,0}$ is a rational multiple of π^{6g-6}.
- §5.6, where we find the volumes of moduli spaces not covered by [Mir07].

2 Understanding local coordinates

2.1 The moduli space of the torus

The moduli space of the torus is the simplest nontrivial case, so we begin with this example.

Any genus 1 Riemann surface has fundamental group isomorphic to $\mathbb{Z}^2$, and from §1.1 we know it also has universal cover isomorphic to $\mathbb{C}$. Hence the surface can be written as a quotient $\mathbb{C}/\Lambda$, where

$$\Lambda = \{mw_0 + mw_1 : m, n \in \mathbb{Z}\},$$

for complex numbers $w_0, w_1 \in \mathbb{C}$ with $\frac{w_0}{w_1} \notin \mathbb{R}$. We use this to describe the moduli space, following the second strategy in [Don11, §6.3.2] (slightly expanded).

First, we can rescale the lattice without changing the isomorphism class of the torus, so we rescale (w_0, w_1) to $(\zeta, 1) := (\frac{w_0}{w_1}, 1)$. By swapping w_0, w_1 if necessary, we can assume that $\zeta\mathbb{H}$.

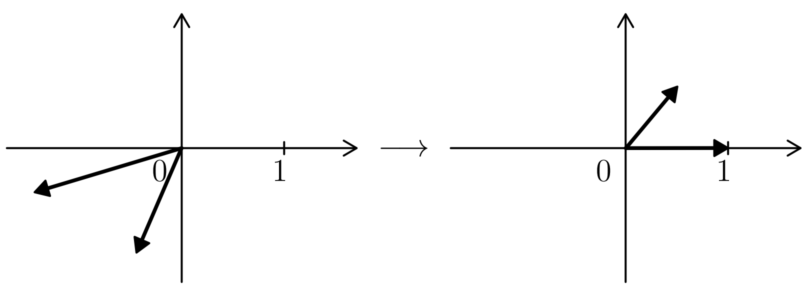

Figure 2.1: Rescaling (w_0, w_1)

Next, we note that different elements of $\mathbb{H}$ can represent the same lattice (eg. $\zeta = i$ and $\zeta = i + 1$ both give $\Lambda = \mathbb{Z} + i\mathbb{Z}$). Now (w_0, w_1) and (z_0, z_1) generate the same lattice if and only if $\left(\begin{smallmatrix} w_0 \\ w_1 \end{smallmatrix}\right) = A\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right)$ for some $A \in GL(2, \mathbb{Z})$, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \zeta \\ 1 \end{pmatrix} = \begin{pmatrix} a\zeta + b \\ c\zeta + d \end{pmatrix} \text{ which rescales to } \left(\frac{a\zeta + b}{c\zeta + d}, 1\right).$$

So we're left with the action of $PSL(2, \mathbb{Z})$ on $\mathbb{H}$ via Möbius maps (since scaling a matrix doesn't affect its action on $\mathbb{H}$). Hence $\mathcal{M}(T^2) \cong \mathbb{H}/PSL(2, \mathbb{Z})$ as sets.

A fundamental domain for the action of $PSL(2, \mathbb{Z})$ on $\mathbb{H}$ is[1]

$$\{z \in \mathbb{H} : |z| > 1 \text{ and } \operatorname{Re}(z) \in (-\tfrac{1}{2}, \tfrac{1}{2})\}$$

with the edge identifications shown in Figure 2.2.

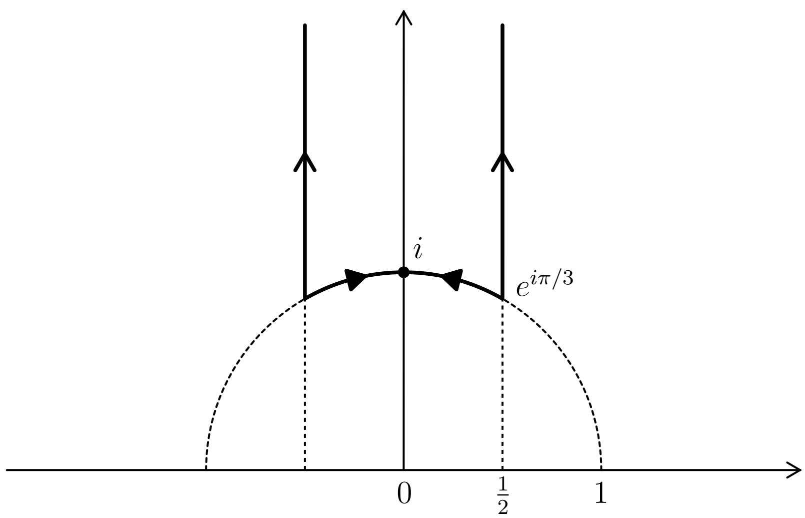

Figure 2.2: A fundamental domain for the action of $PSL(2, \mathbb{Z})$ on $\mathbb{H}$

Finally, we can give $\mathcal{M}(T^2)$ a topology by declaring the bijection with $\mathbb{H}/PSL(2, \mathbb{Z})$ to be a homeomorphism.

2.2 Teichmüller space

One of the key steps in §2.1 was to introduce a larger space, and take the quotient by a group action to get $\mathcal{M}(T^2)$. Additionally, as remarked in [Don11, §11.2.4], all compact oriented hyperbolic surfaces have a compatible complex structure, so for genus $g \geq 2$ we can speak of compact oriented hyperbolic surfaces instead of compact Riemann surfaces. These ideas lead us to the following definitions from [FM12, §10.1].

Definition 2.1. Fix a compact surface S. A *marked hyperbolic surface* is a pair (X, φ), where X is a hyperbolic surface and $\varphi : S \to X$ is a diffeomorphism.

[1]See [Don11, §6.3.2] for proof.

Two marked hyperbolic surfaces $(X_1, \varphi_1), (X_2, \varphi_2)$ are *homotopic* if there is an isometry $f : X_1 \to X_2$ such that $f \circ \varphi_1, \varphi_2$ are homotopic.

The *Teichmüller space*[2] $\mathcal{T}(S)$ of S is the set of homotopy classes of marked hyperbolic surfaces.

This is similar to the role played by $\mathbb{H}$ in §2.1, with the marking playing the role of a choice of basis for Λ.

2.3 Pants

We aim to understand hyperbolic surfaces by cutting them into simpler pieces. In order to understand these pieces, we present the following definitions and a lemma from [FM12, §10.5].

Definition 2.2. A *pair of pants* is a surface homeomorphic to a sphere with 3 boundary components. We require hyperbolic pairs of pants to have totally geodesic boundary.

A *pants decomposition* of a compact surface S with Euler characteristic $\chi(S) < 0$ is a collection of disjoint simple closed curves in S, such that cutting along the curves splits S into a disjoint union of pairs of pants.

Assigning an orientation to each curve gives an *oriented pants decomposition.*

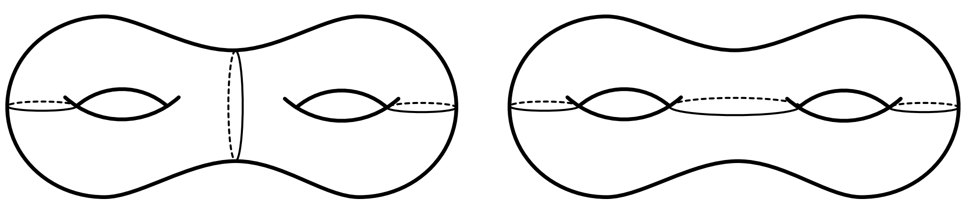

Figure 2.3: The two types of pants decomposition for $g = 2$

It is shown in [FM12, §8.3] that a collection of curves in S_g is a pants decomposition if and only if it is a collection of $3g-3$ disjoint essential simple

[2]As remarked in [JP13], this space was introduced by Oswald Teichmüller, along with the idea of reframing Riemann's notion of the existence of $3g-3$ parameters (or "moduli") associated with a closed genus $g \geq 2$ Riemann surface, into the number of complex dimensions of a "moduli space". Riemann was able to give heuristic counts of the number of moduli (for examples of heuristic counts see [FM12, §10.4]), but Teichmüller's construction allowed for a formal proof.

closed curves such that no two curves are isotopic. Here, a curve is *essential* if it is not homotopic to a point or boundary component.

Lemma 2.1. *Given any triple $(a, b, c) \in \mathbb{R}^3_+$, there is a unique (up to isometry) hyperbolic right-angled hexagon with a marked vertex such that alternate sides of the hexagon have lengths a, b, c, moving anticlockwise from the marked vertex.*

Proof. For $t > 0$, choose geodesics α_t, β_t in $\mathbb{H}$ a distance t apart, with γ'_t the unique geodesic segment realising this distance. Then take perpendicular geodesic rays β'_t, α'_t at distances a, b along α_t, β_t from γ'_t as shown in Figure 2.4.

There is $t_0 > 0$ such that $\alpha'_{t_0}, \beta'_{t_0}$ meet at the boundary of $\mathbb{H}$. If $t > t_0$ then α'_t, β'_t are a distance $C(t) > 0$ apart, and there is a unique geodesic segment γ_t realising this distance. Then $C(t)$ increases continuously from 0 to infinity as t increases from t_0 to infinity, so there is a unique t such that $C(t) = c$. The geodesics constructed above then form the edges of the desired hyperbolic hexagon.

To see uniqueness, note that we made no more choices after the choice of α_t, β_t, which was unique up to isometries of $\mathbb{H}$. □

As noted in [FM12], two copies of a right-angled hyperbolic hexagon can be glued to give a pair of pants with boundary components of any length (see Figure 2.5). Conversely, a hyperbolic pair of pants has three unique geodesic arcs connecting the boundary components pairwise, which we call *seams* (as in [Do13, lemma 3]). Cutting along the seams gives two hyperbolic hexagons, which by Lemma 2.1 must be uniquely specified. This proves the following proposition.

Proposition 2.2. *Given any triple $(a, b, c) \in \mathbb{R}^3_+$, there is a unique (up to isometry) hyperbolic pair of pants with boundary components of lengths a, b, c.*

2.4 Length and twist parameters

Now we focus on the surface S_g for $g \geq 2$, and fix an oriented pants decomposition $\mathcal{P} = \{\gamma_1, \ldots, \gamma_{3g-3}\}$ of S_g. If $\mathcal{X} = [(X, \varphi)] \in \mathcal{T}(S_g)$, then Proposition 2.2 shows that the lengths of the curves $\varphi(\gamma_i)$ fully determines the pairs of pants which make up the surface. This motivates the following definitions.

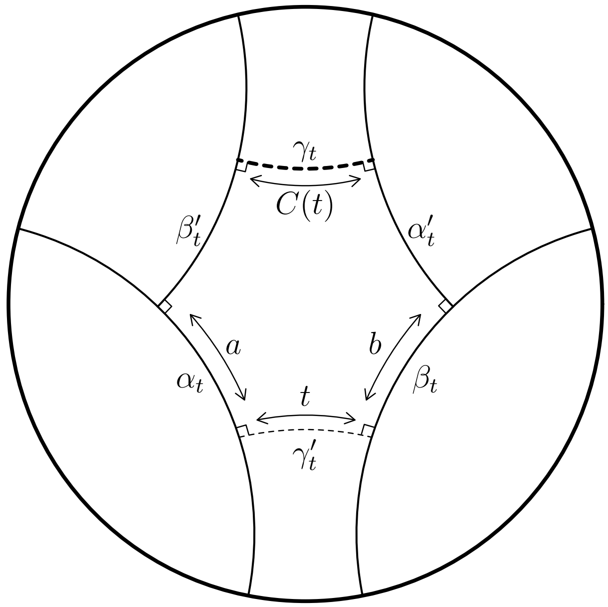

Figure 2.4: The picture for the proof of Lemma 2.1
(similar to [FM12, Figure 10.4])

Definition 2.3. For any surface S, if $\mathcal{X} = [(X, \varphi)] \in \mathcal{T}(S)$ is the equivalence class of (X, φ) and $\gamma \subseteq S$ is a simple closed curve, then $\ell_{\mathcal{X}}(\gamma)$ is the length of the unique[3] geodesic in the isotopy class of $\varphi(\gamma)$.

Given $\mathcal{X} = [(X, \varphi)] \in \mathcal{T}(S_g)$, the ith *length parameter* of $\mathcal{X}$ is $\ell_i(\mathcal{X}) = \ell_{\mathcal{X}}(\gamma_i)$.

We also need to know how the pairs of pants are glued together, so we define *twist parameters*, following the definition in [Do13].

When fixing $\mathcal{P}$, also fix a collection of disjoint simple closed curves $\mathcal{B} = \{\beta_j, \ldots, \beta_n\}$ in S_g, such that the restriction of $\mathcal{B}$ to any pair of pants P (determined by $\mathcal{P}$) gives three disjoint arcs connecting the boundary components of P in pairs (see Figure 2.6).

Now given $\mathcal{X} = [(X, \varphi)] \in \mathcal{T}(S_g)$, assume that each $\varphi(\gamma_i)$ is a geodesic (otherwise use the unique geodesic in its isotopy class). Then each curve $\varphi(\beta_j)$ can be uniquely modified by homotopy to a (not necessarily embedded) length-minimising curve $\hat{\beta}_j$ entirely contained within the $\varphi(\gamma_i)$ and the seams

[3] See [FM12, §1.2.1] for proof.

