Notes on tameness

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0. Introduction.

The aim of this paper is to give an account of the Tameness Theorem for complete hyperbolic 3-manifolds. Our presentation makes use of ideas from a number of different sources. We aim to give an account that is readily accessible from fairly standard geometrical and topological arguments, and adaptable to variable curvature.

Let M be a complete hyperbolic 3-manifold (without boundary) with $\pi_1(M)$ finitely generated. Marden [Ma] asked if this implies that M is topologically finite, i.e. homeomorphic (or equivalently diffeomorphic) to the interior a compact manifold with boundary. This was proven in the "indecomposable" case by Bonahon [Bon2], and in general independently by Agol [Ag] and Calegari and Gabai [CalG]. This is generally referred to as the "Tameness Theorem":

Theorem 0.1 : (Bonahon, Agol, Calegari, Gabai) *Let M be a complete hyperbolic 3-manifold with $\pi_1(M)$ finitely generated. Then M homeomorphic to the interior of a compact manifold with boundary.*

(It is well known that such a compactification is unique up to homeomorphism.)

In view of [Tu] it is easily seen that a 3-manifold is topologically finite if and only if its orientable cover is. For this reason we will assume henceforth that all our 3-manifolds are orientable. However, all the theorems stated in this section remain valid in the non-orientable case. We also note that, using [MeS], Theorem 0.1 extends to orbifolds, as we explain in Section 6.6.

To set Theorem 0.1 in context, we note that (in contrast to the analogous statement for 2-manifolds) one certainly needs the geometrical hypothesis in Theorem 0.1. For example Whitehead [Wh] described a simply connected 3-manifold which is not topologically finite in the above sense (see also [H]). We do however have the following positive topological result due to Scott [Sc1,Sc2]:

Theorem 0.2 : (Scott) *Let M be a (topological) 3-manifold with $\pi_1(M)$ finitely generated. Then there is a compact submanifold of M whose inclusion into M induces an isomorphism of fundamental groups.*

Such a compact submanifold is referred to as a *Scott core*.

This is a key ingredient in the proof of the Tameness Theorem. It effectively reduces it to asserting that the ends of M are just topological products.

In fact, we are only interested here in irreducible 3-manifolds (that is, where every embedded 2-sphere bounds a ball). This always holds for a hyperbolic 3-manifold. Such a manifold is aspherical and it follows that we can obtain a homotopy equivalence in Scott's theorem (after capping off any 2-spheres in the boundary by 3-balls). Moreover, irreducibility allows us to bypass discussion of the Poincaré conjecture, as proven by Perelman, since no such manifold can contain a fake 3-ball.

The Tameness Theorem is related to another well known result, namely Ahlfors's Finiteness Theorem [Ah]. This is usually phrased analytcally in terms of Riemann surfaces. However (in the torsion-free case) it is easily seen to be equivalent to the following geometrical statement. Let M be a hyperbolic 3-manifold, which we assume non-elementary. (This rules out certain trivial cases where $\pi_1(M)$ is trivial, $\mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z}$.) The "convex core" of M is essentially the unique smallest submanifold which carries the whole of $\pi_1(M)$ and has convex boundary. (It follows that its inclusion into M in a homotopy equivalence.) We say "essentially" since it might degenerate to a totally geodesic surface in M. In any case, for any $t > 0$, the metric t-neighbourhood of the convex core is a C^1-submanifold. (Unlike the Scott core, the convex core need not be compact.) The boundary, F_t, of this neighbourhood is an embedded C^1-surface (possibly empty or disconnected). As we note in Section 8, Ahlfors's Finiteness Theorem (in the torsion-free case) is equivalent to:

Theorem 0.3 : *Let M be a non-elementary complete hyperbolic 3-manifold with $\pi_1(M)$ finitely generated. Let $t > 0$, and let F_t be the boundary of the t-neighbourhood of the convex core of M. Then F_t has finite area.*

In fact, we can make similar definitions when M is complete riemannian 3-manifold of pinched negative curvature. We have versions of both tameness and the Ahlfors Finiteness Theorem respectively generalising Theorems 0.1 and 0.3:

Theorem 0.4 : *Let M be a complete riemannian 3-manifold of pinched negative curvature with $\pi_1(M)$ finitely generated. Then M is homeomorphic to the interior of a compact manifold with boundary.*

Theorem 0.5 : *Let M be a complete riemannian 3-manifold of pinched negative curvature with $\pi_1(M)$ finitely generated. Let $t > 0$, and let F_t be the boundary of the t-neighbourhood of the convex core of M. Then F_t has finite area.*

These are proven in Sections 7 and 8 respectively. As far as I know, these have not been made explicit before, though it seems to have been folklore that tameness techniques could be applied to give results of this type in variable curvature.

Since the work of Agol and Calegari and Gabai, other accounts of tameness have been given by Choi [Ch] and Soma [Som] (in the case without cusps). Our exposition is inspired by that of Soma, which gives another perspective on various constructions of [CalG]. The strategy we follow is broadly similar. However we phrase things differently, so as to remove the dependence on the end reduction theory [BrinT,My] and we include an independent argument for the relevant part of Souto's result [Sou]. In this way, almost all our arguments, are built from first principles from results essentially known prior to

Thurston's work [Th1]. Apart, that is, from one critical appeal to the hyperbolisation theorem for atoroidal Haken manifolds [Th2,O,Ka], which is here used to deduce a purely topological statement (Theorem 2.8.1). There are a couple of places where the argument could be shortened a little by appeal to other more sophisticated results or machinery, as we point out in the relevant places.

Originally, the term "tameness" was used by Thurston [Th1] to refer to a certain geometric property of M. He showed that this implies topological finiteness. Subsequently Canary [Can1] showed that tameness was, in fact, equivelant to topological finiteness. Since then, the two terms have commonly been regarded as synonymous, though many of the consequences of tameness follow via Thurston's geometrical interpretation.

One consequence of tameness is the Ahlfors measure conjecture, which states that the limit set of a finitely generated kleinian group has either zero or full Lebesgue measure (see [Can1]). Tameness is also a critical step in the classification of finitely generated kleinian groups. Other key ingredients in this classification are Thurston's Ending Lamination Conjecture, [Mi,BrocCM] (see also [Bow2,Bow3]), as well as a description of those end invariants which are realisable (see for example [KlS,NS]). Furthermore, in combination with various other results, it leads to a proof of the density conjecture of Bers, Sullivan and Thurston, namely that geometrically finite kleinian groups are dense among finitely generated ones. Arguments to deal with many (but not all) cases have been given by Bromberg and Brock [Brom,BrocB].

A key notion we use is that of a "polyhedral" surface, as in [Som]. The basic idea is that, appropriately interpreted, many of the essential constructions of "pleated surfaces" in hyperbolic 3-manifolds adapt to a more general context. Such surfaces have also been used, for example, in [Bon2] (see Lemme 1.7) and [Can2] (Section 3 thereof), where they are termed "simplicially hyperbolic". They also feature in [Sou].

As seems to be the tradition, we simplify the exposition by dealing first with the case where there are no parabolics (Section 5), and later discuss how this generalises when there are parabolics (Section 6). We go on to discuss variable curvature (Section 7) and Ahlfors's Finiteness Theorem (Section 8).

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1. Outline of the proof.

In this section, we give a brief informal outline of the proof of tameness. We restrict the discussion here to the case without cusps. The adaptations necessary when there are cusps will be discussed in Section 6.

Let M be a complete hyperbolic 3-manifold with $\pi_1(M)$ finitely generated. Using Scott's theorem, it is enough to show that each end, e, of M is topologically finite; in other words, it has a neighbourhood that is homeomorphic to a closed surface times $[0, \infty)$. As discussed in Section 2.4, e has a certain "genus", $g = \mathrm{genus}(e)$, associated to it, which is determined a-priori. Using Waldhausen's cobordism theorem, we can show that e is topologically finite if we can find a sequence of embedded surfaces of genus g going out the end, which separate the end from a fixed compact core the manifold and which are all homotopic to each other in the complement of the core — see Corollary 2.5.2. In fact we can make do with a sequence of maps of a fixed closed surface of genus g into M which homologically separate the end e from a core (since such maps can be replaced by embeddings using [FrHS] as described in Section 2.6). As discussed in Section 2.4, after passing to a cover, we can assume that all of $\pi_1(M)$ is carried by (any neighbourhood of) the end, e.

So far, the discussion has been topological. We now assume that M is hyperbolic without cusps. One can split into two cases. If e is "geometrically finite", then topological finiteness follows easily. We therefore assume that e is "degenerate", i.e. not geometrically finite. We construct a compact polyhedral subset $\Phi = \phi(K)$, of M by taking a suitable finite simplicial 2-complex, K, and mapping it into M by a piecewise geodesic map $\phi : K \longrightarrow M$, where ϕ is homotopy equivalence. In fact, we can extend this to a larger complex, $\phi : K_\alpha \longrightarrow M$, to give a larger polyhedral subset $\Phi_\alpha \subseteq M$. Since e is assumed degenerate, we can arrange that Φ_α enters arbitrarily far into e (see Lemma 5.2). Its purpose will be to act as a barrier to the homotopies we want to perform. We can arrange that the completion of the complements of Φ and Φ_α are locally CAT(-1) spaces.

Now let F be a closed embedded surface in $M \setminus \Phi_\alpha$ separating e from Φ_α. If it happens that its genus equals $g = \text{genus}(e)$ then its inclusion in $M \setminus \Phi_\alpha$ will be π_1-injective (Corollary 2.4.5), and we can homotope it to a "balanced" polyhedral map in (the metric completion of) $M \setminus \Phi_\alpha$. Via a Gauss-Bonnet argument, we get geometric control on the diameter of the image of this map. We end up with a map of Σ into M which homologically separates e from Φ and (since Φ_α enters deeply into e) stays a long way from Φ. If we could find such an F for every such Φ_α, then we get a sequence of surfaces, which we can replace by embeddings using [FrHS], as described in Section 2.6. One can then give an argument (cf. [Sou]) to show that these lie in finitely many homotopy classes in $M \setminus \Phi$ (see Lemma 5.3). We can therefore assume they all lie in the same homotopy class and we are done by Waldhausen as described in the previous paragraph.