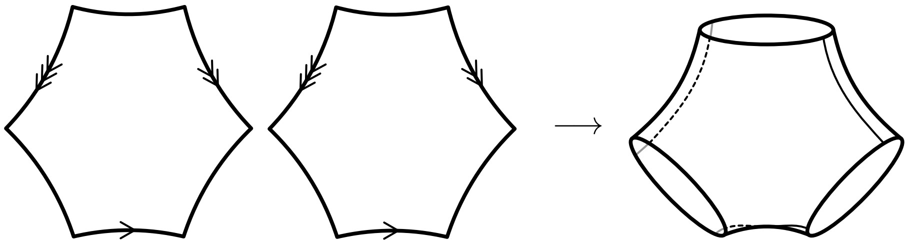

Figure 2.5: Gluing hyperbolic hexagons into pairs of pants

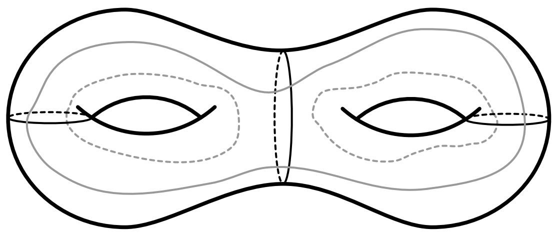

Figure 2.6: The extra curves used to define twist parameters

of the pairs of pants. Then for each γ_i, take some $\hat{\beta}_j$ which crosses it,[4] and define the ith *twisting parameter*[5] τ_i to be the signed distance travelled by $\hat{\beta}_j$ along $\varphi(\gamma_i)$, with the sign determined by the orientation of γ_i.

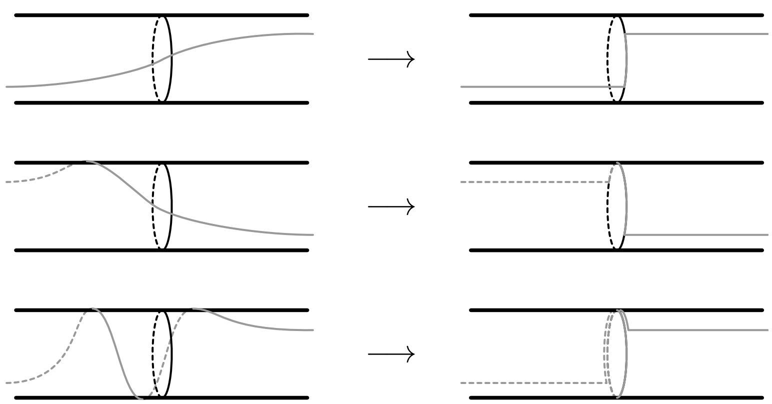

Figure 2.7: Modifying β_j

[4]The final definition is independent of the choice of $\hat{\beta}_j$ (see [FM12, §10.6] for proof)

[5]This definition of twisting parameter matches the one used in [Wol85] and [Mir07], where a full twist adds ℓ_i to τ_i, but differs from that in [FM12] and [IT92], where a full twist adds 2π to τ_i.

The numbers $(\ell_1, \tau_1, \ldots, \ell_{3g-3}, \tau_{3g-3})$ are the *Fenchel–Nielsen coordinates* of $\mathcal{X}$. If we take

$$FN : \mathcal{T}(S_g) \to (\mathbb{R}_+ \times \mathbb{R})^{3g-3}$$

defined by

$$FN(\mathcal{X}) = (\ell_1, \tau_1, \ldots, \ell_{3g-3}, \tau_{3g-3}),$$

then we can see that this is a bijection by constructing an explicit inverse. (This is done in [FM12, §10.6] by taking the pairs of pants uniquely specified by the ℓ_i, and gluing them in the way uniquely specified by the τ_i.)

Finally, there is a natural topology on $\mathcal{T}(S_g)$, and it is shown in [IT92, §3.2.4] that the map FN is a homeomorphism with respect to this topology.

2.5 Fenchel–Nielsen coordinates for genus 1

The coordinates for $\mathcal{T}(T^2)$ arising from the identification $\mathcal{T}(T^2) \cong \mathbb{H}$ in §2.1 are quite different to the Fenchel–Nielsen coordinates for genus $g \geq 2$. However, the following construction from [IT92, §7.3.5] (expanded to give more detail) gives an analogue of length and twist parameters, which comes from cutting T^2 into a cylinder instead of a pair of pants.

By Gauss–Bonnet, all hyperbolic surfaces have fixed area given by their genus, so we mirror this by scaling the torus $\mathbb{C}/\Lambda$ to have area 1. If Λ has basis $(\zeta, 1)$, then the area of our torus is $\mathrm{Im}(\zeta)$, so scaling the basis gives $\left(\frac{\zeta}{\sqrt{\mathrm{Im}(\zeta)}}, \frac{1}{\sqrt{\mathrm{Im}(\zeta)}}\right)$.

Now if we take the curve γ for our cylinder decomposition to be the image of the x-axis, then we see from Figure 2.8 that we can fix an orientation for γ such that the resulting Fenchel–Nielsen coordinates are

$$(\ell, \tau) = \left(\frac{1}{\sqrt{\mathrm{Im}(\zeta)}}, \frac{\mathrm{Re}(\zeta)}{\sqrt{\mathrm{Im}(\zeta)}}\right).$$

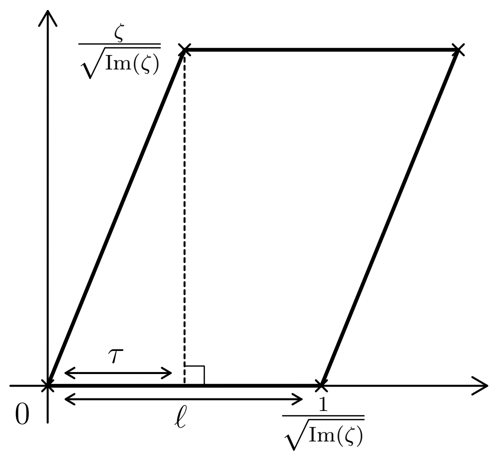

Figure 2.8: Finding length and twist parameters for a torus

2.6 Getting the moduli space from the Teichmüller space

As in §2.1, we can define the moduli space as the quotient of the Teichmüller space by a group action. In the genus 1 case, the group $PSL(2, \mathbb{Z})$ acts on the space of lattices with bases by changing the basis, so we want a group that acts on $\mathcal{T}(S)$ by changing the marking.

Consider the group $\mathcal{D}(S)$ of orientation-preserving diffeomorphisms $f : S \to S$. As noted in [IT92, §1.3], this acts on $\mathcal{T}(S)$ by pullback:

$$f \cdot [(X, \varphi)] = [(X, \varphi \circ f^{-1})] \quad \text{for } f \in \mathcal{D}(S).$$

Further, if $f \in \mathcal{D}(S)$ is homotopic to the identity, then $(X, \varphi), [(X, \varphi \circ f^{-1})]$ represent the same point of $\mathcal{T}(S)$, so we are really looking at the action of $\mathcal{D}(S)/\mathcal{D}_0(S)$, where $\mathcal{D}_0$ is the subgroup of $\mathcal{D}(S)$ of diffeomorphisms homotopic to the identity. We call $\mathrm{Mod}(S) := \mathcal{D}(S)/\mathcal{D}_0(S)$ the *mapping class group* of S, and we can now give the following definition.

Definition 2.4. The *moduli space* of a topological surface S is $\mathcal{M}(S) := \mathcal{T}(S)/\operatorname{Mod}(S)$.

Since the action of $\mathrm{Mod}(S)$ 'removes the marking' from elements of $\mathcal{T}(S)$, we will identify X with $[[(X, \varphi)]] \in \mathcal{M}(S)$, where φ is any marking of X.

We also have the following theorem from [FM12, §12.3].

Theorem 2.3. *For $g \geq 1$, the action of* $\mathrm{Mod}(S_g)$ *on* $\mathcal{T}(S_g)$ *is properly discontinuous.*

The properly discontinuous action of a group by isometries on any metric space induces a metric on the quotient space (see [DV97, corollary 2] for proof), so Theorem 2.3 means that any metric on $\mathcal{T}(S_g)$ induces a metric on $\mathcal{M}(S_g)$.

3 Investigating compactness

3.1 What does the moduli space of the torus look like?

In §2.1, we found a fundamental domain for the action of $\mathrm{Mod}(T^2)$ on $\mathcal{T}(T^2)$ (Figure 2.2). If we give $\mathbb{H} \cong \mathcal{T}(T^2)$ the Poincaré metric,[6] then after gluing we get the picture in Figure 3.1 (Figure 3.2 shows why the point $e^{i\pi/3}$ gives the hexagonal torus).

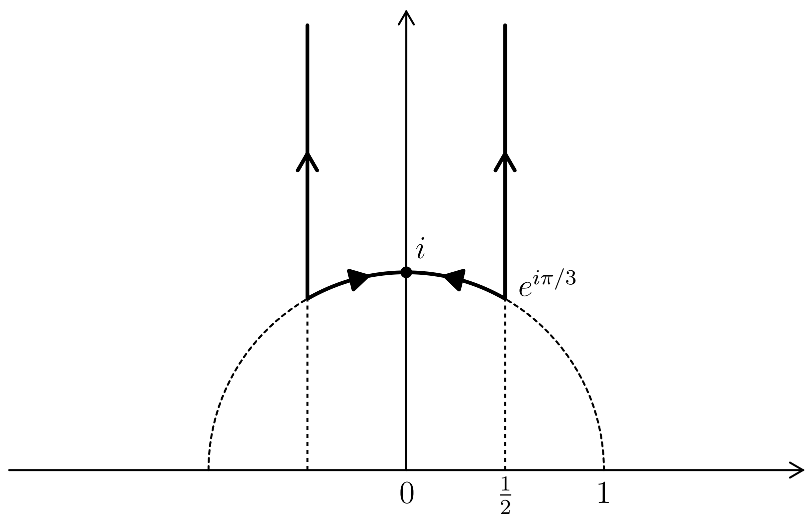

Figure 2.2: A fundamental domain for the action of $\mathrm{Mod}(T^2)$ on $\mathcal{T}(T^2)$

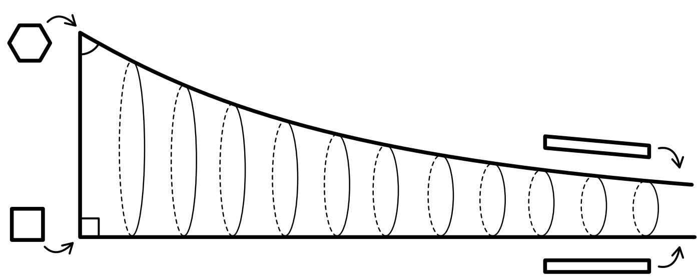

Figure 3.1: A picture of $\mathcal{M}(T^2)$ with sample points labelled (similar to [FM12, Figure 12.2])

[6]See §4.6 for justification.

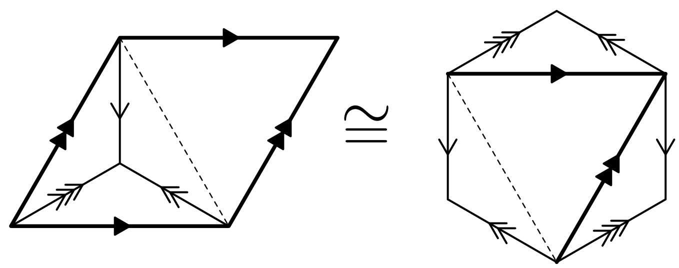

Figure 3.2: Why $e^{i\pi/3}$ gives a hexagonal torus
(similar to [FM12, Figure 12.3])

We see that $\mathcal{M}(T^2)$ is noncompact, with a single 'spike' which corresponds to shortening a geodesic. We will see that this picture remains true for the higher genus case as well.

3.2 The statement of Mumford's compactness criterion

Recall from §2.4 that the length of a simple closed curve is the length of the unique geodesic in its isotopy class. This allows us to give the following definition.

Definition 3.1. For $g \geq 1$ and $\varepsilon > 0$, the *ε-thick part* of $\mathcal{M}(S_g)$ is $\mathcal{M}_\varepsilon(S_g) = \{X \in \mathcal{M}(S_g) : \ell_X(\gamma) \geq \varepsilon$ for any essential simple closed curve γ in $X\}$

It is shown in [FM12, lemma 12.4] that for $g \geq 2$, every $X \in \mathcal{M}(S_g)$ is contained in some $\mathcal{M}_\varepsilon(S_g)$, which means that $\mathcal{M}(S_g) = \bigcup_{\varepsilon>0} \mathcal{M}_\varepsilon(S_g)$. The same result for the genus 1 case is an immediate consequence of proposition 3.3 below.

We state a theorem which we will prove in §3.3 and §3.4.

Theorem 3.1 (Mumford's compactness criterion)**.** *For $g \geq 1$ and $\varepsilon > 0$, $\mathcal{M}_\varepsilon(S_g)$ is compact.*

When combined with the following result from [FM12, §12.5], this lets us 'see' what the moduli space looks like.

Proposition 3.2. *For $g \geq 1$, $\mathcal{M}(S_g) \setminus \mathcal{M}(S_g)$ is path-connected.*

This is intuitively clear: if γ_1, γ_2 are 'short' in X_1, X_2 respectively, then we can make a path between them in $\overline{\mathcal{M}(S_g)} \setminus \mathcal{M}(S_g)$ by first contracting γ_2, then relaxing γ_1.

So we can see that for any genus, the moduli space has a single 'spike' similar to the case for $g = 1$.

3.3 Mumford's compactness criterion for genus 1

In this section, we work with unit-area tori, so write Λ_ζ for the lattice with basis $\left(\frac{\zeta}{\sqrt{\mathrm{Im}(\zeta)}}, \frac{1}{\sqrt{\mathrm{Im}(\zeta)}}\right)$, and $\mathbb{T}_\zeta := \mathbb{C}/\Lambda_\zeta$. Using this notation, we give our own result.

Proposition 3.3. *The set of lengths of essential simple closed geodesics in $\mathbb{T}_\zeta$ is equal to*

$$\left\{ \frac{1}{\sqrt{\mathrm{Im}(\zeta)}} |m\zeta + n| : m, n \in \mathbb{Z},\ \gcd(m, n) = 1 \right\}$$

Proof. Any closed geodesic (not necessarily simple) in $\mathbb{T}_\zeta$ can be represented by an element of $\pi_1(\mathbb{T}_\zeta) \cong \mathbb{Z}^2$, so it suffices to check the geodesic in each equivalence class $[\gamma] \in \pi_1(\mathbb{T}_\zeta)$. Take generators a, b for $\pi_1(\mathbb{T}_\zeta)$ as shown in Figure 3.3, and consider $[\gamma] = a^m b^n$ for $m, n \in \mathbb{Z}$.

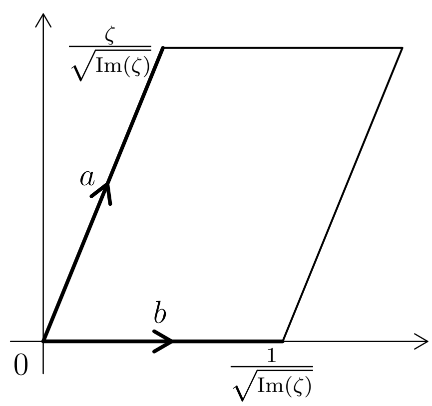

Figure 3.3: Generators for $\pi_1(\mathbb{T}_\zeta)$

We see that the geodesic $\widehat{\gamma} \in [\gamma]$ corresponds to the line segment joining 0 to $m \cdot \frac{1}{\sqrt{\mathrm{Im}(\zeta)}} + n \cdot \frac{\zeta}{\sqrt{\mathrm{Im}(\zeta)}}$ in $\mathbb{C}$. Clearly $\widehat{\gamma}$ is essential and simple if and only if $\gcd(m,n)=1$ (see Figure 3.4), and its length is

$$\frac{1}{\sqrt{\mathrm{Im}(\zeta)}}|m+n\zeta|,$$

so we're done.

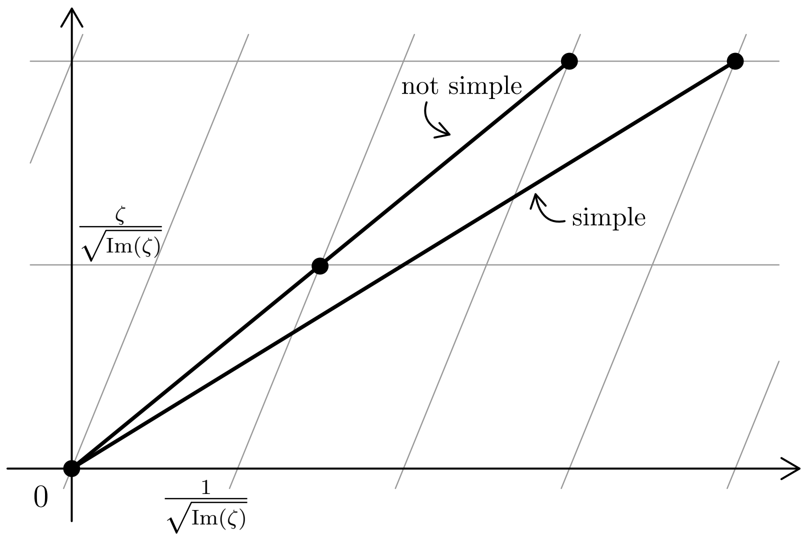

Figure 3.4: Geodesics in $\mathbb{T}_\zeta$

$\square$

By Proposition 3.3, the length of the shortest closed geodesic in $\mathbb{T}_\zeta$ is $(\mathrm{Im}\,\zeta)^{-1/2}\cdot\mathrm{Min}\{|\zeta|,1\}$. Restricting our attention to the fundamental domain given in §2.1, where $|\zeta|\geq 1$, this gives the picture for $\mathcal{M}_\varepsilon(T^2)$ in Figure 3.5. Hence Mumford's compactness criterion is true for $g=1$.

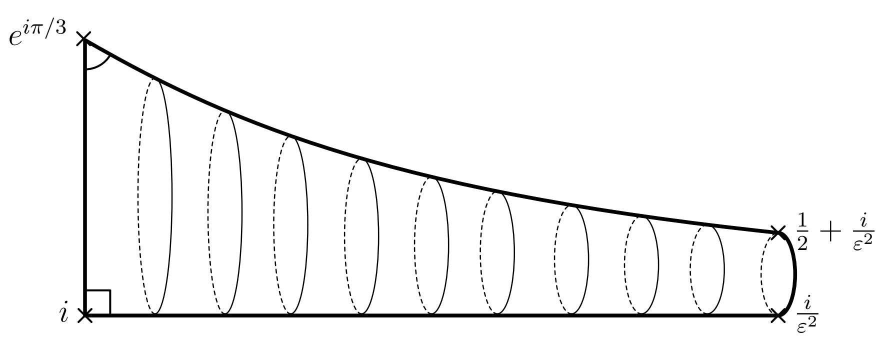

Figure 3.5: The ε-thick moduli space of the torus

3.4 Mumford's compactness criterion for higher genus

The proof of Mumford's compactness criterion for the torus relies on computing $\mathcal{M}_\varepsilon(T^2)$ explicitly. This isn't practical for genus $g \geq 2$, so we need a different strategy.

Note that $\mathcal{T}(S_g) \cong \mathbb{R}^{6g-6}$ is metrisable, so $\mathcal{M}(S_g)$ is metrisable by the remark in §2.6. This means that we can prove sequential compactness instead of compactness, so given a sequence in $\mathcal{M}_\varepsilon(S_g)$, we aim to lift it into a closed bounded subset of $\mathcal{T}(S_g)$. This requires an upper bound on the length parameters, which motivates the following lemma from [FM12, §12.4].

Theorem 3.4 (Bers' constant)**.** *Let $g \geq 2$. There is a constant $L = L(S_g)$ such that any hyperbolic surface homeomorphic to S_g has a pants decomposition $\mathcal{P} = \{\gamma_1, \dots, \gamma_{3g-3}\}$ with $\ell_X(\gamma_i) \leq L$ for each i.*

Strictly speaking, *Bers' constant* is the smallest such L, but we just need that such an L exists.

Proof. Our aim is to inductively build the collection $\{\gamma_1, \dots, \gamma_{3g-3}\}$.

To get the first curve, take any $x \in X$, and consider the disc

$$D(x, r) := \{\widetilde{x} \in X : d(\widetilde{x}, x) \leq r\}$$

where the distance $d(\widetilde{x}, x)$ is given by the hyperbolic metric on X. For small r, this is an embedded disc isometric to a disc of radius r in $\mathbb{H}$, and hence has area

$$\int_0^{2\pi} \int_0^r \sinh \rho \, d\rho \, d\theta = 2\pi(\cosh(r) - 1).$$

So if $r_x = \sup\{r : D(x, r) \text{ is an embedded disc in } X\}$, then we see that

$$2\pi(\cosh(r_x) - 1) = \operatorname{Area}(D(x, r_x)) \leq \operatorname{Area}(X) = -2\pi\chi(S_g),$$

so $r_x \leq \cosh^{-1}(1 - \chi(S_g))$. In particular, r_x is finite and bounded above by a function of S_g.

It must be true that $D(x, r_x)$ isn't embedded in X, so there are two radii of $D(x, r_x)$ which meet at their endpoints (see Figure 3.6). Their union is a simple essential closed geodesic γ_1 of length at most $L_1 := 2\cosh^{-1}(1 - \chi(S_g))$.

We repeat a similar process inductively to generate the other curves. Say k curves with length less than L_k have already been constructed, cut X along

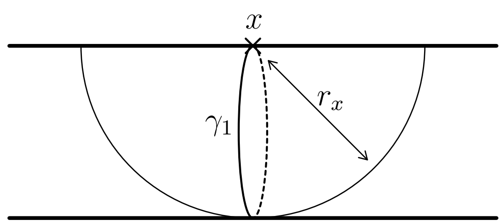

Figure 3.6: The case when $D(x, r_x)$ isn't embedded in X

them, and choose a component Y not homeomorphic to a pair of pants. Now take $x \in Y$ to be a point furthest from ∂Y, and set

$$r_x = \sup\{r : D(x, r) \text{ is embedded in } Y \text{ and disjoint from } \partial Y\}.$$

If $D(x, r_x)$ isn't embedded in X, then we proceed as above. Otherwise, $D(x, r_x)$ intersects ∂Y. This gives two radii ρ_1, ρ_2 connecting boundary components δ_1, δ_2 of Y (possibly $\delta_1 = \delta_2$). The boundary α of a neighbourhood of $\rho_1 \cup \rho_2 \cup \delta_1 \cup \delta_2$ is a simple closed curve (see Figure 3.7).

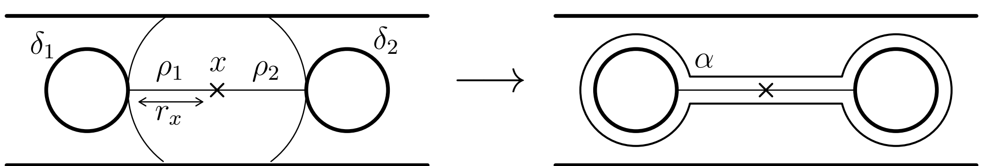

Figure 3.7: The case when $D(x, r_x)$ intersects ∂Y

The length of α can be made arbitrarily close to

$$2\ell_Y(\rho_1) + 2\ell_Y(\rho_2) + \ell_Y(\delta_1) + \ell_Y(\delta_2) \leq 4\cosh^{-1}(1 - \chi(S_g)) + 2L_k,$$

so the geodesic in its isotopy class has length at most this. Further, Y is not a pair of pants or annulus so α is essential.

Hence we can take $L_{k+1} = 4\cosh^{-1}(1 - \chi(S_g)) + 2L_k$. Once we've constructed $3g - 3$ curves, we're done, and we can take $L = L_{3g-3}$. □

Now we can finish the proof of Mumford's compactness criterion by checking the case $g \geq 2$, following the proof in [FM12, §12.4].

Proof of Theorem 3.1. Take any sequence (X_i) in $\mathcal{M}_\varepsilon(S_g)$, and let $\mathcal{X}_i \in \mathcal{T}(S_g)$ be lifts of the X_i.

For each $\mathcal{X}_i$, there is a pants decomposition $\mathcal{P}_i = \{\gamma_{i,1}, \ldots, \gamma_{i,3g-3}\}$ of S_g such that $\ell_{\mathcal{X}_i}(\gamma_{i,j}) \in [\varepsilon, L]$ for each j. There are only finitely many different topological types of pants decomposition of S_g, so by passing to a subsequence we can assume that each $\mathcal{P}_i$ is the same type.

For each i, we can now choose $f_i \in \mathrm{Mod}(S_g)$ such that $f_i(\mathcal{P}_i) = \mathcal{P}_1 := \mathcal{P}$, and let $\mathcal{Y}_i = f_i \cdot \mathcal{X}_i$. Then the $\mathcal{Y}_i$ all have length parameters in the interval $[\varepsilon, L]$ in Fenchel–Nielsen coordinates adapted to $\mathcal{P}$.

Further, full twists about any $\gamma_j \in \mathcal{P}$ don't change the element of $\mathcal{M}(S_g)$. Hence, by composing with elements of $\mathrm{Mod}(S_g)$ again, we obtain new lifts $\mathcal{Z}_i$ of the X_i with length parameters $\ell_j(\mathcal{Z}_i) \in [\varepsilon, L]$, and twist parameters $\tau_j(\mathcal{Z}_i) \in [-\frac{1}{2}\ell_j(\mathcal{Z}_i), \frac{1}{2}\ell_j(\mathcal{Z}_i)] \subseteq [-\frac{L}{2}, \frac{L}{2}]$. So

$$(\ell_1, \tau_1, \ldots, \ell_{3g-3}, \tau_{3g-3})(\mathcal{Z}_i) \in \left([\varepsilon, L] \times \left[-\tfrac{L}{2}, \tfrac{L}{2}\right]\right)^{3g-3}.$$

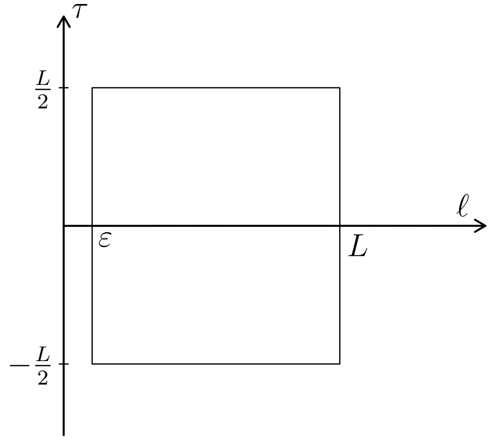

Figure 3.8: Lifting a subsequence into the Teichmüller space

This is a compact space, so $(\mathcal{Z}_i)$ has a convergent subsequence $(\mathcal{Z}_{i_j})$. Hence X_i has a convergent subsequence X_{i_j} converging to $X \in \mathcal{M}(S_g)$.

Finally, we need to check that $X \in \mathcal{M}_\varepsilon(S_g)$.[7] It is shown in [FM12, §10.3] that the length maps $\ell_\gamma(\mathcal{X}) : \mathcal{T}(S_g) \to \mathbb{R}$ are continuous, so we write

$$\mathcal{T}_\varepsilon(S_g) = \bigcap_{\gamma \in \pi_1(S_g)} \ell_\gamma^{-1}([\varepsilon, \infty))$$

where $\mathcal{T}_\varepsilon(S_g)$ is the ε-thick part of the Teichmüller space. This is obviously closed, so contains $\lim_{j\to\infty} \mathcal{Z}_{i_j}$. Hence X is contained in $\mathcal{M}_\varepsilon(S_g)$. $\square$

[7] This is our own addition, as it isn't checked in [FM12].

3.5 Compactifying the moduli space for genus 1

From the description in §3.1, we see that the only way for a sequence in $\mathcal{M}(T^2)$ to leave every compact set is if $\mathrm{Im}(\zeta_n) \to \infty$ for the corresponding sequence (ζ_n) in $\mathbb{H}$. We aim to compactify the space by adding the 'limits' of such sequences.