The problem is that, a-priori, there is no reason to suppose that we can choose F with the right genus. In general its genus will be greater than g (Lemma 2.4.4). Using Dehn's Lemma, we can certainly assume that its inclusion into $M \setminus \Phi_\alpha$ is π_1-injective. The trick now (at least in the compressible case) is to consider the component, W, of $M \setminus F$, containing Φ_α. Let X be the cover of W corresponding to the image of $\pi_1(K_\alpha) \cong \pi_1(M)$ in W. We lift ϕ to a map $\tilde{\phi} : K_\alpha \longrightarrow M$ and set $\tilde{\Phi}_\alpha = \tilde{\phi}(K_\alpha) \subseteq W$, and $\tilde{\Phi} = \tilde{\phi}(K) \subseteq \tilde{\Phi}_\alpha$. Using an inductive hypothesis on genus, and Theorem 2.8.1, one can show that X is topologically finite. It has an "outer" end which corresponds to e. Removing a collar neighbourhood of this outer end, we obtain a manifold $P \subseteq X$, with boundary ∂P, so that $X \setminus P$ is just a product $\partial P \times [0, \infty)$. We can assume $\tilde{\Phi}_\alpha \subseteq P$. Moreover, it turns out that ∂P has the correct genus g. This surface, ∂P, can now play a similar role to F as in the previous paragraph.

It would be nice to homotope ∂P in $P \setminus \tilde{\Phi}_\alpha$ to a balanced polyhedral map there, and then map down to M to give us a surface homologically separating e from Φ, and with controlled diameter. There are a couple of complications however.

First, the homotopy might get snagged on ∂P, over which we have no geometric control. To get around this, we first embed P via a homotopy equivalence larger manifold, Z, with polyhedral boundary. This space Z will be locally $\mathrm{CAT}(-1)$, and the natural map of P to M extends to Z. We can then carry out our straightening homotopy in Z instead. To constrain the amount this homotopy moves the surface, we consider its homological intersection with a certain ray, τ, going out the end.

Another complication is that, although our homotopy is disjoint from $\tilde{\Phi}_\alpha$ in Z, its projection to M may sweep through Φ_α, or indeed through Φ. We therefore need some other argument to show that the polyhedral surface we end up with does indeed separate e from Φ. The intuitive reason for this is that the homotopy does not meet the preimage, $\Psi \cap \operatorname{int} Z$, of Φ in $\operatorname{int} Z$. In fact, $\Psi \cap \operatorname{int} Z$ has only one compact component, and this maps homeomorphically to Φ (cf. Lemma 4.6). The homotopy cannot sweep through any non-compact component. We shall approach this more formally, in the form of Lemmas 4.7 and 4.8.

We now choose larger and larger sets Φ_α, to give us our sequence of surfaces, and we are done (in the compressible case without cusps) using [FrHS], Lemma 5.3 and Waldhausen as described above.

A slight variation on the above is called for in the case where the end e is incompressible. Most of the essential core of the above argument is to be found in Sections 4 and 5.

2. Topological constructions.

In this section, we review or modify various standard 3-manifold constructions. The arguments are purely topological, apart from one appeal to Thurston's hyperbolisation theorem in Section 2.8, albeit to prove a purely topological statement. We divide the discussion into a number of largely independent topics. We confine our discussion to the "non-relative" case — sufficient to deal with the case without cusps. Some generalisations to the relative case will be discussed in Section 6.

Convention : We assume all 3-manifolds to be aspherical (every 2-sphere bounds a ball) and orientable.

The former assumption holds in any hyperbolic 3-manifold. The latter will simplify the exposition, and as observed in the introduction, is justified by [Tu].

We will reserve the letter M to refer to 3-manifolds with empty boundary. A "complete hyperbolic 3-manifold" is assumed to have empty boundary. In general, topological 3-manifolds allowed to have non-empty boundary.

2.1. Basic notions.

We refer to [H] for the general theory. In particular, we note the following.

Let P be a 3-manifold with (possibly empty) boundary ∂P.

Definition : A *compressing disc* is a properly embedded disc, $D \subseteq P$, with $\partial D = D \cap \partial P$ an essential closed curve in ∂P.

We have "Dehn's Lemma" due to Papakyriakopoulos:

Lemma 2.1.1 : [Papakyriakopoulos] *If $F \subseteq \partial P$ is any subsurface of ∂P with $\pi_1(F) \longrightarrow \pi_1(P)$ not injective, then there is a compressing disc, $D \subseteq P$ with $\partial D \subseteq F$.*

We also have the following special case of work of Stallings. It can be proven directly by the argument given for example in "Stage 3" of [Sc2]:

Lemma 2.1.2 : *If P is compact and $\pi_1(P)$ is a non-trivial free product, then P has a compressing disc.*

We also note from Papakyriakopoulos's Sphere Theorem that (since P is assumed irreducible) $\pi_2(P) = 0$.

We can also define compressing discs for closed surfaces in P:

Definition : If $S \subseteq P$ is an embedded 2-sided closed surface, a *compressing disc* for S is an embedded disc, $D \subseteq P$, with $\partial D = D \cap S$ an essential closed curve in S.
We say that S is *incompressible* if it admits no compressing disc.

Using Dehn's lemma (applied to P cut along S) one can show that this is equivalent to saying that $\pi_1(S) \longrightarrow \pi_1(P)$ is injective.

Definition : We say that P is *atoroidal* if every incompressible torus in P is boundary parallel (i.e. can be homotoped into a boundary component).

2.2. Compression bodies.

Let H be a handlebody and let $\Sigma = \partial H$. If we imagine H as embedded unknotted in $\mathbf{R}^3$, then a *compression body*, P, is obtained by removing the interior of a (possibly empty) set of unknotted positive genus handlebodies $H_1, H_2, \dots, H_n$ from the interior of H, such that the H_i are separated by a collection of disjoint compressing discs in H. (Here "positive genus" means not a 3-ball.) Write $\Sigma_i = \partial H_i$. Thus, $\partial P = \Sigma \cup \Sigma_1 \cup \cdots \Sigma_n$. We refer to $\partial_0 P = \Sigma$ as the *outer boundary*, and to the Σ_i as the *inner boundary* components. Each inner boundary component is incompressible.

(One can equivalently define compression bodies inductively. Start with a disjoint collection of handlebodies and products of surfaces with the interval, and then glue them together along discs in their respective outer boundaries so as to form a connected manifold.)

Let $g = \text{genus}(\Sigma)$, $g_i = \text{genus}(\Sigma_i)$ and $g_0 = g - \sum_{i=1}^{n} g_i \geq 0$. We refer to $(g_0, g_1, \dots, g_n)$ as the *type* of P. It determines P up to homeomorphism. Note that special cases are those of a handlebody (type (g)) or a product $\Sigma \times [0,1]$ (type $(0, g)$). In the latter case, there is no natural distinction between the inner and outer boundaries. In all other cases, the outer boundary is the unique maximal genus boundary component.

Note that

$$\pi_1(P) = F_{g_0} * \pi_1(\Sigma_1) * \cdots * \pi_1(\Sigma_n)$$

where F_p denotes the free group of rank p. Recall that the non-cyclic factors in a maximal free product decomposition are unique up to conjugation. We refer to a group of this sort as a *compression group* of *type* $(g_0, g_1, \ldots, g_n)$. (This is termed a "free/surface group" in [CalG].) We refer to $g = g_0 + \cdots + g_n$ as its *genus*. One can show (cf. [CalG] Lemma 5.6):

Lemma 2.2.1 : *If P is a compact (orientable and irreducible) 3-manifold with $\partial P \neq \emptyset$, with $\pi_1(P)$ a compression group, and with each surface factor is conjugate to the fundamental group of boundary component, then P is a compression body.*

Note that the genus of the group is the genus of the outer boundary component. The clause about surface groups factors can be omitted if $\pi_1(P)$ is itself a surface group. (The only purpose of the hypothesis $\partial P \neq \emptyset$ is to rule out a simply connected closed 3-manifold.)

Proof : (Sketch) We refer to the boundary components homotopic to the surface factors as "inner" boundary components. From the hypotheses these are all π_1-injective. (A priori, there might be any finite number of non-inner boundary components.) If $\pi_1(P)$ is trivial then each component of ∂P is a 2-sphere and so P is a 3-ball. If $\pi_1(P)$ is a surface group, then P is a closed surface times an interval [H]. Otherwise P splits as a non-trivial free product, and so Lemma 2.1.2 gives us a compressing disc $D \subseteq P$. Thus ∂D lies in a non-inner boundary component of ∂P. We cut P along D, and verify that the hypotheses also hold for the resulting manifold, or pair of manifolds. By induction on the genus of the compression group, we can suppose these pieces are compression bodies. The original manifold P therefore consists either of a compression body with a handle attached to the outer boundary, or two compression bodies connected along a disc in their respective outer boundaries. Thus, P is itself a compression body. $\diamondsuit$

We also have the following (a stronger version of which can be found in [BrinJS]).

Lemma 2.2.2 : *If P is a compact 3-manifold and $\Sigma \subseteq \partial P$ is a boundary component such that the induced map $\pi_1(\Sigma) \longrightarrow \pi_1(P)$ surjects, then P is a compression body with outer boundary $\partial_0 P = \Sigma$.*

Proof : (Sketch) The argument is similar to that of Lemma 2.2.1, This time, the outer boundary is determined a-priori by the hypotheses, and we use Dehn's Lemma (2.1.1) in place of Lemma 2.1.2 to find a compressing disc. $\diamondsuit$

Let P be a compression body of type $(g_0, g_1, \ldots, g_n)$. Abelianising, we see that $H_1(P)$ (with $\mathbf{Z}_2$ coefficients) decomposes as $H_1(P) \equiv \mathbf{Z}_2^{g_0} \oplus \bigoplus_i H_1(\Sigma_i)$, where $H_1(\Sigma_i) = \mathbf{Z}_2^{2g_i}$.