First we consider the sequence $(ni)_{n\in\mathbb{N}}$. We can see from Figure 3.9 that the 'limit' is[8] $T_\infty := \mathbb{C}/\mathbb{Z}$, where the equivalence class of $z \in \mathbb{C}$ is $\{z + n : n \in \mathbb{Z}\}$.

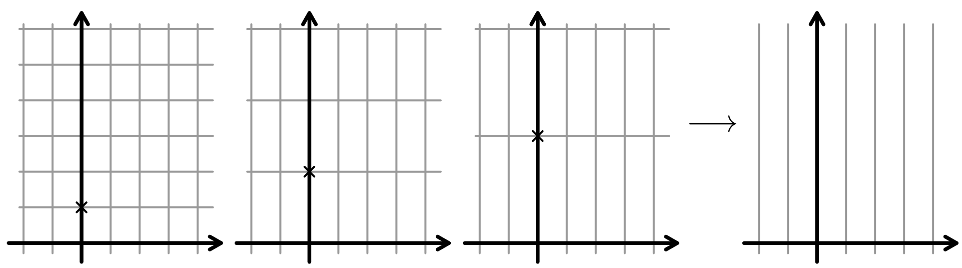

Figure 3.9: Lattices corresponding to $(ni)_{n\in\mathbb{N}}$

Further, if we change to a different sequence, for example $(n\zeta)_{n\in\mathbb{N}}$ for some $\zeta \in \mathbb{H}$, then the resulting 'limit' is biholomorphic to T_∞. Hence the one-point compactification is the natural compactification to choose for $\mathcal{M}(T^2)$.

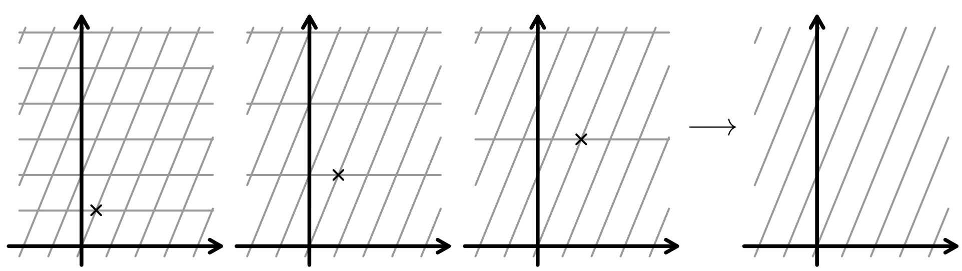

Figure 3.10: Lattices Corresponding to $(n\zeta)_{n\in\mathbb{N}}$

We will get a result more consistent with the construction in §3.6 (where the 'extra surfaces' are compact) if we consider our additional point to be the one-point compactification of T_∞.

[8]See [IT92, §B.1].

Note also that if $(\zeta_n)_{n\in\mathbb{N}}$ has $\mathrm{Im}(\zeta_n) \to \infty$, then in the Fenchel–Nielsen coordinates from §2.5 we have that the corresponding length parameters are

$$\ell(\mathbb{T}_{\zeta_n}) = \frac{1}{\sqrt{\mathrm{Im}(\zeta_n)}} \to 0 \text{ as } n \to \infty.$$

3.6 A compactification of the moduli space

In §3.5, we added a single point to $\mathcal{M}(T^2)$ to be the limit of all sequences (X_n) in $\mathcal{M}(T^2)$ with $\ell(X_n) \to 0$, independently of the behaviour of $\tau(X_n)$. Here we extend this idea to the case $g \geq 2$, to give the *Deligne–Mumford Compactification.*[9]

We give the following definitions from [HK14] (simplified to remove references to marked points).

Definition 3.2. A *stable curve* is a surface X which has a finite subset N (the set of *nodes* of X) such that:

- The components of $X \setminus N$ are all hyperbolic Riemann surfaces.
- Each node has a neighbourhood isomorphic to a neighbourhood of the origin in the curve with equation $xy = 0$ in $\mathbb{C}^2$.

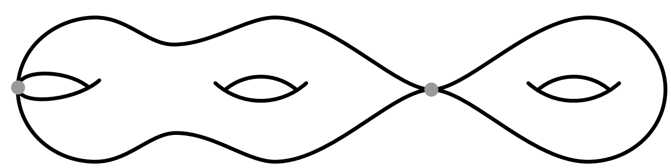

Figure 3.11: A stable curve

Definition 3.3. A *multicurve* Γ in a surface S is a set of disjoint curves $\{\gamma_1, \ldots, \gamma_n\}$, where no two curves are homotopic.

We saw in §2.3 that a multicurve in S_g is a pants decomposition if and only if it has $3g-3$ curves.

[9]Note the definition below is that of the *augmented moduli space* $\widehat{\mathcal{M}}(S)$. This is shown in [HK14] to be canonically isomorphic to the Deligne–Mumford compactification, so we will use the two terms interchangeably.

Definition 3.4. Take a compact surface S. A *marked stable curve* is a pair (X, φ) where X is a stable curve, $\varphi : S \to X$ is continuous, and there is a multicurve $\Gamma \subseteq S$ such that φ induces an orientation-preserving homeomorphism $\varphi_* : S/\Gamma \to X$. (Here S/Γ is the space obtained by collapsing the elements of Γ to points.) We say that X is *marked by S*.

Two marked stable curves $(X_1, \varphi_1), (X_2, \varphi_2)$ are *isotopic* if there exists an isomorphism $\alpha : X_1 \to X_2$, and a homeomorphism $\beta : S \to S$ which is isotopic to the identity, such that $\alpha \circ \varphi_1 = \beta \circ \varphi_2$. ($\alpha$ is the analogue of f in Definition 2.1, and β sends the multicurve collapsed by φ_1 to the multicurve collapsed by φ_2.)

The *augmented Teichmüller space* $\widehat{\mathcal{T}}(S)$ of S is the set of isotopy classes of stable curves marked by S.

Note that $\operatorname{Mod}(S_g)$ still acts on $\widehat{\mathcal{T}}(S_g)$, so we can define the *augmented moduli space* to be the quotient $\widehat{\mathcal{M}}(S_g) := \widehat{\mathcal{T}}(S_g)/\operatorname{Mod}(S_g)$.

We can give $\widehat{\mathcal{T}}(S_g)$ a topology which doesn't refer to Fenchel–Nielsen coordinates (see [HK14, §2]), but for simplicity we instead give the topology defined in [Mon09, §5].

Consider $\mathcal{X} = [(X, \varphi)] \in \widehat{\mathcal{T}}(S_g)$. If $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is the multicurve collapsed by φ, then add enough curves to get a pants decomposition $\mathcal{P} = \{\gamma_1, \ldots, \gamma_{3g-3}\}$ of S_g. Consider Fenchel–Nielsen coordinates adapted to $\mathcal{P}$, with the length parameters $\ell_1, \ldots, \ell_n$ all taken to be 0 and the twist parameters $\tau_1, \ldots, \tau_n$ left undefined.

We declare that a sequence $\mathcal{X}_i = [(X_i, \varphi_i)]$ of marked stable curves converges to $\mathcal{X}$ if and only if:

- There is some i_0 such that for each $i \geq i_0$, each curve collapsed by φ_i is homotopic to some curve in $\mathcal{P}$,
- For each $j = 1, \ldots, 3g-3$, we have that $\ell_j(\mathcal{X}_i) \to \ell_j(\mathcal{X})$ as $i \to \infty$, and
- For each $j = n+1, \ldots, 3g-3$, we have that $\tau_j(\mathcal{X}_i) \to \tau_j(\mathcal{X})$ as $i \to \infty$.

This defines a topology on $\widehat{\mathcal{T}}(S_g)$, and it is noted in [Mon09, §5] that the conditions above are equivalent to two conditions which are independent of the choice of $\gamma_{n+1}, \ldots, \gamma_{3g-3}$. Hence this topology is independent of the choice of pants decomposition. Further, we can see that this is the 'correct' topology because sequences which would converge in $\mathcal{M}(S_g)$ still converge to the same point, and the new convergent sequences correspond to 'convergence to a node'.

3.7 Compactness of the augmented moduli space

In §3.4, we saw the proof from [FM12] that $\mathcal{M}_\varepsilon(S_g)$ is sequentially compact. We give our own adaptation of this proof to give sequential compactness of $\widehat{\mathcal{M}}(S_g)$, beginning with an analogue of Theorem 3.4.

Theorem 3.5 (Bers' constant for stable curves). *Let $g \geq 2$. There is a constant $L = L(S_g)$ such that given any $\mathcal{X} = [(X, \varphi)] \in \widehat{\mathcal{T}}(S_g)$, there is a pants decomposition $\mathcal{P} = \{\gamma_1, \ldots, \gamma_{3g-3}\}$ of S_g such that:*

- *The multicurve Γ collapsed by φ is contained in $\mathcal{P}$.*
- *$\ell_X(\gamma_i) \leq L$ for each i.*

Proof. Given $\mathcal{X}$ as above, take any pants decomposition $\widetilde{\mathcal{P}}$ of S_g containing the curve Γ collapsed by φ. Then the multicurve $\varphi(\widetilde{\mathcal{P}})$ cuts $\mathcal{X}$ into pairs of pants, where we allow one or more boundary components of a pair of pants to have length 0.

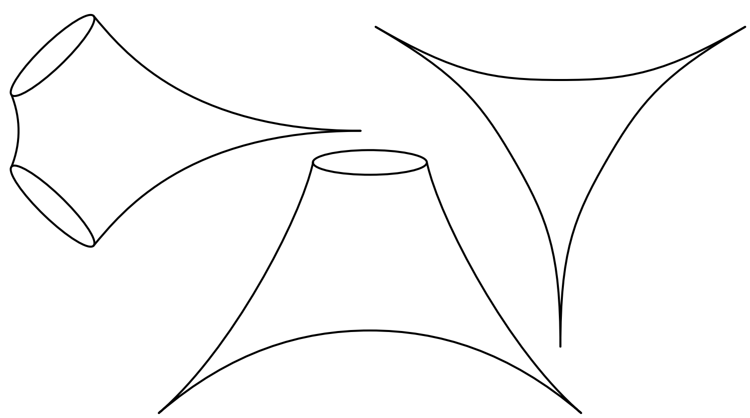

Figure 3.12: Degenerate Pants

Note that if a pair of pants has a length 0 boundary component (a *cusp*), then by considering the corresponding hyperbolic hexagon (which has a side of length 0), we can find a curve of length at most 1 which we can cut along to remove the cusp. In Figure 3.13, this is done by moving the curve δ upwards until it has length at most $\frac{1}{2}$.

This means that we can cut along curves in X to remove the nodes, giving a hyperbolic surface $\widetilde{X}$ with (non-geodesic) boundary components all of length at most 1.

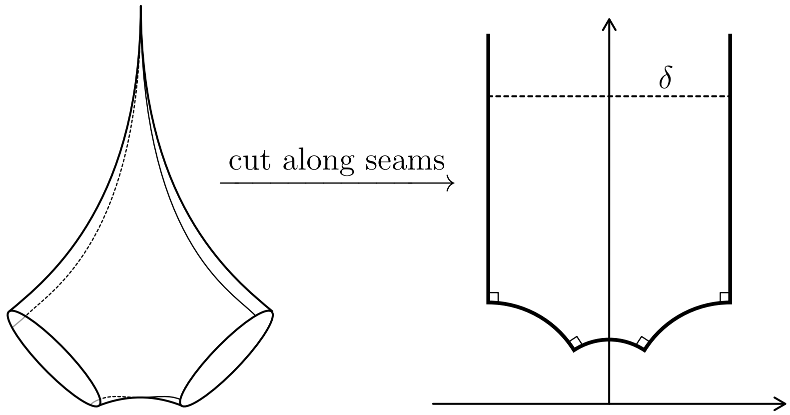

Figure 3.13: Finding a short curve around a node in a pair of pants

We also note that by [HK14, proposition 1.8], if X is a stable curve marked by S_g with set of nodes N, then $X \setminus N$ has area $-2\pi\chi(S_g)$. Hence $\mathrm{Area}(\widetilde{X}) \leq -2\pi\chi(S_g)$.

This allows us to repeat the procedure in the second step of the proof of Theorem 3.4 to get a constant $L = L(S_g)$ and $3g - 3 - |\Gamma|$ essential simple closed geodesics in $\widetilde{X}$, each with length at most L. Taking the preimage of these curves under φ gives a set of curves in S_g, and together with Γ these give the required pants decomposition of S_g. $\square$

This allows us to prove the following theorem.

Theorem 3.6. *If $g \geq 1$, then $\widehat{\mathcal{M}}(S_g)$ is sequentially compact.*

Proof. The case $g = 1$ was checked in §3.5, so assume that $g \geq 2$.

Take a sequence (X_i) in $\widehat{\mathcal{M}}(S_g)$. As in Theorem 3.4, we pass to a subsequence, use Theorem 3.5 and act with elements of $\mathrm{Mod}(S_g)$ to get lifts $\mathcal{Z}_i = [(X_i, \varphi_i)] \in \widehat{\mathcal{T}}(S_g)$ of the X_i with the following properties:

- The $\mathcal{Z}_i$ all have length parameters less than L and twist parameters between $-\frac{L}{2}$ and $\frac{L}{2}$, in Fenchel–Nielsen coordinates with respect to the same pants decomposition $\mathcal{P}$ of S_g.
- For each $\mathcal{Z}_i$, the multicurve collapsed by φ_i is contained in $\mathcal{P}$.

If we write

$$z_i := (\ell_1(\mathcal{Z}_i), \tau_1(\mathcal{Z}_i), \ldots, \ell_{3g-3}(\mathcal{Z}_i), \tau_{3g-3}(\mathcal{Z}_i)),$$

then (z_i) is a bounded sequence in $(\mathbb{R}_{\geq 0} \times \mathbb{R})^{3g-3}$, so has a subsequence (z_{i_j}) converging to some $z = (\ell_1, \tau_1, \ldots, \ell_{3g-3}, \tau_{3g-3}) \in (\mathbb{R}_{\geq 0} \times \mathbb{R})^{3g-3}$. Let $\mathcal{Z}$ be the stable curve with Fenchel–Nielsen coordinates $(\ell_1, \tau_1, \ldots, \ell_{3g-3}, \tau_{3g-3})$ with respect to $\mathcal{P}$, and X the corresponding element of $\widehat{\mathcal{M}}(S_g)$. Then we see from the description of the topology in §3.6 that $\mathcal{Z}_{i_j} \to \mathcal{Z}$ in $\widehat{\mathcal{T}}(S_g)$, so $X_{i_j} \to X$ in $\widehat{\mathcal{M}}(S_g)$.

$\square$

Interestingly, in the proof above we found a subsequence of $(\mathcal{Z}_i)$ such that all length and twist parameters converge. This indicates that if a version of $\widehat{\mathcal{M}}(S_g)$ exists where twisting around a node is recorded, then this space is also sequentially compact.

Finally, to deduce from Theorem 3.6 that $\widehat{\mathcal{M}}(S_g)$ is compact, we trace the following trail of references.

- From [Mon09, §5], $\widehat{\mathcal{T}}(S_g)$ is the completion of $\mathcal{T}(S_g)$ with respect to the Weil–Petersson metric (see §5).
- From [Bri10, Theorem A], $\mathrm{Mod}(S_g)$ acts on this completion by isometries.
- From [Him68, Theorem 4], if a group acts on a metric space by isometries, then the resulting quotient is pseudometrisable. (A pseudometric is a generalisation of a metric which allows distinct points to have distance 0.)
- From [Oec17, Proposition 1.50], compactness and sequential compactness are equivalent in a pseudometric space.

4 Measuring distance

In the rest of this essay, we aim to find the volume of the moduli space. We begin by giving $\mathcal{T}(S)$ the *Weil–Petersson metric*. This is a *Riemannian metric*, which associates an inner product on the tangent space to each point of $\mathcal{T}(S)$ in a smooth way. This allows us to measure distance in $\mathcal{T}(S)$ and hence in $\mathcal{M}(S)$, as noted in §2.6.

For most of this chapter, our smooth surface S will have $\chi(S) < 0$. In this case, we write $\mathcal{M}$, $\mathcal{T}$ and Mod in place of $\mathcal{M}(S)$, $\mathcal{T}(S)$ and Mod(S) for convenience.

4.1 A new way of looking at the Teichmüller space

In §1 we gave an informal definition of the moduli space, and in §2.6 we defined the moduli space as a quotient of the Teichmüller space, but we now need a different definition, for which we follow the definition in [Tro92].

Definition 4.1. A 1-1 *tensor* is a map H which assigns to each point $x \in S$ a linear map $H_x : T_xS \to T_xS$.

$C^\infty(T_1^1S)$ is the vector space of all smooth 1-1 tensors.

An *almost-complex structure* is a 1-1 tensor $J \in C^\infty(T_1^1S)$ such that $J_x^2 = -I_x$ for each $x \in S$, and (u, J_xu) forms an ordered basis for T_xS for each $u \in T_xS$.

$\mathcal{A}$ is the set of smooth almost-complex structures on S.

An almost-complex structure tells us what multiplication by i looks like on S: the first condition ensures that J_x is a quarter-turn rotation, and the second ensures the rotation is anticlockwise.

It is a fact that there is a bijection between complex structures on S (defined in terms of charts) and almost-complex structures on S. Hence we want the moduli space to be the set of almost-complex structures, modulo diffeomorphisms of S, as in the following definition.

Definition 4.2. $\mathcal{M} = \mathcal{A}/\mathcal{D}$ where the action is by pullback, that is

$$(f^*J)_x := (df_x)^{-1}J_{f(x)}df_x \text{ for } f \in \mathcal{D}, J \in \mathcal{A}.$$

Recall from §2.6 that $\mathcal{D}$ is the group of orientation-preserving diffeomorphisms of S, $\mathcal{D}_0$ is the subgroup of diffeomorphisms homotopic to the identity, and Mod $= \mathcal{D}/\mathcal{D}_0$.

Now almost-complex structures should only represent the same point of $\mathcal{T}$ if they differ by a diffeomorphism of S homotopic to the identity, so we take $\mathcal{T} = \mathcal{A}/\mathcal{D}_0$. Then we still have that $\mathcal{M} = \mathcal{A}/\mathcal{D} = (\mathcal{A}/\mathcal{D}_0)/(\mathcal{D}/\mathcal{D}_0) = \mathcal{T}/\operatorname{Mod}$, at least as sets (see [Tro92, §0]).

4.2 A natural metric on the space of almost complex structures

We want to define a Riemannian metric on $\mathcal{A}$ which we can pass to the quotient $\mathcal{T} = \mathcal{A}/\mathcal{D}_0$, so we need to understand its tangent space. We can get some insight by relaxing the C^∞ constraint on $C^\infty(T^1_1 S)$ and $\mathcal{A}$, to allow tensors which are k times differentiable (in the sense of distributions), giving the spaces $\mathcal{H}^k(T^1_1 S)$ and $\mathcal{A}^k$. Then $\mathcal{H}^k(T^1_1 S)$ is Banach, which gives access to the implicit function theorem for the following proposition from [Tro92, §1.1].

Proposition 4.1. *$\mathcal{A}^k$ is a smooth submanifold of $\mathcal{H}^k(T^1_1 S)$ with tangent space*

$$T_J\mathcal{A}^k = \{H \in \mathcal{H}^k(T^1_1 S) : HJ = -JH\}$$

Proof (sketch). If $J \in \mathcal{H}^k(T^1_1 S)$ then

$$J^2 = -I \Leftrightarrow \operatorname{tr} J = 0 \text{ and } \det J = 1,$$

so $\mathcal{A}^k = \mathcal{N}_0 \cap \mathcal{N}_1$, where $\mathcal{N}_0 := \operatorname{tr}^{-1}(0)$ and $\mathcal{N}_1 := \det^{-1}(1)$.

(For the forwards direction, if v is a nonzero vector field on a neighbourhood then check the form of J with respect to the basis $\{v_x, J_x v_x\}$ for $T_x S$. The reverse direction follows from the Cayley–Hamilton theorem.)

Then $\mathcal{N}_0$ is a subspace of $\mathcal{H}^k(T^1_1 S)$, and hence a C^∞ submanifold of $\mathcal{H}^k(T^1_1 S)$ with $T_J\mathcal{N}_0 = \{H : \operatorname{tr} H = 0\}$. The implicit function theorem gives that $\mathcal{N}_1$ is a C^∞ submanifold with tangent space $T_J\mathcal{N}_1 = \{H : \operatorname{tr} J^{-1}H = 0\}$.

Another application of the implicit function theorem gives that $\mathcal{A}^k = \mathcal{N}_0 \cap \mathcal{N}_1$ is a C^∞ submanifold of $\mathcal{H}^k(T^1_1 S)$ with tangent space

$$\begin{aligned} T_J\mathcal{A}^k &= T_J\mathcal{N}_0 \cap T_J\mathcal{N}_1 \\ &= \{H : \operatorname{tr} H = 0, \operatorname{tr} J^{-1}H = 0\} \\ &= \{H : \operatorname{tr} H = 0, \operatorname{tr} JH = 0\} \\ &= \{H : HJ = -JH\} \end{aligned}$$

where the third equality uses the fact that $J^2 = -I$, and the final equality is a straightforward point calculation.

□

This result extends to the C^∞ case—as remarked in [Dau05], $\mathcal{A}$ is indeed a smooth manifold with tangent space $T_J\mathcal{A} = \{H \in C^\infty(T_1^1 S) : HJ = -JH\}$.

As stated in [Tro92, §2.5], there is a natural L^2 metric on $\mathcal{A}$, given at a point $J \in \mathcal{A}$ by

$$\langle\langle H, K\rangle\rangle_J := \int_S \mathrm{tr}(HK)\, d\mu_{g(J)} \tag{1}$$

Here, $g(J)$ is the unique Riemannian metric with constant curvature -1 associated to J (see §4.3), and μ_g is the area element associated to a Riemannian metric g: if (u, v) is a positively oriented basis for T_xS then

$$\mu_g(u, v) := \sqrt{\det\begin{pmatrix} g_x(u,u) & g_x(u,v) \\ g_x(v,u) & g_x(v,v)\end{pmatrix}}, \tag{2}$$

and $\mu_g(v, u) = -\mu_g(u, v)$.

We now carry out our own calculations to justify why these are the natural definitions to choose. We work on $\mathbb{R}^2$ and choose coordinates, and check the definitions above match the resulting expressions.

Area element

Take a positively oriented basis $u = \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right), v = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$ for $\mathbb{R}^2$. Then the area of the parallelogram spanned by u, v is the area of the image of the unit square under the map $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} u_1 & v_1 \\ u_2 & v_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. This is

$$\begin{aligned}
\det\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} &= u_1v_2 - u_2v_1 \\
&= \sqrt{u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2} \\
&= \sqrt{\det\begin{pmatrix} u_1^2 + u_2^2 & u_1v_1 + u_2v_2 \\ v_1u_1 + v_2u_2 & v_1^2 + v_2^2 \end{pmatrix}} \\
&= \sqrt{\det\begin{pmatrix} u\cdot u & u\cdot v \\ v\cdot u & v\cdot v \end{pmatrix}}.
\end{aligned}$$

Replacing the dot product with an inner product g gives an expression similar to (2).

Matrix Inner Product[10]

Note that if $J : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and $J^2 = -I$, then we can choose a basis such that J is represented by $\widehat{J} := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. And if $H\widehat{J} = -\widehat{J}H$, $K\widehat{J} = -\widehat{J}K$ then

$$\begin{pmatrix} h_{12} & -h_{11} \\ h_{22} & -h_{21} \end{pmatrix} = - \begin{pmatrix} -h_{21} & -h_{22} \\ h_{11} & h_{12} \end{pmatrix}$$

so in particular H, K are symmetric. Hence $\operatorname{tr}(HK) = \sum_{i,j=1}^{2} h_{ij}k_{ji} = \sum_{i,j=1}^{2} h_{ij}k_{ij}$. This is indeed what we expect a 'dot product' on matrices to look like.

The expression in (1) can therefore be interpreted as taking the dot product of H and K, and integrating the outcome with respect to the area form given by J.

4.3 The space of Riemannian metrics

We take a brief diversion to investigate the relationship between almost-complex structures and Riemannian metrics explained in [Tro92, §§1–2], beginning with the following definition.

Definition 4.3. S_2 is the space of symmetric 0-2 tensors on S, that is the set of smooth maps which assign to each point $x \in S$ a symmetric bilinear form on T_xS.

$\mathcal{G} := \{g \in S_2 : g(x)(u,u) > 0 \text{ if } u \neq 0\}$ is the set of Riemannian metrics.

$\mathcal{G}_{-1} \subseteq \mathcal{G}$ is the set of Riemannian metrics with constant curvature -1.

$\mathcal{F}$ is the set of smooth positive functions on S.

We define a map $\Phi : \mathcal{G} \to \mathcal{A}$ using the formula

$$g(x)(u, \Phi(g)v) = -\mu_g(x)(u,v) \text{ for } u, v \in T_xS.$$

To justify this formula, we give our own calculation.

[10] After carrying this calculation out independently, we found a similar calculation in [FT84b, theorem 2.2].

If g is the dot product on $\mathbb{R}^2$, then Φ is an anticlockwise quarter-turn rotation. Now if $u, v \in \mathbb{R}^2$ and θ is the signed angle between them, then

$$u \cdot \Phi(g)v = \|u\| \, \|v\| \cos(\theta + \tfrac{\pi}{2}) = - \|u\| \, \|v\| \sin\theta = -\mu_g(u, v),$$

so the formula makes sense.

It is shown in [Tro92, §1.3] that Φ is a well-defined surjective map $\mathcal{G} \to \mathcal{A}$, which restricts to a bijection $\mathcal{G}/\mathcal{F} \to \mathcal{A}$.

To get the bijection between $\mathcal{G}_{-1}$ and $\mathcal{A}$ which we need to define the metric in §4.2, we seek a bijection between $\mathcal{G}/\mathcal{F}$ and $\mathcal{G}_{-1}$. This follows as an immediate consequence of Poincaré's theorem, stated below.

Theorem 4.2. *Suppose S is a compact oriented surface of genus at least 2. If $g \in \mathcal{G}(S)$, then there is a unique $\lambda \in \mathcal{F}(S)$ such that $\lambda g \in \mathcal{G}_{-1}(S)$.*

These bijections are very important in [FT84a], since in the k-times-differentiable case they turn out to be diffeomorphisms between smooth manifolds. This allows Fisher and Tromba to consider $\mathcal{T}$ as the quotient $\mathcal{G}_{-1}/\mathcal{D}_0$, which gives access to Poincaré's theorem, facilitating the proof that $\mathcal{T}$ is a smooth manifold. Further, the definition of $\mathcal{T}$ in §2.2 is essentially $\mathcal{T} := \mathcal{G}_{-1}/\mathcal{D}_0$, so this reconciles the definitions in §2.2 and §4.1.

It is also shown in [Tro92] that the bijection $\mathcal{A} \to \mathcal{G}_{-1}$ is $\mathcal{D}_0$-equivariant, that is $g(f^*J) = f^*g(J)$ for any $J \in \mathcal{A}, f \in \mathcal{D}_0$.

4.4 The Weil–Petersson metric

For $\mathcal{T} = \mathcal{A}/\mathcal{D}_0$ to inherit the metric on $\mathcal{A}$, we certainly need the following proposition from [FT84b].

Theorem 4.3. *The L^2 metric on $\mathcal{A}$ is $\mathcal{D}_0$-invariant.*

Proof. Take $f \in \mathcal{D}_0$, $J \in \mathcal{A}$, $H, K \in T_J\mathcal{A}$.