We also note that $H_2(P) \equiv \bigoplus_i H_2(\Sigma_i)$. In particular, $H_2(\partial P) \longrightarrow H_2(P)$ surjects, and so $H_2(P, \partial P) = 0$. Thus any closed surface in P separates.

2.3. Cores.

Here we shall take homology with $\mathbf{Z}_2$ coefficients, though the discussion applies more generally.

Let M be a non-compact 3-manifold (without boundary). An end of M is *isolated* if it is seperated from all other ends by a compact subset of M. (If M has only finitely many ends then all of them are isolated.) Any isolated end, e, of M has an associated (possibly trivial) second homology class, $h(e) \in H_2(M)$ — induced by any surface that separates the end.

If $P \subseteq M$ is a compact connected submanifold with $H_2(P) \longrightarrow H_2(M)$ surjective, then P separates the ends of M. In particular, M has finitely many ends (all isolated). If $H_2(P) \longrightarrow H_2(M)$ is also injective, then each component of $M \setminus \operatorname{int} P$ is non-compact. We see that there is a bijection between the ends of M and the components of $M \setminus \operatorname{int} P$. If $H_1(P) \rightarrow H_1(M)$ is surjective, then each such component has connected boundary, and so there is a bijection between the ends of M and the boundary components of P. In particular, this applies if $P \hookrightarrow M$ is a homotopy equivalence, or more generally, if P carries all of H_1 and H_2 and no component of $M \setminus \operatorname{int} P$ is compact.

We can identify the image of $H_2(\partial P)$ in $H_2(P) \equiv H_2(M)$ with the span of $h(e)$ as e varies over the set of ends. If $H_2(P) = 0$, there is precisely one relation arising from the fact that $H_3(P, \partial P) = \mathbf{Z}_2$ is the kernel of $H_2(\partial P) \longrightarrow H_2(P)$.

We recall again, the key result (Theorem 0.2):

Theorem 2.3.1 : (Scott [Sc1,Sc2]) *Suppose M is a 3-manifold with $\pi_1(M)$ finitely generated. Then there is a compact submanifold P whose inclusion into M is a homotopy equivalence.* $\diamondsuit$

We refer to P as a *Scott core* of M.

2.4. Ends.

Let M be a 3-manifold with $\pi_1(M)$ finitely generated, and let e be an end of M. Let $E \subseteq M$ be a submanifold with ∂E compact, containing the end e. This will be a "neighbourhood" of the end e. Given a subset, $Q \subseteq M$, we write $G(Q) \leq \pi_1(M)$ for the image of $\pi_1(Q)$ in $\pi_1(M)$.

Definition : We say that E is *full* if ∂E is connected and we can homotope M into $M \setminus E$.

(For example, this holds if $M \setminus E$ contains a Scott core.) In particular, $\pi_1(M \setminus E) \longrightarrow \pi_1(M)$ is surjective (i.e. $G(M \setminus E) = \pi_1(M)$). It also follows that E contains no other end of M. We have maps $\pi_1(\partial E) \longrightarrow \pi_1(E) \longrightarrow \pi_1(M)$. Since $\pi_1(M) = \pi_1(M \setminus E) *_{G(\partial E)} G(E)$, this splitting must be trivial, and so $G(\partial E) = G(E)$ (with respect to a basepoint in ∂E). In particular, $G(E)$ is finitely generated.

Let $N = N(E)$ be the cover of M corresponding to $G(E)$. Now E lifts to a one-ended submanifold, $\hat{E} \subseteq N$ which carries all of $\pi_1(N)$. This is a neighbourhood of an end, $\hat{e}$, of N. In fact, the homotopy of M into $M \setminus E$ lifts to a homotopy of N into $N \setminus E$, and so $\hat{E}$ is a full neighbourhood of $\hat{e}$ in N.

We claim that $N(e)$ is independent of the choice of full neighbourhood of e:

Lemma 2.4.1 : *Suppose E and E' are full neighbourhoods of e in M. Taking any basepoint in the unbounded component of $E \cap E'$, the subgroups $G(E)$ and $G(E')$ of $\pi_1(M)$ are equal.*

Proof : Since there is a base of full neighbourhoods of e, we can assume that $E \subseteq E'$. Clearly $G(E) \subseteq G(E')$. Let $\lambda : N \longrightarrow M$ be the cover of M corresponding to $G(E)$. As observed above, E lifts to a (full) neighbourhood, $\hat{E}$, of an end $\hat{e}$ of N. Since E' is full in M, we can homotope ∂E into $M \setminus E'$ in M. We can lift this to a homotopy in N, starting at $\partial \hat{E}$ and finishing with a map into N, with image Q say, disjoint from $\hat{E}$. Let $R \subseteq N$ be the set of point to which this homotopy maps with degree 1. Thus, R is compact, and $\partial \hat{E} \subseteq \partial R \subseteq Q \cup \partial \hat{E}$. Moreover, R lies on the opposite side of $\partial \hat{E}$ to $\hat{e}$. Now $\lambda(Q) \subseteq M \setminus \hat{E}$, and so $\partial E' \subseteq \lambda(R)$. Since λ is covering map, it follows that $\lambda|(R \cap \lambda^{-1}(\partial E'))$ is a homeomorphism. In other words we can lift $\partial E'$ to N. It follows that $G(E') = G(\partial E') \subseteq G(E)$, so $G(E') = G(E)$ as claimed. $\diamondsuit$

In particular, we see that the isomorphism type of $G(E)$ depends only on e, and we denote it by $G(e)$ — the *end group*. The cover $N(e) = N(E)$ only depends on E. We will denote the corresponding end of $N(e)$ also by e.

Let $N = N(e)$. Let P be a Scott core in N, and let E be the component of $N \setminus \operatorname{int} P$ containing e. Thus, E is a full neighbourhood, and so $G(E) = G(e) = \pi_1(N)$. Thus $\pi_1(\partial E) \longrightarrow \pi_1(N)$ is surjective. But this factors through $\pi_1(P) \longrightarrow \pi_1(N)$ which is an isomorphism, and so $\pi_1(\partial E) \longrightarrow \pi_1(P)$ is also surjective. By Lemma 2.2.2, P is a compression body with outer boundary $\partial_0 P = \partial E$. Thus, $G(e) \cong \pi_1(P)$ is a compression group. We have shown:

Lemma 2.4.2 : *The end group, $G(e)$, of any end e of M is a compression group.* $\diamondsuit$

Its type, $(g_0, g_1, \ldots, g_n)$ is well defined, and we refer to $g = g_0 + \cdots + g_n$ as the *genus* of the end e of M (or of N). We denote it by $\operatorname{genus}(e)$.

Remark : One can go on to show that $\operatorname{genus}(e)$ is equal to the genus of the corresponding boundary component of any Scott core of M, though we won't be needing this fact here.

Definition : We say that an end, e, of M is *incompressible* if it has type $(0, g)$ for some $g \in \mathbf{N}$ (the genus of e).

In other words, the cover $N(e)$, has the homotopy type of a closed surface, Σ, of genus g. Suppose $S \subseteq N(e)$ is any surface separating the ends of $N(e)$. We get a degree-1 map of S to Σ. If $\operatorname{genus}(S) = g$, then this is a homotopy equivalence (using the Hopf property of $\pi_1(\Sigma)$). Suppose E is the neighbourhood of e with $\partial E = S$. Then the inclusion of $S = \partial E$ into E is also a homotopy equivalence (for example, via van-Kampen's theorem and using asphericity). It follows that E deformation retracts onto ∂E, and that E is a full neigbourhood of e in $N(e)$. Note that (from the earlier discussion) such a surface, S, always exists in $N(e)$.

The following is now easily verified:

Lemma 2.4.3 : *If e has a neighbourhood, E, in M with ∂E incompressible M, then e is incompressible, and E deformation retracts onto ∂E.* ♢

Remark : In fact, such a neighbourhood will always exist in M if e is incompressible — take a complementary component of a Scott core — though we won't need this fact here.

We need the following result about surfaces separating an end, e, of M.

Lemma 2.4.4 : *Suppose that E is a full neighbourhood of the end e. Then* $\mathrm{genus}(\partial E) \geq \mathrm{genus}(e)$.

Proof : We can pass to the cover, $N = N(e)$, of M corresponding to e. As observed earlier, E is also full in N. Let $e_1, \ldots, e_n$ be the set of other "inner" ends of N. (This set may be empty.) These ends are all incompressible in N. (Take a Scott core of N and apply Lemma 2.4.3.) We now take a maximal compression of ∂E in $N \setminus \mathrm{int}\, E$, so as to give us a compression body $Q \subseteq N \setminus \mathrm{int}\, E$ with outer boundary $\partial_0 Q = \partial E$, and incompressible inner boundary components, $S_1, \cdots, S_n$. (Take a maximal collection, $D_1, \ldots, D_k$, of disjoint compressing discs for ∂E in $N \setminus \mathrm{int}\, E$, thicken up $\partial E \cup \bigcup_i D_i$ in $N \setminus E$, and cap off with a 3-ball any 2-sphere boundary components that arise.)

Recall that $\pi_1(N)$ is a compression group, $F_{g_0} * (*_i \pi_1(\Sigma_i))$, where Σ_i is a surface of genus g_i, embedded in N, and separating the end e_i. Now $\pi_1(S_i)$ does not split as a free product, and is thus conjugate in $\pi_1(N)$ to one of the surface groups $\pi_1(\Sigma_j)$. Thus, S_i is homotopic in N to Σ_j and so separates the end e_j. No two S_i can correspond to the same end e_j, since they would then bound a compact region. Moreover, we see that the S_i must account for all of the inner ends of N (since each such end is contained in some component of $N \setminus \mathrm{int}\, Q$ bounded by some S_i). We can thus assume that S_i separates the end e_i. Let E_i be the component of $N \setminus \mathrm{int}\, Q$ containing e_i, so that $\partial E_i = S_i$. Since e_i is incompressible, E_i deformation retracts onto ∂E_i. Thus $\pi_1(Q) \longrightarrow \pi_1(N \setminus \mathrm{int}\, E)$ is surjective. Since E is full, $\pi_1(N \setminus E) \longrightarrow \pi_1(N)$ is surjective, and so $\pi_1(Q) \longrightarrow \pi_1(N)$ is also surjective. Now $\pi_1(Q)$ is a compression group of type $(h, g_1, \ldots, g_n)$ for some $h \geq 0$, and we have a surjective map $F_h * (*_i \pi_1(S_i)) \longrightarrow F_{g_0} * (*_i \pi_1(\Sigma_i))$ which sends $\pi_1(S_i)$ bijectively to $\pi_1(\Sigma_i)$. Quotienting out by the normal closures of the surface factors, we get a surjective map between free groups, $F_h \longrightarrow F_{g_0}$. It follows that $h \geq g_0$.

But $\mathrm{genus}(e) = g_0 + g_1 + \cdots + g_n$ and $\mathrm{genus}(\partial E) = h + g_1 + \cdots + g_n$, and so $\mathrm{genus}(\partial E) \geq \mathrm{genus}(e)$ as required. ♢

Corollary 2.4.5 : *Suppose that $P \subseteq M$ is a closed subset and that $S \subseteq M \setminus P$ is a connected surface separating P from e. Suppose that the component of $M \setminus S$ containing e can be homotoped into P. Then* $\mathrm{genus}(S) \geq \mathrm{genus}(e)$. *Moreover, if* $\mathrm{genus}(S) = \mathrm{genus}(e)$, *then S is π_1-injective in $M \setminus P$.*

Proof : Note that S bounds a full neighbourhood, E, of e in M, and so the inequality follows immediately from Lemma 2.4.4. For the second statement, note that if the conclusion fails, then we could surger S along a compressing disc in $M \setminus P$ to give surfaces of smaller genus, one of which must also separate e from P giving a contradiction. $\diamondsuit$

2.5. Waldhausen's cobordism theorem.

Theorem 2.5.1 : (Waldhausen [Wa]) *Suppose $S_1, S_2 \subseteq M$ are disjoint embedded π_1-injective surfaces of positive genus in a 3-manifold M, and suppose that S_1 can be homotoped in M to S_2. Then there is an embedded submanifold $R \subseteq M$, homeomorphic to $S_1 \times [0,1]$, with $\partial R = S_1 \cup S_2$.* $\diamondsuit$

Definition : We say that an end, e, of M is *topologically finite* if it has a neighbourhood homeomorphic to $\Sigma \times [0, \infty)$ for some surface Σ.

Corollary 2.5.2 : *Suppose that $P \subseteq M$ is closed, and $(S_i)_i$ is a sequence of surfaces in $M \setminus P$ separating P from an end e of M, and tending out the end. Suppose the S_i are all homotopic in $M \setminus P$ and π_1-injective in $M \setminus P$. Then e is topologically finite.* $\diamondsuit$

Note that M is topologically finite if and only if it has finitely many ends and each end is topogically finite.

2.6. Replacing singular surfaces by embedded ones.

The following constructions are based on an argument in [FrHS]. In that paper, the results are expressed in terms of minimal surfaces, though they can also be formulated in terms of normal surface theory [JR]. In any case, the parts of the argument relevant to Theorem 2.6.2 below are purely combinatorial.

Theorem 2.6.1 : *Suppose $f : \Sigma \longrightarrow M$ is π_1-injective and homotopic in M to an embedding. Given any neighbourhood, U, of $f(\Sigma)$ in M, there is an embedding, $f' : \Sigma \longrightarrow U$, with f' homotopic to f in M.* $\diamondsuit$

This is a consequence of a result in [FrHS], as observed by Bonahon [Bon1] (Troisième Partie, Lemme 1.22). (We note that [FrHS] only talks explicitly about immersed surfaces, though the argument can be applied to general singular maps.) For more details, see [CanM]. $\diamondsuit$

Another means of obtaining embedded surfaces is via the Thurston norm. The following can be deduced using the result of Gabai [G] that the singular Thurston norm equals the non-singular norm. (See also [P].) This uses some fairly sophisticated machinery. With an additional (unnecessary) assumption on first homology, one can give a more direct argument. This is based on the proof of Theorem 2.1 of [FrHS] which is a relatively straightforward, though ingenious, tower argument. We give this as Theorem 2.6.2 below.

In what follows, we take homology coefficients in $\mathbf{Z}_2$.

Theorem 2.6.2 : *Suppose that $f : \Sigma \longrightarrow M$ is a map inducing injections $H_1(\Sigma) \longrightarrow H_1(M)$ and $H_2(\Sigma) \longrightarrow H_2(M)$. Given any neighbourhood, U, of $f(\Sigma)$ in M, there is an embedding $S \hookrightarrow U$, of another surface S, with* $\mathrm{genus}(S) \leq \mathrm{genus}(\Sigma)$ *and with $H_2(S) \longrightarrow H_2(M)$ injective.*

Proof : Following [FrHS] we construct a tower of double covers so that at the top of the tower we have maps $h : \Sigma \longrightarrow N$ and $\lambda : N \longrightarrow M$, with $f = \lambda \circ h$, where N is a 3-manifold that is a regular neighbourhood of $h(\Sigma)$, where $H_1(\Sigma) \longrightarrow H_1(N)$ is surjective, and where λ is locally injective. (The last statement follows from the construction, since it is assumed that $h(\Sigma)$ does not lift to any double cover of N.) Moreover, we can assume that $\lambda(N) \subseteq U$. Under our hypotheses, $h_* : H_1(\Sigma) \longrightarrow H_1(N)$ is in fact an isomorphism.

Now (as with any compact 3-manifold) we have $\dim H_1(\partial N) \leq 2 \dim H_1(N)$. But $\dim H_1(N) = \dim H_1(\Sigma) = 2\,\mathrm{genus}(\Sigma)$, and so $\dim H_1(\partial N) \leq 4\,\mathrm{genus}(\Sigma)$. As in [FrHS], we can write $\partial N = A \sqcup B$, where $h(\Sigma)$ homologically separates A and B in N. (In [FrHS] it was assumed that $h : \Sigma \longrightarrow M$ is a homotopy equivalence, but only the bijectivity of $h_* : H_1(\Sigma) \longrightarrow H_1(N)$ is needed for that part of the argument. It was also assumed there that f, and hence h, is an immersion. However, the only new situation we have to consider for a general position map is the possibility that there are two paths, α, β in Σ, connecting the same pair of cross-caps in N, with $h(\alpha) = h(\beta)$. In this case, $\alpha \cup \beta$ is null-homologous in Σ. We see that this situation is covered by the same argument as a disjoint pair of closed curves identified under h.)

Now the images of $H_2(A)$, $H_2(B)$ and $H_2(S)$ are all equal in $H_2(N)$ and hence in $H_2(M)$. In particular, they are all non-zero. Also, $\dim H_1(A) + \dim H_1(B) = \dim H_1(\partial N) \leq 4\,\mathrm{genus}(\Sigma)$, and so we can assume that $\dim H_1(A) \leq 2\,\mathrm{genus}(\Sigma)$. Thus, each component of A has genus at most $\mathrm{genus}(\Sigma)$. Let $F \subseteq A$ be such a component with $H_2(F)$ non-zero in $H_2(M)$.

We now proceed down the tower through a sequence of double covers. If two curves, $\alpha, \beta \subseteq F$, get identified at a given stage to a curve $\gamma \subseteq M$, then we can surger F in a neighbourhood of γ so as to eliminate the intersection and in such a way that the resulting surface is connected. Note that this preserves the class of the surface in $H_2(M)$, and also preserves the Euler characteristic and hence the genus of the surface.

At the bottom of the tower, we arrive at the required embedding in U. $\diamondsuit$

2.7. An incompressibility condition.

Lemma 2.7.1 : *Suppose that S is a compact boundary component of a 3-manifold, M, and that $P \subseteq M$ is a closed subset. Suppose that S can be homotoped into P. Then S is π_1-injective in $M \setminus P$.*

Proof : Passing to the cover corresponding to S, and replacing P by its preimage, we can suppose that $\pi_1(M)$ is supported on S. We can also suppose that P is connected (taking the appropriate component). Suppose that S is compressible in $M \setminus P$. Let $D \subseteq M \setminus P$ be a compressing disc for S. Let A be the component of $M \setminus D$ containing P. Now $\pi_1(M)$ splits as a free product with $\pi_1(A)$ as a vertex group. But $\pi_1(A)$ is all of $\pi_1(M)$ and so this splitting must be trivial. In other words, $M \setminus D$ has another component which is simply connected. Its closure in M is a 3-manifold with only 2-sphere boundary components. It follows that ∂D bounds a disc in S giving a contradiction. $\diamondsuit$

2.8. A covering theorem.

The following argument is due to Thurston. Another account of it can be found in [Can1].

Theorem 2.8.1 : *Let N be a non-compact topologically finite aspherical atoroidal 3-manifold with empty boundary. Let X be a cover of N with $\pi_1(X)$ finitely generated. Then X is topologically finite.*

Proof : Let $\Gamma = \pi_1(N)$. By Thurston's hyperbolisation theorem for Haken manifolds (see [O,Ka]), there is a convex cocompact action of Γ on $\mathbf{H}^3$ with $N \cong \mathbf{H}^3/\Gamma$. Let $Y \subseteq \mathbf{H}^3$ be the 1-neighbourhood (say) of convex hull of the limit set. Now Y/Γ is compact, and $\partial Y \neq \emptyset$ (since N is non-compact) so there is some $r \geq 0$ such that $Y \subseteq N(\partial Y, r)$. Let $G = \pi_1(X) \leq \pi_1(N) = \Gamma$, and let Z be the 1-neighbourhood of the convex hull of the limit set of G. Thus $Z \subseteq Y$, and so $Z \subseteq N(\partial Z, r)$. Now $\partial Z/G$ is homeomorphic to a component of the quotient of the discontinuity domain of G, which is compact, by Ahlfors's Finiteness Theorem [Ah]. Thus, $Z/G \subseteq N(\partial Z/G, r)$ is compact. But $\mathrm{int}(Z/G)$ is homeomorphic to $\mathbf{H}^3/G$ and hence to X. $\diamondsuit$

3. Polyhedral constructions.

In this section, we describe the main geometric tool used in the proof. The basic idea is that if one removes certain polyhedral subsets from a hyperbolic 3-manifold the result will be negatively curved (locally CAT(-1)) in the induced path metric. In practice, these polyhedral subsets can be thought of as barriers in the sense that we can realise certain homotopy classes of maps in their complement. In this way, we construct "balanced maps" of surfaces. One can then draw various conclusions (bounded diameter lemmas etc.) much as in the case of hyperbolic 3-manifolds. Some messing around is necessary to find the appropriate spaces in which to perform these constructions to prove tameness. The details of this are discussed in Section 4. Similar ideas have been used in [Som] and in [Bow3]. (We remark that while some of the results generalise, others are specific to ambient dimension 3.) An alternative, but less intuitive, variation of these ideas is described in Section 7, where they are more readily adapted to variable curvature.