Then

$$f^*\left(\langle\langle H, K\rangle\rangle_J\right) = \langle\langle f^*H, f^*K\rangle\rangle_{f^*J} = \int_S \mathrm{tr}(f^*Hf^*K)\, d\mu_{g(f^*J)}.$$

From §4.3 we have $g(f^*J) = f^*g(J)$, so this is equal to

$$\int_S \mathrm{tr}(f^*Hf^*K)\, d\mu_{f^*g(J)} = \int_S f^*\left(\mathrm{tr}(HK)\right)\, d\mu_{f^*g(J)}.$$

Now by the change of variables theorem, this is equal to

$$\int_S \mathrm{tr}(HK)\, d\mu_{g(J)} = \langle\langle H, K\rangle\rangle_J.$$

$\square$

It turns out that $\mathcal{T}$ is also a smooth manifold, and the quotient map $\pi : \mathcal{A} \to \mathcal{T}$ is 'nice' enough that we can define $\langle \cdot, \cdot \rangle$ to be the Riemannian metric on $\mathcal{T}$ induced by $\langle\langle \cdot, \cdot \rangle\rangle$ (see [Tro92]).

As noted in [FT84b, p.335], Weil's definition of the Weil–Petersson metric $\langle \cdot, \cdot \rangle_{WP}$ is quite different to this, and involves identifying the tangent space of $\mathcal{T}$ with the space of holomorphic quadratic differentials on S. However, a straightforward computation in [Tro92, §2.6] gives that $\langle \cdot, \cdot \rangle = 2\langle \cdot, \cdot \rangle_{WP}$, so we will refer to $\langle \cdot, \cdot \rangle$ as the *Weil-Petersson metric*.

4.5 Wolpert's magic formula

It is also possible to give $\mathcal{T}$ a complex structure (see [Tro92, §4]), although we do not give details here. It is shown in [FT84b] that the Weil–Petersson metric is Hermitian with respect to this structure, that is if J is the associated almost-complex structure on $\mathcal{T}$, then we always have $\langle JX, JY \rangle = \langle X, Y \rangle$ ("multiplying by i doesn't change the angle"). This means that we can define an area form on $\mathcal{T}$, given by[11]

$$\omega(X, Y) = \langle JX, Y \rangle.$$

The motivation behind this definition is similar to that for Φ in §4.3.

In [Wol85], Wolpert shows that

$$\omega = \sum_{i=1}^{3g-3} d\ell_i \wedge d\tau_i$$

in Fenchel–Nielsen coordinates for $\mathcal{T}(S_g)$ with respect to any pants decomposition.

As noted in [Wri20, §2.8], this result is often referred to as Wolpert's Magic Formula, since it's surprising that the Weil–Petersson metric has such a simple relationship with Fenchel–Nielsen coordinates, and also that $\sum_{i=1}^{3g-3} d\ell_i \wedge d\tau_i$ is independent of the choice of coordinates.

To get a clearer idea of what $\sum_{i=1}^{3g-3} d\ell_i \wedge d\tau_i$ means,[12] note that $dx \wedge dy$ is

[11] See [Bal06].

[12] We found [Col11] helpful for this.

the usual oriented area form on $\mathbb{R}^2$, so $(dx \wedge dy)(a, b)$ is the area of the parallelogram spanned by a, b (up to sign, which is determined by orientation). So given a parallelogram in $\mathcal{T}$, the form ω tells us to sum the signed areas of its projections onto each of the (ℓ_i, τ_i) planes.

Wolpert also showed that ω is invariant under the action of the mapping class group, so it defines an area form on $\mathcal{M}$, and that this can be extended to the compactification $\widehat{\mathcal{M}}$.

4.6 The Weil–Petersson metric for genus 1

In [IT92, §7.3.5], an analogue to the Weil–Petersson metric is defined for the genus 1 case by mimicking the definition given in terms of holomorphic quadratic differentials. We present our own approach to the problem, starting instead from the metric in §4.2.

Fix an explicit description of T^2 as the square torus $\mathbb{T} := \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, so we can identify the tangent space at any point with $\mathbb{R}^2$, equipped with the usual basis. Then $C^\infty(T^1_1\mathbb{T})$ can be identified with the space of smooth maps from $\mathbb{T}$ into the space $M_2(\mathbb{R})$ of real 2×2 matrices. This gives that

$$\begin{aligned}\mathcal{A} = \{J \in C^\infty(T^1_1\mathbb{T}) : J(z)^2 = -I \text{ for each } z \in \mathbb{T}, \text{ and}\\ (u, J(z)u) \text{ is an oriented basis for } T_z\mathbb{T} \text{ for each } u \in T_z\mathbb{T}\}.\end{aligned}$$

We know from [Dau05] that $\mathcal{A}$ is a smooth manifold with tangent space

$$T_J\mathcal{A} = \{H \in C^\infty(T^1_1\mathbb{T}) : H(z)J(z) = -J(z)H(z) \text{ for each } z \in \mathbb{T}\},$$

so we define the L^2 metric

$$\langle\langle H, K\rangle\rangle_J := \int_S \operatorname{tr}(HK)\, d\mu_{g(J)}$$

where $g(J)$ is some flat unit-area metric associated with J. We don't have uniqueness of $g(J)$, so we need to either specify $g(J)$ uniquely or show that the metric is independent of this choice. We also need that $\langle\langle \cdot, \cdot\rangle\rangle$ is $\mathcal{D}_0$-equivariant. We leave these problems and move on to examine the Teichmüller space.

We have a description of $\mathcal{T}(T^2)$ as the upper half plane from §2.1, where $\zeta \in \mathbb{H}$ corresponds to the torus $\mathbb{T}_\zeta = \mathbb{C}/\Lambda_\zeta$ (using the notation from §3.3). If we use the description of T^2 and its tangent space fixed above, then the almost-complex structure corresponding to $\zeta \in \mathbb{H}$ is given by moving the point $x + iy$ to $x + \zeta y$, carrying out multiplication by i, and moving back (see Figure 4.1).

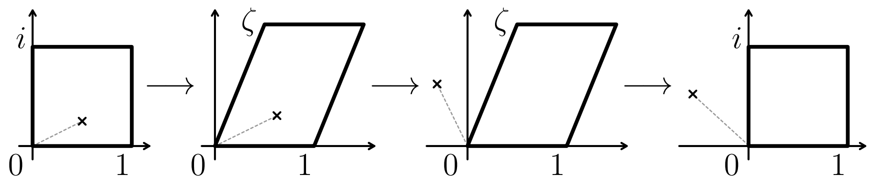

Figure 4.1: Finding the almost-complex structure associated with ζ

If we write $\zeta = a + ib$, then a change of basis calculation gives

$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} = \frac{1}{b} \begin{pmatrix} -a & -|\zeta|^2 \\ 1 & a \end{pmatrix}.$$

Hence the almost-complex structure associated with ζ is the constant map $z \mapsto J^{(\zeta)}$, where

$$J^{(\zeta)} = \frac{1}{b} \begin{pmatrix} -a & -|\zeta|^2 \\ 1 & a \end{pmatrix},$$

and another short calculation gives that any constant map in $\mathcal{A}$ has this form. This gives the identification

$$\mathcal{T} \cong \{\text{constant maps in } \mathcal{A}\} \cong \{J^{(\zeta)} : \zeta \in \mathbb{H}\}.$$

The canonical basis for the tangent space is

$$H_0 := \frac{\partial}{\partial a} \begin{pmatrix} -\frac{a}{b} & -\frac{a^2}{b} - b \\ \frac{1}{b} & \frac{a}{b} \end{pmatrix} = \begin{pmatrix} -\frac{1}{b} & -\frac{2a}{b} \\ 0 & \frac{1}{b} \end{pmatrix},$$

$$H_1 := \frac{\partial}{\partial b} \begin{pmatrix} -\frac{a}{b} & -\frac{a^2}{b} - b \\ \frac{1}{b} & \frac{a}{b} \end{pmatrix} = \begin{pmatrix} \frac{a}{b^2} & \frac{a^2}{b^2} - 1 \\ -\frac{1}{b^2} & -\frac{a}{b^2} \end{pmatrix}.$$

Note that H_0, H_1 are linearly independent and anticommute with $J^{(\zeta)}$, and the space of matrices which anticommute with $J^{(\zeta)}$ can be shown to be 2-dimensional, so

$$T_{J^{(\zeta)}}\mathcal{T} = \{H \in GL(2, \mathbb{R}) : HJ^{(\zeta)} = -J^{(\zeta)}H\}.$$

We know that $\mathcal{T}$ is the quotient $\mathcal{A}/\mathcal{D}_0$, so take $\pi : \mathcal{A} \to \{J^{(\zeta)} : \zeta \in \mathbb{H}\}$ to be the quotient map. Note that π sends constant maps $z \mapsto J^{(\zeta)}$ to $J^{(\zeta)}$, and we hope that it is sufficiently 'nice' for our purposes, in particular that it has similar identifications on the tangent space.

Then the metric induced by $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathcal{T}$ is given by taking lifts $\widetilde{J}$ of $J \in \mathcal{T}$ and $\widetilde{H}, \widetilde{K}$ of $H, K \in T_J\mathcal{T}$, and calculating $\langle\langle \widetilde{H}, \widetilde{K}\rangle\rangle_{\widetilde{J}}$. Taking the lifts to be the constant maps mentioned above gives

$$\langle H, K\rangle := \int_S \operatorname{tr}(\widetilde{H}\widetilde{K})\, d\mu_{g(\widetilde{J})} = \int_S \operatorname{tr}(HK)\, d\mu_{g(\widetilde{J})} = \operatorname{tr}(HK).$$

Then

$$\langle H_0, H_0\rangle = \frac{2}{b^2}, \quad \langle H_0, H_1\rangle = 0, \quad \langle H_1, H_1\rangle = \frac{2}{b^2},$$

so the metric is $\frac{2}{b^2}(da^2 + db^2)$, which is the Poincaré metric (up to a constant).

The result in [IT92] is $\frac{1}{2b^2}(da^2 + db^2)$, which only differs from ours by a constant.[13] As observed there, if we take the coordinates

$$(\ell, \tau) = (\frac{1}{\sqrt{b}}, \frac{a}{\sqrt{b}}),$$

from §2.5, then from their result we recover the area form $d\ell \wedge d\tau$, which is analogous to Wolpert's formula.

[13] This indicates that the issues with uniqueness and $\mathcal{D}_0$-equivariance of $\langle\langle\cdot,\cdot\rangle\rangle$ and 'niceness' of the map π could be resolvable.

5 Computing volume

We can use ω to define a volume form on $\mathcal{M}(S)$ by taking the m-fold wedge product

$$\frac{1}{m!}\,\omega \wedge \ldots \wedge \omega,$$

where $m = \frac{1}{2}\dim(\mathcal{M}(S))$.

Our aim is to find the volume of $\mathcal{M}(S_g)$ with respect to this volume form. The obvious approach is to find a fundamental domain for the action of $\mathrm{Mod}(S_g)$ on $\mathcal{T}(S_g)$, but in [Mir07], Mirzakhani was able to find a recurrence relation for volumes of moduli spaces of surfaces with boundary using a different method,[14] which we will investigate below.

5.1 Introducing boundary components

Mirzakhani's recursion only gives the volumes of moduli spaces of surfaces with boundary, so we introduce the following notation from [Mir07]. Note we require the boundary components of a hyperbolic surface to be geodesic and of fixed length, where a length-0 boundary component is viewed as a puncture.

Definition 5.1. Write $S_{g,n}$ for the topological surface of genus g with n boundary components, and call the boundary components $\beta_1, \ldots, \beta_n$.

Given $L = (L_1, \ldots, L_n) \in \mathbb{R}^n_{\geq 0}$, we define $\mathcal{T}_{g,n}(L)$ to be the Teichmüller space of marked hyperbolic surfaces homeomorphic to $S_{g,n}$ with $\ell_{\mathcal{X}}(\beta_i) = L_i$ for each i. (Markings are defined as in §2.2.)

Then $\mathcal{M}_{g,n}(L) := \mathcal{T}_{g,n}(L)/\operatorname{Mod}(S_{g,n})$, where elements of $\mathrm{Mod}(S_{g,n})$ are required to fix each boundary component setwise.

Finally, we write $V_{g,n}(L)$ for the Weil–Petersson volume of $\mathcal{M}_{g,n}(L)$.

Note that a pants decomposition of $S_{g,n}$ consists of $3g-3+b$ curves and splits it into $2g-2+b$ pairs of pants, so using Fenchel–Nielsen coordinates gives $\mathcal{T}_{g,n}(L) \cong (\mathbb{R}_+ \times \mathbb{R})^{3g-3+b}$.

[14] As remarked in [Wri20, §6], before Mirzakhani's work these volumes were only known in the cases $L = (0, \ldots, 0)$, $\mathcal{M}_{0,4}(L)$ and $\mathcal{M}_{1,1}(L)$.

5.2 The volume of the moduli space of a once-punctured torus

Mirzakhani begins [Mir07] with a calculation of $V_{1,1}(0)$, since the calculation of the volumes of other moduli spaces follows a similar structure. We reproduce the calculation here (slightly expanded to give more detail), beginning with the following identity of McShane.

Theorem 5.1. *If X is a hyperbolic once-punctured torus, then*

$$\sum_{\gamma} \frac{1}{1+e^{\ell_X(\gamma)}} = \frac{1}{2} \tag{3}$$

where the sum is over all simple closed geodesics γ on X.