We make use of the theory of CAT(k)-spaces, see for example [BridH]. Here k will be ± 1. For constant curvature, all our spaces will be locally compact. In variable curvature (Section 7) we will allow non-locally compact spaces. All the relevant facts apply equally well in that situation.

3.1. Polyhedra.

Let M be a complete hyperbolic 3-manifold. Given $x \in M$, let $\Delta_x(M)$ be the unit tangent space at x.

Definition : A *polyhedron*, Φ, in M is a locally finite embedded simplicial complex, all of whose simplices are (embedded) totally geodesic simplices.

Here, locally finite means that only finitely many simplices meet any compact subset of M, and so Φ is closed in M.

Any point $x \in \Phi$ determines a closed polyhedral subset, $\Delta_x(M, \Phi) \subseteq \Delta_x(M)$, of tangent vectors lying in Φ.

Definition : We say that Φ is *balanced* at $x \in \Phi$, if $\Delta_x(M, \Phi)$ is not contained in any open hemisphere of $\Delta_x(M)$.

Definition : We say that Φ is *fat* at x if $\Delta_x(M, \Phi)$ is connected.

Definition : We say that Φ is *balanced* (respectively *fat*) if it is balanced (respectively fat) at every point.

We remark that both of these properties are closed under locally finite union.

The definition of "balanced" is essentially the same as the definition of "NLSC" given in [Can2]. It will be used here mainly to ensure that the complementary components are negatively curved. Fatness is a somewhat technical condition that ensures the complex still acts as a barrier when we take the metric completion of the complement. Without it, we could lose part of the complex altogether. For example, if Φ were a simple closed geodesic, then taking the metric completion of $M \setminus \Phi$ would just give us back M. We want to avoid this phenomenon.

Given a polyhedron $\Phi \subseteq M$, write $\Pi = \Pi(\Phi)$ for the metric completion of $M \setminus \Phi$, and write $\mathrm{int}\,\Pi(\Phi) = M \setminus \Phi$. The inclusion $\mathrm{int}\,\Pi \hookrightarrow M$ extends to a natural map, $\omega : \Pi \longrightarrow M$. We write $\partial \Pi = \Pi(\Phi) \setminus \Pi$.

If Φ is fat, then one can construct a topological (in fact, PL) collar of $\partial\Pi$ in Π; that is, an arbitrarily small neighbourhood homeomorphic to $\partial\Pi \times [0, 1]$ with $\partial\Pi$ identified with $\partial\Pi \times \{0\}$. We see that $\Pi(\Phi)$ is a topological (indeed PL) manifold with boundary $\partial\Pi$. In particular, we conclude:

Lemma 3.1.1 : *If Φ is fat, then $\mathrm{int}\,\Pi \hookrightarrow \Pi$ is a homotopy equivalence.* ◊

We also have:

Lemma 3.1.2 : *If Φ is fat and balanced, then Φ is locally CAT(-1).*

Proof : This follows from the fact that for all $x \in \Phi$, the completion, $\Delta_x(M, \Pi)$, of $\Delta_x(M) \setminus \Delta_x(M, \Phi)$ is globally CAT(1) (cf. [Bow3]). ◊

We have the following, slightly technical, "fattening" procedure.

Lemma 3.1.3 : *Suppose that Φ is a connected compact balanced polyhedron in M, not contained in any embedded closed geodesic or geodesic segment in M. If U is any neighbourhood of Φ in M, then there is a fat balanced polyhedron, Φ', with $\Phi \subseteq \Phi' \subseteq U$.*

Proof : If $I \subseteq \Phi$ is any maximal geodesic segment whose interior is open in Φ, we can adjoin the convex hull of $V \cap \Phi$, where V is a small closed polyhedral neighbourhood of I in M (interpreting in the universal cover). We apply this to all such intervals. If $x \in \Phi$ is any point remaining with $\Delta_x(M, \Phi)$ not connected, then we adjoin the convex hull of $V \cap \Phi$ for a small closed polyhedral neighbourhood, V, of x in M. $\diamondsuit$

For the sake of notational convenience, we will frequently assume that the map $\omega : \Pi \longrightarrow M$ is injective, and identify Π as a subset of M (namely the closure of $M \setminus \Phi$). All our constructions are readily interpreted in the general case.

3.2. Polyhedral maps.

Suppose that K is a finite simplicial complex.

Definition : A map $\phi : K \longrightarrow M$ is polyhedral if the image of every simplex is a (possibly degenerate) totally geodesic simplex in M.

Here, we do not assume ϕ to be injective on simplices. However, is easily seen that the image, $\phi(K)$, has the structure of a polyhedron in M (possibly after subdividing, and replacing degenerate simplices by lower dimensional ones).

Definition : We say that ϕ is *balanced* at $x \in K$ if there is some neighbourhood, V, of x in K such that $\phi(V)$ is a balanced polyhedron in M at $\phi(x)$. (Here, $\phi|V$ need not be injective.)

This is, of course, really a property of the induced map of the link of x into $\Delta_{\phi(x)}(M)$.

Definition : We say that ϕ is *balanced* if it is balanced at every point.

We note that if ϕ is balanced, then $\phi(K)$ is a balanced polyhedron. Also, since it is a local property, the lift of any balanced map to any cover of M is also balanced.

We have the following simple criterion for recognising that certain maps are balanced (cf. [Bon2] Lemme 1.8). Suppose that $\phi : K \longrightarrow M$ is a polyhedral map. Suppose that $x \in K$ lies in the interior of an embedded interval $I \subseteq K$, and that $\phi|I$ maps I injectively to a geodesic segment in M. Then Φ is balanced at x.

One can construct balanced maps in abundance, following the basic idea of Bonahon [Bon2] (Lemme 1.7) which elaborates an earlier idea of Thurston. We can formalise this as follows:

Lemma 3.2.1 : *Suppose that K is a finite simplicial 2-complex and that we are given a homotopy class of maps from K into M. Suppose that $\gamma \subseteq K$ is an embedded closed polygonal curve whose image in M is homotopically non-trivial and non-parabolic. Then (perhaps after subdividing) we can realise the homotopy class by a balanced simplicial map, $\phi : K \longrightarrow M$, with $\phi(\gamma)$ a closed geodesic.*

Proof : After subdividing, we can assume that γ lies in the 1-skeleton of K. Let ϵ be any edge of γ, and let τ be any maximal tree in the 1-skeleton of K containing $\gamma \setminus \epsilon$. Let $\bar{\gamma}$ be the corresponding closed geodesic in M, and choose any $y \in \bar{\gamma}$. (In general, we should allow $\bar{\gamma}$ to wrap a number of times around a primitive geodesic in M.) We now fix a relative homotopy class $(K, \tau) \longrightarrow (M, y)$ in the right free homotopy class, such that $(\tau \cup \epsilon, \tau)$ gets sent to the class $(\bar{\gamma}, y)$. (There is some choice in "spinning" around $\bar{\gamma}$, but this need not concern us here.) We now define ϕ by sending τ to y, and sending each remaining edge of the 1-skeleton to the corresponding geodesic loop based at y (so that $\phi(\epsilon) = \bar{\gamma}$). We send each 2-simplex to a geodesic simplex in M. $\diamondsuit$

Of course, $\phi(K)$ will not be in any sense convex, though we can include additional geodesic loops into the construction by the following procedure (the eventual aim of which is to exhaust the convex core up to bounded distance, see Section 5).

Suppose that K is a 2-complex and α is a polygonal path in K (not necessarily embedded). After subdivison of K, we can suppose that α lies in the 1-skeleton. We form another complex K_α by gluing a disc to α along a subarc of its boundary. We write $\hat{\alpha}$ for the unattached part of the boundary of the disc. Thus, we can think of the disc as realising a homotopy from $\hat{\alpha}$ to α in K_α, relative to their common endpoints. We refer to the disc we have added as a "fin" attached to K.

Lemma 3.2.2 : *Suppose that $\phi : K \longrightarrow M$ be a balanced polyhedral map, and let α be a path in K. We can extend ϕ to a balanced simplicial map $\phi : K_\alpha \longrightarrow M$, so that $\phi(\hat{\alpha})$ is geodesic in M.*

Proof : We can realise our fin by taking triangulation whose 1-skeleton is a zigzag between the paths α and $\hat{\alpha}$ (cf. the construction of [Bon2] Section II). $\diamondsuit$

3.3. Polyhedral maps into $\Pi(\Phi)$.

In the above, we assumed the codomain of our maps to be M. However many of the essential principles apply when this is replaced by a locally finite locally CAT(-1) polyhedral complex. Here we restrict our attention to spaces of the form $\Pi(\Phi)$ as described above. In order to quote results from elsewhere, we note that such spaces can be triangulated as polyhedral complexes, though this construction is a little artificial, given that the essential arguments involving such complexes adapt directly to this set up.

For simplicity of exposition, we shall assume $\Pi \subseteq M$ to be embedded, so that for all $y \in \Pi$, we have $\Delta_y(M, \Pi) \subseteq \Delta_y(M)$. Since Φ is balanced, $\Delta_y(M, \Pi)$ is globally CAT(-1) in the induced path metric.