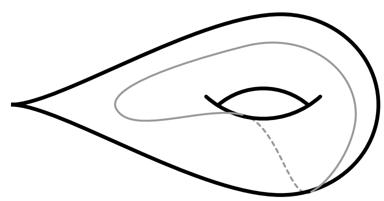

Figure 5.1: A path in a once-punctured torus
(similar to [Hoe19, Figure 12])

The key step is to introduce the space

$$\mathcal{M}_{1,1}^* = \{(X,\gamma) : X \in \mathcal{M}_{1,1}(0), \gamma \subseteq X \text{ a simple closed geodesic}\}.$$

Fix a simple closed curve $\alpha \subseteq S_{1,1}$. Then given $(X,\varphi) \in \mathcal{M}_{1,1}^*$, there is some diffeomorphism $\varphi : S_{1,1} \to X$ such that $\varphi(\alpha) = \gamma$, and $\varphi_1(\alpha) = \varphi_2(\alpha)$ if and only if $\varphi_1 = \varphi_2 \circ f$ for some $f \in \operatorname{Stab}(\alpha)$. Hence we can identify $\mathcal{M}_{1,1}^*$ with $\mathcal{T}_{1,1}(0)/\operatorname{Stab}(\alpha)$, so $\mathcal{M}_{1,1}^*$ is a space 'between' $\mathcal{T}_{1,1}(0)$ and $\mathcal{M}_{1,1}(0)$.

We give $\mathcal{M}_{1,1}^*$ Fenchel–Nielsen coordinates about α, where any $(X,\gamma) \in \mathcal{M}_{1,1}^*$ is determined by (ℓ,τ), the length and twist of X about γ. Then the only redundancy is full twists of X about γ, which correspond to adding multiples of ℓ to τ. Hence[15]

$$\mathcal{M}_{1,1}^* \cong \{(\ell,\tau) : 0 \leq \tau \leq \ell\}/(x,0) \sim (x,x)$$

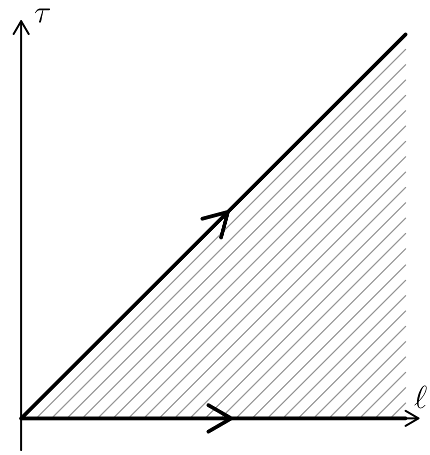

Figure 5.2: A picture of $\mathcal{M}_{1,1}^*$

as in Figure 5.2.

Next, we aim to rewrite (3). If we define the maps

$$\begin{array}{ll} \pi : \mathcal{M}_{1,1}^* \to \mathcal{M}_{1,1}(0) & (X,\gamma) \mapsto X, \\ \ell : \mathcal{M}_{1,1}^* \to \mathbb{R} & (X,\gamma) \mapsto \ell_X(\gamma), \end{array}$$

then

$$\sum_{\pi(Y)=X} f(\ell(Y)) = \frac{1}{2},$$

where $f(x) = (1+e^x)^{-1}$.

Finally, using Wolpert's formula we can calculate

$$\begin{aligned} V_{1,1}(0) = 2\int_{\mathcal{M}_{1,1}} \frac{1}{2}\, dX &= 2\int_{\mathcal{M}_{1,1}} \sum_{\pi(Y)=X} f(\ell(Y))\, dX = 2\int_{\mathcal{M}_{1,1}^*} f(\ell(Y))\, dY \\ &= 2\int_{\ell=0}^{\infty}\int_{\tau=0}^{\ell} f(\ell)\, d\tau\, d\ell = 2\int_{\ell=0}^{\infty} \ell f(\ell)\, d\ell = 2\int_{\ell=0}^{\infty} \frac{\ell}{1+e^{\ell}}\, d\ell = \frac{\pi^2}{6} \end{aligned}$$

5.3 Mirzakhani's strategy for calculating the volume of the moduli space

To calculate $V_{g,n}(L)$, Mirzakhani generalises the calculation in §5.2 in two steps.

[15]Note we have corrected a typo from [Mir07] below.

Step 1: Generalise the identity (3) to arbitrary hyperbolic surfaces with geodesic boundary.

Step 2: Develop a method for integrating functions given in terms of hyperbolic length.

We follow a summary from [Wri20, §5] of the strategy for step 1 (with additional exposition). First, choose some boundary curve β_1, and consider the set F of points $x \in \beta_1$ where the geodesic ray γ_x starting at x and perpendicular to β_1 continues forever without hitting itself or the boundary. It can be shown that F cuts β_1 into a countable union of disjoint intervals (a_k, b_k). Mirzakhani shows that for each interval, the geodesics $\gamma_{a_k}, \gamma_{b_k}$ both spiral towards either a simple closed curve, or a boundary component that isn't β_1. Further, there is a unique pair of pants P with geodesic boundary containing $\gamma_{a_k}, \gamma_{b_k}$.

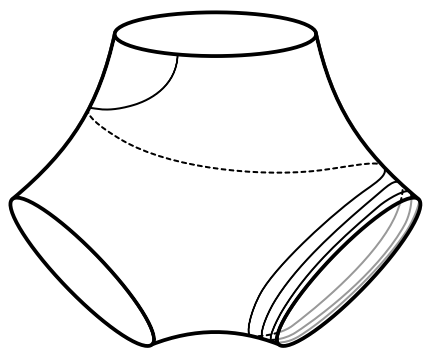

Figure 5.3: A spiralling geodesic on a pair of pants
(similar to [Wri20, Figure 5.2])

The length $|b_k - a_k|$ can be calculated explicitly. This calculation depends on whether $\gamma_{a_k}, \gamma_{b_k}$ spiral towards the same curve α (so P is bounded by α, β_1, and another boundary component β_i), or different curves α_{a_k}, α_{b_k} (so P is bounded by α_{a_k}, α_{b_k}, and β_1). The final identity states that $L_1 = \sum |b_k - a_k|$, and the two cases give rise to two terms. The first is a sum over all simple closed curves bounding a pair of pants with β_1 and another boundary component, and the second is over all unordered pairs of simple closed curves bounding a pair of pants with β_1.

Now we look at step 2, following the summary in [Do13, §4.3]. We split the second term in the identity from step 1 according to whether removing the relevant pair of pants leaves the surface connected (the disconnected case is further split according to the resulting connected components). Then each term is a sum over a mapping class group orbit, which allows for the definition of a similar 'space of pairs' to the one in §5.2, and a similar use of Fenchel–Nielsen coordinates and Wolpert's formula.

After integrating, this gives an expression for $L_1 V_{g,n}(L)$. By taking a partial derivative, the final recursion can be given explicitly (see §5.4).

5.4 The recursion formula

To give the statement of Mirzakhani's recursion, we introduce some helpful notation from [Mir07].

Define $H : \mathbb{R}^2 \to \mathbb{R}$ by

$$H(x,y) = \frac{1}{1+e^{\frac{x+y}{2}}} + \frac{1}{1+e^{\frac{x-y}{2}}}.$$

This function comes from computing lengths of intervals on the boundaries of pairs of pants.

We take

$$m(g,n) = \begin{cases} 1 & g = 1 \text{ and } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This is needed because the surfaces arising when a curve separates off a handle have extra symmetry (consider a half-turn rotation about the boundary).

Given $L = (L_1, \ldots, L_n)$, we write

$$\widehat{L} = (L_2, \ldots, L_n), \quad \widehat{L}_j = (L_2, \ldots, L_{j-1}, L_{j+1}, \ldots, L_n),$$

and we allow any combination of numbers and tuples to be arguments in $V_{g,n}$, for example

$$V_{g,n}(x, \widehat{L}) = V_{g,n}(x, L_2, \ldots, L_n).$$

This is convenient because removing a pair of pants from a surface gives a new surface where most of the boundary lengths are unchanged.

We define $\mathcal{I}_{g,n}$ to be the set of ordered pairs

$$a = ((g_1, I_1), (g_2, I_2)),$$

where I_1, I_2 are disjoint sets with $I_1 \sqcup I_2 = \{2, 3, \ldots, n\}$, and $0 \leq g_1, g_2 \leq g$ are numbers such that $2g_j + |I_j| \geq 2$ for each j. These pairs correspond to the resulting components when removing a pair of pants disconnects a surface.

Then given $L = (L_1, \ldots, L_n)$ and $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, we write

$$L_I = (L_{i_1}, \ldots, L_{i_k})$$

for notational convenience.

We are now ready to give Mirzakhani's recursion.

Theorem 5.2. *If $\chi(S_{g,n}) < 0$, $n \neq 0$, then $V_{g,n}(L)$ is determined by*

$$V_{0,3}(L_1, L_2, L_3) = 1 \qquad \text{and} \qquad V_{1,1}(L_1) = \frac{L_1^2}{24} + \frac{\pi^2}{6},$$

and if $(g, n) \neq (0, 3), (1, 1)$, then

$$\begin{aligned}
\frac{\partial}{\partial L_1} L_1 V_{g,n}(L) &= \frac{1}{2} \int_0^\infty \int_0^\infty xy \, \frac{V_{g-1,n+1}(x, y, \widehat{L})}{2^{m(g-1,n+1)}} \, H(x+y, L_1) \, dx \, dy \\
&\quad + \frac{1}{2} \sum_{\mathcal{I}_{g,n}} \int_0^\infty \int_0^\infty xy \, \frac{V_{g_1,|I_1|+1}(x, L_{I_1})}{2^{m(g_1,|I_1|+1)}} \, \frac{V_{g_2,|I_2|+1}(x, L_{I_2})}{2^{m(g_2,|I_2|+1)}} \, H(x+y, L_1) \, dx \, dy \\
&\quad + \frac{1}{2} \sum_{j=2}^{n} \int_0^\infty x \, \frac{V_{g,n-1}(x, \widehat{L_j})}{2^{m(g,n-1)}} \, (H(x, L_1 + L_j) + H(x, L_1 - L_j)) \, dx
\end{aligned}$$

Note the volume $V_{1,1}(L_1)$ matches the result in §5.2 when $L_1 = 0$. We take $V_{0,3}(L_1, L_2, L_3) = 1$ because $\mathcal{M}_{0,3}$ is a single point (the volume of the product space is the product of the volumes, and taking the product of a space with a single point doesn't change the volume).

It is possible (if somewhat unwieldy—see [Do13, §4.4]) to use this recursion for explicit calculations by looking at certain integrals involving H. This approach gives the following corollary from [Mir07].

Theorem 5.3. *The function $V_{g,n}(L)$ is a polynomial in L of the form*

$$V_{g,n}(L) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq 3g-3+n}} \pi^{6g-6+2n-2|\alpha|} c_\alpha \cdot L^{2\alpha},$$

where $c_\alpha \in \mathbb{Q}_+$ for each α.

5.5 Returning to surfaces without boundary

We are interested in $V_{g,0}$, but Theorem 5.2 does not apply in this case. Fortunately, [DN09, theorem 3] gives the following formula for $V_{g,0}$ in terms of $V_{g,1}(L_1)$.

Theorem 5.4. *For $g \geq 2$, we have*

$$V_{g,0} = \frac{V'_{g,1}(2\pi i)}{2\pi i(2g-2)}$$

We give our own calculation to examine a consequence of this.

From Theorem 5.3, we have

$$V_{g,1}(L_1) = \sum_{k=0}^{3g-2} \pi^{6g-4-2k} c_k L_1^{2k}$$

where $c_k \in \mathbb{Q}_+$ for each k.

Then Theorem 5.4 gives

$$\begin{aligned} V_{g,0} &= \frac{1}{2\pi i(2g-2)} V'_{g,1}(2\pi i) \\ &= \frac{1}{2\pi i(2g-2)} \sum_{k=0}^{3g-2} \pi^{6g-4-2k} \cdot 2kc_k(2\pi i)^{2k-1} \\ &= \sum_{k=1}^{3g-2} \frac{2kc_k}{2g-2} \cdot (2\pi i)^{2k-2} \cdot \pi^{6g-4-2k} \\ &= \left(\sum_{k=1}^{3g-2} (-1)^{k-1} \frac{2^{2k-2} kc_k}{g-1} \right) \cdot \pi^{6g-6} \end{aligned}$$

So if $g \geq 2$, then $V_{g,0}$ is a rational multiple of π^{6g-6}.

5.6 Finding the missing volumes

Theorems 5.2 and 5.4 give expressions for $V_{g,n}(L)$ whenever $\chi(S_{g,n}) < 0$. This misses four cases for (g, n), so we present our own working to find these volumes.

<u>$V_{0,0}$</u>

In §1.1 we saw that $\mathcal{M}_{0,0}$ contains only one point, so $V_{0,0} = 1$.

$\underline{V_{1,0}}$

From §4.6, we know that the Weil–Petersson metric is a multiple of the Poincaré metric on $\mathbb{H} \cong \mathcal{T}_{1,0}$. (Here we use half the Poincaré metric since this matches Wolpert's formula.) From §2.1, we also have the fundamental domain

$$\{z \in \mathbb{H} : |z| > 1 \text{ and } \operatorname{Re}(z) \in (-\tfrac{1}{2}, \tfrac{1}{2})\}$$

for the action of $\operatorname{Mod}(T^2)$ on $\mathcal{T}(T^2)$.

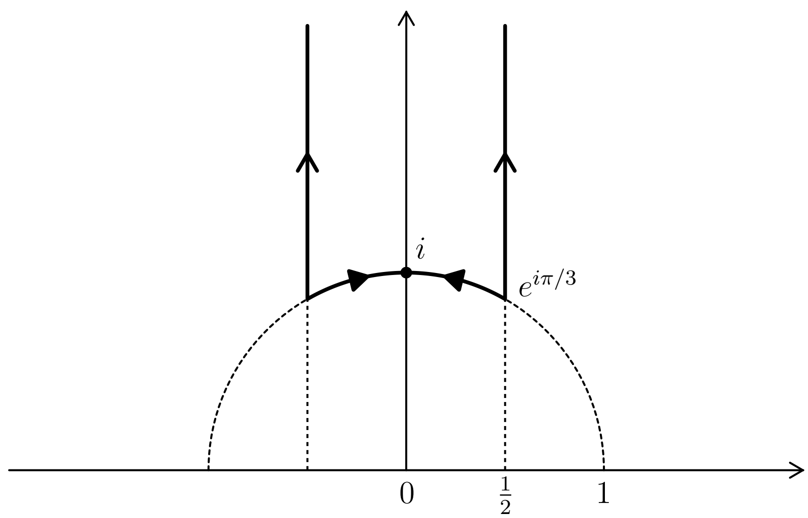

Figure 2.2: A fundamental domain for the action of $\operatorname{Mod}(T^2)$ on $\mathcal{T}(T^2)$

This is a hyperbolic triangle with angles $0, \frac{\pi}{3}, \frac{\pi}{3}$, so has area

$$\pi - 0 - \frac{\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$$

in the Poincaré metric. Accounting for the factor of $\frac{1}{2}$ gives $V_{1,0} = \frac{\pi}{6}$.

$\underline{V_{0,1}(L_1)}$

Note that $\chi(S_{0,1}) > 0$, so we require any surface in $\mathcal{M}_{0,1}(L_1)$ to have a metric with constant curvature 1 and totally geodesic boundary.

If $L_1 > 0$, then given $X \in \mathcal{M}_{0,1}(L_1)$, we can glue two copies of X along their boundaries to give a surface X' such that the curve corresponding to the boundary of X is a geodesic. But then X' is homeomorphic to a sphere, so from §1.1 we know that X' is the Riemann sphere. All geodesics on the Riemann sphere are great circles and hence have length 2π, so $\mathcal{M}_{0,1}(L_1)$ is empty (and has volume 0) unless $L_1 = 2\pi$. Further, since any great circle on the Riemann sphere can be mapped to any other great circle by an isometry, it is clear that $\mathcal{M}_{0,1}(2\pi)$ is a single point and hence has volume 1.

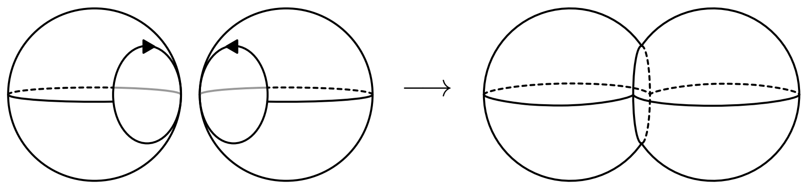

Figure 5.4: Gluing two copies of X

When $L_1 = 0$, an element of $\mathcal{M}_{0,1}(L_1)$ is a sphere with a marked point. This can be moved to any other point of the sphere by an isometry, so $\mathcal{M}_{0,1}(0)$ is a single point.

$\underline{V_{0,2}(L_1, L_2)}$

We repeat a similar strategy for the final case. First, $\chi(S_{0,2}) = 0$, so we require any surface in $\mathcal{M}_{0,1}(L_1)$ to have a flat unit-area metric and totally geodesic boundary.

If $L_1, L_2 > 0$ and $Y \in \mathcal{M}_{0,2}(L_1, L_2)$, then gluing two copies of Y along matching boundary components gives an area-2 torus Y' with geodesic curves γ_1, γ_2 corresponding to the boundary components. Further, if we take Fenchel–Nielsen coordinates for Y' about γ_1, then by adjusting the way we glue the copies of Y we can ensure that Y' has twist parameter 0. Then Y' is uniquely described by the length parameter L_1, and by noting that Y' is a quotient $\mathbb{C}/\Lambda$ we see that $L_2 = L_1$. Hence $\mathcal{M}_{0,2}(L_1, L_2)$ is empty unless $L_2 = L_1$, in which case it has a single point.

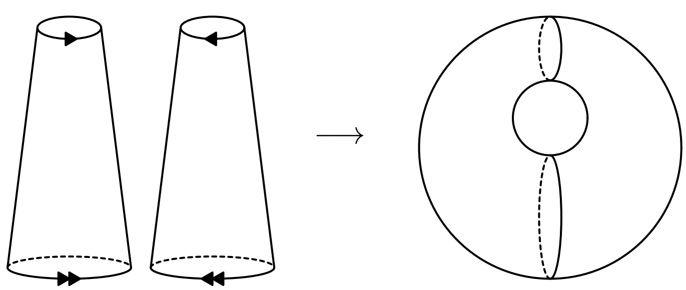

Figure 5.5: Gluing two copies of Y

If $L_1 = 0$ or $L_2 = 0$, then by symmetry we can assume $L_1 = 0$. Then $Y \in \mathcal{M}_{0,2}(L_1, L_2)$ is an element of $\mathcal{M}_{0,1}(L_2)$ with an additional marked point, so $\mathcal{M}_{0,2}(L_1, L_2)$ is empty unless $L_2 \in \{0, 2\pi\}$.

If $L_2 = 2\pi$, then we can identify $Y \in \mathcal{M}_{0,2}(L_1, L_2)$ with

$$\{z \in \mathbb{C} \cup \{\infty\} : \mathrm{Im}(z) \geq 0\},$$

with a marked point $z_0 = x_0 + y_0$ in the interior (note $y_0 > 0$). But the Möbius map

$$z \mapsto \frac{x_0 z - |z_0|^2}{y_0 z}$$

maps z_0 to i and maps $\mathbb{R} \cup \{\infty\}$ to itself, so we can move the marked point anywhere in Y.

If $L_2 = 0$, then $Y \in \mathcal{M}_{0,2}(L_1, L_2)$ is a sphere with two marked points. These can be mapped to any two other points using Möbius maps.

Hence in both cases there is one point in $\mathcal{M}_{0,2}(L_1, L_2)$.

So if we take δ to be the indicator function $\mathbb{1}_{\{0\}}$, then we can write

$$\begin{aligned}
V_{0,0} &= 1 \\
V_{0,1}(L_1) &= \delta(L_1) + \delta(L_1 - 2\pi) \\
V_{0,2}(L_1, L_2) &= \delta(L_1 - L_2) + \delta(L_1)\delta(L_2 - 2\pi) + \delta(L_2)\delta(L_1 - 2\pi) \\
V_{1,0} &= \frac{\pi}{6}
\end{aligned}$$

This is quite different to the hyperbolic case, where the formulae are all polynomials and $V_{g,0}$ is a rational multiple of π^d where d is the dimension of the Teichmüller space.

In [Do13, §4.3], the convention that $V_{0,1}(L_1) \equiv 0$, $V_{0,2}(L_1, L_2) \equiv 0$ is adopted to simplify the statement of Mirzakhani's recursion by allowing $\mathcal{I}_{g,n}$ to contain pairs where $2g_j + |I_j| < 2$ for one or both j. It is interesting to see how close this is to our result.

6 Conclusion

In this essay, we explored two notions of the 'size' of the moduli space. We began by introducing the Teichmüller space, which allowed us to give local coordinates for the moduli space. Using these, we saw that the moduli space is very 'close' to being compact, and gave a natural compactification of the space. We then moved on to a second notion of size by defining the Weil–Petersson metric. We outlined Mirzakhani's remarkable calculation of the Weil–Petersson volume of the moduli space, and concluded that the volume of $\mathcal{M}(S_g)$ is a rational multiple of a power of π, and this power is $6g-6$ when $g \geq 2$. Throughout the essay, we continually returned to the non-hyperbolic case, initially as an explicit example to guide our approach, and later as an interesting exception for extending our ideas to.

There were many things which we did not have time to include in this essay, but would nonetheless have been valuable additions. Most prominently, the argument in §4.6 is incomplete, as it is not clear that the metric given for $\mathcal{A}$ is uniquely specified or $\mathcal{D}_0$-equivariant, or that the map from $\mathcal{A}$ to $\mathcal{T}$ is sufficiently 'nice' for our purposes. Completing this argument was sadly not possible in the time required for this essay.

Another potential addition was the further example of genus 2. Any element of $\mathcal{T}(S_2)$ can be identified with a tiling of the hyperbolic plane by octagons (similarly to tori being identified with a tiling of $\mathbb{C}$ by rhombuses)—we were interested in visualising the effect of changing length and twist parameters on this tiling, and the code in [Hoe19] was a potential starting point for this. This could have lead into the explicit description of $\widehat{\mathcal{M}}(S_2) \setminus \mathcal{M}(S_2)$ in [Don11, §14.4.1]. Further, all genus 2 surfaces can be understood as branched double covers of the Riemann sphere, which allows for an explicit description of $\mathcal{M}(S_2)$ (as in [vdV11]). It would be very interesting to find a description of length and twist parameters for points in this space.

Finally, in this essay we only considered one of each of the many possible compactifications of the moduli space and metrics for the Teichmüller space. These choices may seem unrelated to each other, but because $\widehat{\mathcal{M}}$ is the Weil–Petersson completion of $\mathcal{M}$ (see §3.7), the volumes $V_{g,0}$ are the volumes of the spaces $\widehat{\mathcal{M}}(S_g)$, so our two notions of 'size' are in fact closely linked.

References

[Bal06] Werner Ballmann, *Lectures on Kähler manifolds*, ESI Lectures in Mathematics and Physics, European Mathematical Society (EMS), Zürich, 2006. MR2243012

[Bri10] Martin R. Bridson, *Semisimple actions of mapping class groups on* CAT(0) *spaces*, Geometry of Riemann surfaces, London Math. Soc. Lecture Note Ser., vol. 368, Cambridge Univ. Press, Cambridge, 2010, pp. 1–14. MR2665003

[Col11] Susan Jane Colley, *Vector Calculus*, Pearson Education, 2011.

[Dau05] N. A. Daurtseva, *On the manifold of almost complex structures*, Mat. Zametki **78** (2005), no. 1, 66–71, DOI 10.1007/s11006-005-0099-7 (Russian, with Russian summary); English transl., Math. Notes **78** (2005), no. 1-2, 59–63. MR2244870

[DN09] Norman Do and Paul Norbury, *Weil-Petersson volumes and cone surfaces*, Geom. Dedicata **141** (2009), 93–107, DOI 10.1007/s10711-008-9345-y. MR2520065

[Do13] Norman Do, *Moduli spaces of hyperbolic surfaces and their Weil-Petersson volumes*, Handbook of moduli. Vol. I, Adv. Lect. Math. (ALM), vol. 24, Int. Press, Somerville, MA, 2013, pp. 217–258. MR3184165

[Don11] Simon Donaldson, *Riemann surfaces*, Oxford Graduate Texts in Mathematics, vol. 22, Oxford University Press, Oxford, 2011. MR2856237

[DV97] Satya Deo and K. Varadarajan, *Discrete groups and discontinuous actions*, Rocky Mountain J. Math. **27** (1997), no. 2, 559–583, DOI 10.1216/rmjm/1181071925. MR1466157

[FM12] Benson Farb and Dan Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012. MR2850125

[FT84a] Arthur E. Fischer and Anthony J. Tromba, *On a purely "Riemannian" proof of the structure and dimension of the unramified moduli space of a compact Riemann surface*, Math. Ann. **267** (1984), no. 3, 311–345, DOI 10.1007/BF01456093. MR738256

[FT84b] A. E. Fischer and A. J. Tromba, *On the Weil-Petersson metric on Teichmüller space*, Trans. Amer. Math. Soc. **284** (1984), no. 1, 319–335, DOI 10.2307/1999289. MR742427

[Him68] C. J. Himmelberg, *Pseudo-metrizability of quotient spaces*, Fund. Math. **63** (1968), 1–6, DOI 10.4064/fm-63-1-1-6. MR230280

[HK14] John H. Hubbard and Sarah Koch, *An analytic construction of the Deligne-Mumford compactification of the moduli space of curves*, J. Differential Geom. **98** (2014), no. 2, 261–313. MR3263519

[Hoe19] Gage Hoefer, *Visualizing hyperbolic structures on the once punctured torus*, 2019. Unpublished, available at `https://cpb-us-w2.wpmucdn.com/sites.udel.edu/dist/b/10716/files/2021/06/Visualizing_Curves_on_the_Surface_of_the_Once_Punctured_Torus.pdf`.

[IT92] Y. Imayoshi and M. Taniguchi, *An introduction to Teichmüller spaces*, Springer-Verlag, Tokyo, 1992. Translated and revised from the Japanese by the authors. MR1215481

[JP13] Lizhen Ji and Athanase Papadopoulos, *Historical development of Teichmüller theory*, Arch. Hist. Exact Sci. **67** (2013), no. 2, 119–147, DOI 10.1007/s00407-012-0104-y. MR3021551

[Mir07] Maryam Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. **167** (2007), no. 1, 179–222, DOI 10.1007/s00222-006-0013-2. MR2264808

[Mon09] Gabriele Mondello, *A criterion of convergence in the augmented Teichmüller space*, Bull. Lond. Math. Soc. **41** (2009), no. 4, 733–746, DOI 10.1112/blms/bdp052. MR2521369

[Oec17] Robert Oeckl, *Real analysis notes*, Universidad Nacional Autónoma de México, 2017. Lecture notes, available at `https://www.matmor.unam.mx/~robert/cur/2018-1-RA/RA.pdf`.

[Til07] Stephan Tillmann, *Complex Analysis (620-413): Riemann mapping theorem and Riemann surfaces*, University of Melbourne, 2007. Lecture notes, available at `https://www.maths.usyd.edu.au/u/tillmann/2007-complex/ComAna-Lectures.pdf`.

[Tro92] Anthony J. Tromba, *Teichmüller theory in Riemannian geometry*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1992. Lecture notes prepared by Jochen Denzler. MR1164870

[vdV11] Martijn van der Valk, *On moduli of hyperelliptic curves of genus two*, 2011. Unpublished, available at `https://fse.studenttheses.ub.rug.nl/9712/1/Martijn_van_der_Valk_WM_2011.pdf`.

[Wol85] Scott Wolpert, *On the Weil-Petersson geometry of the moduli space of curves*, Amer. J. Math. **107** (1985), no. 4, 969–997, DOI 10.2307/2374363. MR796909

[Wri20] Alex Wright, *A tour through Mirzakhani's work on moduli spaces of Riemann surfaces*, Bull. Amer. Math. Soc. (N.S.) **57** (2020), no. 3, 359–408, DOI 10.1090/bull/1687. MR4108090