We need to define "balanced" maps in this context. One can give a similar definition as before, though we shall restrict here to the case where the domain is a circle or a surface. Let us suppose first that our maps are non-degenerate, i.e. each simplex maps locally injectively.

Suppose that α is a circle, and that $\phi : \alpha \longrightarrow \Pi$ is a (non-degenerate) polyhedral map. Each point $x \in \alpha$ determines a pair of points in $\Delta_{\phi(x)}(M, \Pi)$. The condition that α is locally geodesic at x is equivalent to asserting that these points are distance at least π apart in the induced path metric on $\Delta_{\phi(x)}(M, \Pi)$. In this case we deem ϕ to be *balanced* at x.

Suppose now that Σ is a closed surface, and that $\phi : \Sigma \longrightarrow \Pi$ is a polyhedral map. This time, each $x \in \Sigma$ determines a closed path in $\Delta_{\phi(x)}(M, \Pi)$. For us, the key point is that this curve should have length at least 2π. This is satisfied, for example, if it contains two points distance at least π apart in the induced path metric on $\Delta_{\phi(x)}(M, \Pi)$. This is in turn satisfied if x is contained in the interior of an interval Σ that gets mapped injectively to a local geodesic segment in Π. Such a map will deemed to be "balanced".

If ϕ maps each simplex to a simplex of the same dimension (so that after subdividing, ϕ can be assumed injective on each simplex) then the pull-back metric on Σ is hyperbolic with cone singularities at least 2π. This fact will be used to bound the area of Σ, via the Gauss-Bonnet theorem (see Section 3.4).

In general, we may be faced with the prospect that some simplices will collapse to lower dimensional ones. In this case, we need to consider neighbourhoods of subcomplexes that get collapsed to points. Essentially the same reasoning goes through. In this case we get a pseudometric on Σ whose hausdorffification is singular hyperbolic. Again we will need a bound on the area. Perhaps the simplest way to deal with the technicalities is to view the Gauss-Bonnet theorem in this context as essentially combinatorial: summing angles of each triangle and using Euler's formula directly.

We can construct polyhedral maps of this type exactly as with Bonahon [Bon2] (cf. Lemma 3.2.1 above). To extend over 2-simplices, some care is needed. We can construct a ruled surface by coning an edge over an opposite vertex (modulo some technicalities, this would suffice for our purposes). If we want a bona fide polyhedral map, then we may need to subdivide into smaller simplices.

We remark that the composition of a balanced map $\Sigma \longrightarrow \Pi$ with the inclusion of Π into M need not be balanced as a map into M. (One can construct examples where the closed path in $\Delta_{\phi(x)}(M, \Pi)$ is contained in an open hemisphere in $\Delta_{\phi(x)}(M)$.) However, this need not matter to us, other than as a point of caution.

3.4. The thick-thin decomposition.

First, we recall the thick-thin decomposition of a complete hyperbolic 3-manifold M without parabolics.

Let $\eta > 0$ be some fixed constant less than the Margulis constant. Let $T(M)$ be the set of $x \in M$ such that x lies in some essential curve β_x in M of length at most η. We refer to $T(M)$ as the *thin part* of M. Given a primitive free homotopy class, α, of closed curve in M, let $T(M, \alpha)$ be the set of x for which β_x can be chosen to be homotopic to α^n for some $n \in \mathbf{Z} \setminus \{0\}$. Such a "Margulis tube" is a uniform neighbourhood of the corresponding closed geodesic, α_M, in M. The thin part, $T(M)$, is a disjoint union of $T(M,\alpha)$ as α varies over such primitive classes. The write $\Theta(M)$ for the *thick part* of M, that is, the closure of $M \setminus T(M)$. Let ρ be the riemannian path pseudometric that agrees locally with the hyperbolic metric, d, on $\Theta(M)$, and which is set to be zero on each Margulis tube, often termed the "electric pseudometric".

The following is a generalisation of the well known "bounded diameter lemma" where we allow for surfaces that are not π_1-injective.

Lemma 3.4.1 : *Suppose that $f : (\Sigma, \sigma) \longrightarrow (M, d)$ is a 1-lipschitz map from a closed hyperbolic surface of curvature at most -1 to M. There is a constant, k, depending only on* genus(Σ) *and η, such that either the ρ-diameter of $f(\Sigma)$ is less than k, or else, for any $x \in f(\Sigma)$, there is a homotopically non-trivial simple closed curve, β, of length at most η in Σ, with $f(\beta)$ homotopically trivial in M and with $\rho(x, f(\beta)) < k$.*

Proof : This follows by a standard argument. Briefly, each component of the thick part of Σ has bounded σ-diameter. We can pass between any two points in Σ passing through a bounded number of components of the thin part of Σ. Each component of the thin part either maps into a Margulis tube of M and is therefore not noticed by the pseudometric ρ, or else is homotopically trivial in M, giving rise to our curve β. $\diamondsuit$

4. The wrapping construction.

This section contains the core of the argument. We give our reinterpretation of the shrinkwrapping construction (cf. [CalG,Som]), to equip us with suitable bounded genus surfaces in a 3-manifold. How this construction is used will be explained in Section 5.

Let M be a complete hyperbolic 3-manifold without cusps, and suppose that e is an end of M carrying the fundamental group of M (as in Section 2.4). In other words, $\pi_1(M)$, is a compression group, and any Scott core will be a compression body, with e as the outer end. We write $e_1, \ldots, e_n$ for the (possibly empty) set of inner ends, each of which is incompressible. If e itself is not incompressible, we shall assume that each of the inner ends is topologically finite.

Let $(g_0, g_1, \ldots, g_n)$ be the type of e. Thus, for $i \neq 0$, $g_i = \text{genus}(e_i)$. We set $g = \text{genus}(e) = \sum_{i=0}^{n} g_i$.

Let K be a finite simplicial complex of the homotopy type of M. (For example, take a wedge of n surfaces Σ_i of genus g_i, together with g_0 loops.) By Lemma 3.2.1, there is a balanced polyhedral homotopy equivalence, $\phi : K \longrightarrow M$.

Suppose that α is a path in K, and let K_α be the complex constructed in Section 3.2. We can extend ϕ to a balanced homotopy equivalence $K_\alpha \longrightarrow M$, also denoted ϕ, which sends $\hat{\alpha}$ to a geodesic path.

We will also take as hyphothesis the existence of a proper ray, τ, based in $\phi(K_\alpha)$, and with $\rho(\tau, \phi(K))$ sufficiently large, to be specified later. Here ρ is the electric pseudometric on M as defined in Section 3.

We can now fatten $\phi(K)$ and $\phi(K_\alpha)$ to give fat balanced polyhedral complexes $\Phi \subseteq \Phi_\alpha \subseteq M$ (Lemma 3.1.3). To save overburdening our notation, we shall assume that these complexes are already fat, and write $\Phi = \phi(K)$ and $\Phi_\alpha = \phi(K_\alpha)$. This does not affect our argument. Again, for notational convenience, we regard $\Pi(\Phi)$ and $\Pi(\Phi_\alpha)$ as subsets of M.

Since $H_2(K) \longrightarrow H_2(M)$ surjects, Φ separates the ends of M. The same is true of Φ_α. Let Υ and Υ_α be the components of $\Pi(\Phi)$ and $\Pi(\Phi_\alpha)$ respectively containing the end e. Since $H_1(\Phi) \longrightarrow H_1(M)$ also surjects, its boundary $\partial\Upsilon = \Upsilon \cap \Phi$ is connected. Now Υ is a full neighbourhood of this end, so that the discussion of Section 2.4 applies. This is also true of Υ_α.

The main aim of this section is to prove the following statement:

Proposition 4.1 : *Suppose $\phi : K_\alpha \longrightarrow M$ and $K \subseteq K_\alpha$ are as above. Suppose that τ is a proper ray going out the end e, with basepoint in K_α, and with $\rho(\tau, \phi(K))$ sufficiently large (depending on the geometry of Υ). Then there is an embedded surface $S \subseteq \operatorname{int}\Upsilon$, with $\operatorname{genus}(S) = g$, with S incompressible in Υ, and separating the end e from $\partial\Upsilon$, and such that both $\rho(\tau, S)$ and the ρ-diameter of S are bounded in terms of g and the 3-dimensional Margulis constant.*

We shall explain in Section 5 how this is used to give tameness of e. We now set about the proof of Propostion 4.1.

Let $F \subseteq \operatorname{int}\Upsilon_\alpha$ be an embedded surface separating Φ_α from the end e and with $\pi_1(F) \longrightarrow \pi_1(\Upsilon_\alpha)$ injective. Note that such a surface exists — take any surface that separates of minimal possible genus, and use Dehn's lemma and the fact that $\partial\Upsilon_\alpha$ is connected to see that it must be incompressible in Υ_α. We shall take F to be smooth.

From this point on, we split the discussion into two cases. First we consider the case where e is not incompressible, and that each of the ends e_i is topologically finite. We will describe the modifications necessary for the incompressible case later.

Let W be the component of $M \setminus F$ containing Φ_α. Note that each of the inner ends, e_i, of M is also an end of W. Thus W is topologically finite. Clearly W is aspherical (since M is). In fact:

Lemma 4.2 : *W is atoroidal.*

In fact, this lemma can be bypassed, but at the cost of making appeal to some more sophisticated results, as is done in [Som]. In order not to detract from the main flow of the argument, we postpone the proof until the end of this section.

Recall that we have maps $K \hookrightarrow K_\alpha \xrightarrow{\phi} W \hookrightarrow M$ and that the composition $K \longrightarrow M$ induces an isomorphism on π_1. It follows that $\pi_1(K) \longrightarrow \pi_1(W)$ and $\pi_1(K_\alpha) \longrightarrow \pi_1(W)$ inject. They have the same image $G \leq \pi_1(W)$. Let X be the cover of W corresponding to G. Thus the map $X \longrightarrow M$ obtained by composing $X \longrightarrow W \longrightarrow M$ is a homotopy equivalence.

Lemma 4.3 : *X is topologically finite.*

Proof : Since $G \cong \pi_1(X)$ is finitely generated, this follows from Lemma 4.2 and Theorem 2.8.1. $\diamondsuit$

Since the inner ends of W correspond to surface group factors of G, they lift to ends of X, also denoted by e_i.

We will later need the following observation:

Lemma 4.4 : *Suppose that α, β are primative essential closed curves in X which project to the same closed curve in W. Then $\alpha = \beta$.*

Proof : Since the composition $X \longrightarrow W \longrightarrow M$ is a homotopy equivalence, α and β are freely homotopic in X. The annulus realising this homotopy projects to a (singular) torus in W. Since W is atoroidal (Lemma 4.2), this torus bounds a (singular) solid torus in W with core curve homotopic to the common image of α and β (since these curves are assumed primitive and non-trivial). Thus image of the annulus in W can be homotoped relative to its boundary to the curve that is common image of α and β. Since $X \longrightarrow W$ is a covering space, this homotopy lifts to X showing that $\alpha = \beta$. $\diamondsuit$

(Again, Lemma 4.2 could be bypassed using the observation that α and β are non-trivial in X and hence in M.)

Let $\tilde{\phi} : K_\alpha \longrightarrow X$ be the lift of $\phi : K_\alpha \longrightarrow W$. Thus, $\tilde{\phi}$ and of $\tilde{\phi}|K$ are both homotopy equivalences. We write $\tilde{\Phi} \subseteq \tilde{\Phi}_\alpha$ respectively for the fattenings of $\tilde{\phi}(K_\alpha)$ and of $\tilde{\phi}(K)$. For simplicity we will pretend that these are already fat. (This avoids some technical fussing, but makes no essential difference to the argument.) We write $\Psi \subseteq \Psi_\alpha$ for the preimages of Φ and Φ_α under the covering map $X \longrightarrow W$. Thus $\tilde{\Phi} \subseteq \Psi$ and $\tilde{\Phi}_\alpha \subseteq \Psi_\alpha$. All these polyhedra are fat and balanced in X. (These definitions are local, and so make sense also in an incomplete hyperbolic manifold such as X.) Note that $\tilde{\Phi}$ and $\tilde{\Phi}_\alpha$ are compact.

Since $K_\alpha \longrightarrow X$ is a homotopy equivalence, $\pi_1(X)$ is a compression group. Now X is topologically finite, and we have observed that each surface factor of $G \cong \pi_1(X)$ is conjugate to an end of X. Thus, by Lemma 2.2.1, X must be homeomorphic to the interior of a compression body of type $(g_0, g_1, \dots g_n)$. Its inner ends correspond precisely to the inner ends, $e_1, \dots, e_n$, of M. We can find a manifold $P \subseteq X$, containing $\tilde{\Phi}_\alpha = \tilde{f}(K_\alpha)$, with $X \setminus P$ homeomorphic to $\partial P \times [0, \infty)$. (Imagine pushing the outer boundary of the manifold compactification of X slightly inside X.) Thus, ∂P is a surface of genus g. Note that $\tilde{\phi} : K_\alpha \longrightarrow P$ is again a homotopy equivalence.

While X is topologically nice, its intrinsic geometry may be complicated, since we have not made any geometric assumptions about the boundary, $\partial W = F$, of W, and this issue will arise in the cover, X. For this reason, our next job is will be to embed X in a locally CAT(-1) polyhedral complex Z. In fact, Z will be a hyperbolic 3-manifold with polyhedral boundary, ∂Z. We will carry out the "wrapping" procedure in Z. (The idea is based on a similar construction in [Som].)

Recall that $F = \partial W$ is embedded incompressible surface in Υ_α, where Υ_α is the component of $\Pi(\Phi_\alpha)$ containing the end e. We are assuming that ∂W is smooth (largely for convenience of terminology). Note that $W \cup \partial W$ is the metric completion of W. We write $X \cup \partial X$ for the metric completion of X. This is a manifold with smooth boundary

∂X, and the covering map $X \longrightarrow W$ extends to a covering map $X \cup \partial X \longrightarrow W \cup \partial W$.

Let $\tilde{\Upsilon}_\alpha$ be the universal cover of Υ_α. This has polyhedral boundary, $\partial\tilde{\Upsilon}_\alpha$. Let $\tilde{F}$ be a component of the lift of F. Since $F \hookrightarrow \Upsilon_\alpha$ is π_1-injective, $\tilde{F}$ is a universal cover of F. Now, F is 2-sided, with "outside" on the side of e. Let A be the component of $\tilde{\Upsilon}_\alpha \setminus \tilde{F}$ on the outside of $\tilde{F}$. By van-Kampen, this is also simply connected. Note that it is also aspherical (since Υ_α is). The group $\pi_1(F)$ acts on A.

Suppose that $\hat{F}$ is a any covering space of F. We get a quotient, $A_{\hat{F}}$, of A by the action of $\pi_1(\hat{F})$. We can identify the quotient of $\tilde{F}$ with $\hat{F}$. Thus $A_{\hat{F}}$ is a 3-manifold with smooth boundary, $\hat{F}$, as well as a polyhedral boundary (namely the quotient of $A \cap \partial\tilde{\Upsilon}_\alpha$). Also $A_{\hat{F}}$ is aspherical, so the inclusion $\hat{F} \hookrightarrow A_{\hat{F}}$ is a homotopy equivalence. Thus $A_{\hat{F}}$ deformation retracts onto $\hat{F}$.

Now $\partial X \longrightarrow \partial W = F$ is a union of covering spaces. If $\hat{F}$ is a component of ∂X, we can glue a copy of $A_{\hat{F}}$ to X along $\hat{F}$. Since X lies on the "inside" of $\hat{F}$, we get a seamless gluing. In other words, $X \cup A_{\hat{F}}$ is locally isometric to $\mathbf{H}^3$ in a neighbourhood of $\hat{F}$. Since $A_{\hat{F}}$ is a subset of a covering of Υ_α, there is a natural map $\lambda : \hat{A} \longrightarrow \Upsilon_\alpha \subseteq M$. Note that $\lambda(\operatorname{int}\hat{A}) \cap \Phi_\alpha = \emptyset$.

Performing this construction for each component of $\partial_0 X$, we get our locally CAT(-1) space, Z. Note that ∂X is precisely the topological boundary of X in Z, and that Z deformation retracts onto X. Combining the various maps λ with the covering $X \longrightarrow W \subseteq M$, we get a natural map $Z \longrightarrow M$, also denoted λ. By construction, λ is 1-lipschitz. Writing $\operatorname{int} Z = Z \setminus \partial Z$ for hyperbolic part of Z, we have $\lambda^{-1}(\Phi) \cap \operatorname{int} Z = \Psi \subseteq X$ and $\lambda^{-1}(\Phi_\alpha) \cap \operatorname{int} Z = \Psi_\alpha \subseteq X$. Since $\tilde{\Phi} \subseteq W$ maps surjectively to $\Phi \subseteq W \subseteq M$, we have $\lambda(\tilde{\Phi}) = \Phi$. Similarly, $\lambda(\tilde{\Phi}_\alpha) = \Phi_\alpha$. Note that the inner ends of X can be identified with the inner ends of M, and are not affected by the construction of Z. In particular, λ is injective on the inner ends of Z.

In summary, we have the following spaces and maps:

$$\begin{array}{ccccccc} & & & P & \hookrightarrow & X & \hookrightarrow & Z \\ & & \nearrow & & & \downarrow & & \downarrow \\ K & \hookrightarrow & K_\alpha & & \rightarrow & W & \hookrightarrow & M \end{array}$$

The maps $K \hookrightarrow K_\alpha \longrightarrow P \hookrightarrow X \hookrightarrow Z \longrightarrow M$ are all homotopy equivalences. The space Z is locally CAT(-1), and $\lambda|X : X \longrightarrow W$ is a covering space. Also, Z deformation retracts onto X and X onto P. We have $\lambda^{-1}(\Phi_\alpha) \cap \operatorname{int} Z = \Psi_\alpha \subseteq X$. The map $\operatorname{int} Z \setminus \Psi_\alpha \longrightarrow M \setminus \Phi_\alpha$ is a covering space (with possibly disconnected domain). The map $\lambda : Z \longrightarrow M$ is 1-lipschitz.

The plan now is to homotope ∂P in the completion of $Z \setminus \tilde{\Phi}_\alpha$ to a balanced map, and then project down to M. As observed in Section 1, this homotopy may sweep through Φ, so we need to check that the resulting surface separates Φ from the end e in M.

For this purpose, we bring our ray, τ, into play. Recall that $\tau \subseteq M$ with basepoint in Φ_α and goes out the end e. Moreover, we are assuming that $\rho(\tau, \Phi)$ is sufficiently large (to be specified shortly). We can assume that τ meets Φ_α only in its basepoint, and so lifts to a proper ray $\tilde{\tau}$ in Z with $\lambda(\tilde{\tau}) = \tau$. (Recall that λ restricted to the preimage of $M \setminus \Phi_\alpha$ is a covering space.) Now $\tilde{\tau}$ is a proper ray in Z that eventually leaves X, and hence P.

Thus it crosses ∂P essentially. (More formally, the locally finite $\mathbf{Z}_2$-homology cycle given by $\tilde{\tau}$ has non-trivial intersection with the class of ∂P in $H_2(Z)$.)

Now $\tilde{\Phi}_\alpha$ is fat and balanced in Z, and so we can form the space $\Pi_Z(\tilde{\Phi}_\alpha)$ — the completion of $Z \setminus \tilde{\Phi}_\alpha$ — exactly as for manifolds (see Section 3.2). For notational convenience, we shall assume, as usual, that $\Pi_Z(\tilde{\Phi}_\alpha)$ is a subset of Z. We write Y_α for the component containing ∂P. Thus, Y_α is a hyperbolic 3-manifold with polygonal boundary ∂Y_α. Note that $\tilde{\tau} \subseteq Y_\alpha$. We can similarly define Y as a component of $\Pi_Z(\tilde{\Phi})$. Note that, since $\tilde{\Phi}$ carries all of $H_1(Z)$ and $H_2(Z)$, we see from the discussion of Section 2.3 that the topological boundary of Y in Z is connected. This topological boundary is just $Y \cap \tilde{\Phi}$. Moreover, Y minus its other manifold boundary components has exactly one end, separated from $\tilde{\Phi} \cap Y$ by ∂P.

By construction, ∂P is homotopic in Z into $\tilde{\Phi}_\alpha$. Thus, by Lemma 2.7.1, it is incompressible in $Z \setminus \tilde{\Phi}_\alpha$ and hence also in Υ_α. Note that ∂P is also incompressible in $Z \setminus \tilde{\Phi}$ and hence Y for the same reason.

We can now find a balanced map $\tilde{h} : \Sigma \longrightarrow Y_\alpha$, homotopic in Y_α to ∂P, where Σ is a surface of genus g. The induced pseudometric, σ, on Σ is singular hyperbolic. The composition $h = \lambda \circ \tilde{h}$ is 1-lipschitz. Note that, $\tilde{h}(\Sigma)$ must also intersect $\tilde{\tau}$ homologically in $\tilde{\Upsilon}_\alpha$. In particular, $\tilde{\tau} \cap \tilde{h}(\Sigma) \neq \emptyset$, and so $\tau \cap h(\Sigma) \neq \emptyset$.

Lemma 4.5 : *Provided $\rho(\tau, \Phi)$ is big enough, we have $h(\Sigma) \cap \Phi = \emptyset$, and the ρ-diameter of $h(\Sigma)$ is bounded (depending only on* genus(Σ) *and the Margulis constant).*

Proof : Write η_0 for the 3-dimensional Margulis constant. By Lemma 3.4.1, the only way this might fail would be if there were an essential curve, $\beta \subseteq \Sigma$, of σ-length at most η_0, with $h(\beta)$ homotopically trivial in M, and with $\rho(\tau, h(\beta))$ bounded by some constant k (depending on genus(Σ) and η_0). Now $\lambda : Z \longrightarrow M$ is a homotopy equivalence, and so $\tilde{h}(\beta)$ is homotopically trivial in Z. But Z is locally CAT(-1), and so (coning over some point of $\tilde{h}(\beta)$), it bounds a (singular) disc, D, in Z, of diameter at most η_0. Thus $\lambda(D)$ is a disc of diameter at most η_0 bounding $h(\beta)$ in M. Now, provided that $\rho(\tau, \Phi) > k + 2\eta_0$, then $\rho(h(\beta), \Phi) > \eta_0$ and so $\lambda(D) \cap \Phi = \emptyset$. Since $\lambda(\tilde{\Phi}) = \Phi$, it follows that $D \cap \tilde{\Phi} = \emptyset$, and so $\tilde{h}(\beta)$ is homotopically trivial in $Z \setminus \tilde{\Phi}$. But $\tilde{h}(\Sigma)$ is homotopic to ∂P in $\Pi(\tilde{\Phi}_\alpha)$ and so, in particular, in $\Pi(\tilde{\Phi})$. Thus, $\tilde{h} : \Sigma \longrightarrow Z \setminus \Phi$ is π_1-injective, giving a contradiction. $\diamondsuit$

In other words, $h(\Sigma) \subseteq \operatorname{int} \Upsilon$. It is still not clear that $h(\Sigma)$ is homologically non-trivial in Υ. To make life simpler, we first replace it with another surface, $f(\Sigma)$, as follows.

By [FrHS] (Theorem 2.6.1 here), $\tilde{h}$ is homotopic in Y to an embedding $\tilde{f} : \Sigma \longrightarrow \operatorname{int} Y$ with $\tilde{f}(\Sigma)$ in an arbitrarily small neighbourhood of $\tilde{h}(\Sigma)$. Let $f = \lambda \circ \tilde{f} : \Sigma \longrightarrow M$. From Lemma 4.5, we see that $f(\Sigma) \cap \Phi = \emptyset$, and $f(\Sigma)$ has bounded ρ-diameter.

Let Q be the component of $Z \setminus \tilde{f}(\Sigma)$ containing $\tilde{\Phi} = \tilde{\phi}(K)$. The topological boundary $\partial Q = \tilde{f}(\Sigma)$ is homotopic to ∂P in Y. Now, as observed above, Y has $\tilde{\Phi} \cap Y$ as one boundary component. Writing Y' for Y minus the other (polyhedral) boundary components, we see that Y' has one end, separated from $\Phi \cap Y$ by the surface ∂P. Considering their homology classes in $H_2(Y)$, it follows that $\hat{h}(\Sigma)$ and hence also $\partial Q = \hat{f}(\Sigma)$ separate $\Phi \cap Y$ from the end. Since Σ has genus g — the same genus as the end, it follows by Lemma 2.4.3, that the end of Y' deformation retracts onto ∂Q. Since $K \longrightarrow \operatorname{int} Z$ is a homotopy equivalence, and factors through $\tilde{\phi} : K \longrightarrow Q$, it follows $K \longrightarrow Q \longrightarrow \text{int}\, Z$ are also homotopy equivalences. Thus $Q \longrightarrow Z$ is a homotopy equivalence. In particular, $\pi_1(Q)$ is a compression group, and so Q in the interior of a compression body with outer boundary ∂Q, and inner ends $e_1, \ldots, e_n$.

Recall that $\Psi = \lambda^{-1}(\Phi) \cap \text{int}\, Z$. Since λ is injective on the inner ends of Z, we see that $\Psi \cap Q$ is compact. We claim:

Lemma 4.6 : $\Psi \cap Q = \tilde{\Phi}$.

Proof : By construction, $\tilde{\Phi} \subseteq \Psi \cap Q$. Note that $\Psi \cap \partial Q = \emptyset$, and so $\Psi \cap Q$ is a union of compact components of Ψ. Now Ψ is the preimage of $\Phi = \phi(K)$ under the covering map $X \longrightarrow W \subseteq M$. Thus $\Psi \cap Q$ is a finite union, $\tilde{\Phi}(K) \cup \bigcup_{i=1}^{q} \tilde{\phi}_i(K_i)$, where each K_i is a finite cover of K, and where $\lambda \circ \tilde{\phi}_i$ is equal to the covering $K_i \longrightarrow K$ postcomposed with ϕ.

Suppose $x \in K_i$. We can assume that $\pi_1(K)$ is non-cyclic, and so it's not hard to find a loop β in K_i based at x which maps to a non-trivial primative loop in K under the finite covering $K_i \longrightarrow K$ (for example, using the fact that $\pi_1(K)$ contains a free group of rank 2). Now $\lambda(\tilde{\phi}_i(\beta)) = \lambda(\tilde{\phi}(\gamma))$, and so, by Lemma 4.4, we have $\tilde{\phi}_i(\beta) = \tilde{\phi}(\gamma)$. In particular, $\tilde{\phi}_i(x) \in \tilde{\phi}(K)$. Thus, $\tilde{\phi}_i(K_i) \subseteq \tilde{\Phi}$. This shows that $\Psi \cap Q = \tilde{\Phi}$ as claimed. (In fact, one can see there can be no such $\tilde{\phi}_i$.) $\diamondsuit$

In other words, we have shown that $\tilde{\Phi}$ is the entire preimage of $\Phi = \phi(K)$ under the map $\lambda | Q$, so that $\lambda | \tilde{\Phi} : \tilde{\Phi} \longrightarrow \Phi$ is a homeomorphism.

In what follows, we take homology with $\mathbf{Z}_2$ coefficients.

Lemma 4.7 : *The map* $f : \Sigma \longrightarrow M \setminus \Phi$ *induces an injection* $H_2(\Sigma) \longrightarrow H_2(M \setminus \Phi)$.

Proof : In other words, we claim $f(\Sigma)$ is non-trivial in $H_2(M \setminus \Phi)$. We distinguish two cases.

The easy case is where Q is not a handlebody. Then $\tilde{f}(\Sigma) = \partial Q$ is non-trivial in $H_2(Q)$. Since $\lambda : Q \longrightarrow M$ is a homotopy equivalence, $f(\Sigma) = \lambda(\tilde{f}(\Sigma))$ is also non-trivial in $H_2(M)$ and so, in particular, in $H_2(M \setminus \Phi)$.

Suppose that Q is a handlebody. Choose any point $x \in \tilde{\Phi} \subseteq Q$. Then $\partial Q = \tilde{f}(\Sigma)$ is homologically linked with x in Q. Now $(\lambda | Q)^{-1}(\lambda(x)) = \{x\}$. Since ∂Q is compact, and λ is locally injective, it follows that $\lambda(\partial Q) = f(\Sigma)$ is homologically linked with $\lambda(x)$ in M. (The 3-chain $\lambda(Q)$ has boundary $\lambda(\partial Q)$ and has non-zero $\mathbf{Z}_2$-intersection with $\lambda(x)$.) It follows that $\lambda(\partial Q) = f(\Sigma)$ is non-trivial in $H_2(M \setminus \{\lambda(x)\})$ and hence also in $H_2(M \setminus \Phi)$, since $\lambda(x) \in \lambda(\tilde{\Phi}) = \Phi$. $\diamondsuit$

Lemma 4.8 : *The map* $f : \Sigma \longrightarrow M \setminus \Phi$ *induces an injection* $H_1(\Sigma) \longrightarrow H_1(M \setminus \Phi)$.

Proof : Since $\partial Q = f(\Sigma)$ and $f = \lambda \circ \tilde{f}$, this is the same as saying that the map $\lambda : \partial Q \longrightarrow M \setminus \Phi$ induces an injection to $H_1(M \setminus \Phi)$.

Let $a \in H_1(\partial Q)$ with $\lambda(a)$ trivial in $H_1(M \setminus \Phi)$. Let b be a 2-chain in $M \setminus \Phi$ with $\partial b = \lambda(a)$.

Now since $\lambda : Q \longrightarrow M$ is a homotopy equivalence, and $\lambda(a)$ is trivial in M, a must be trivial in $H_1(Q)$. It thus bounds a non-trivial relative 2-cycle, c, in $H_2(Q, \partial Q)$. By Alexander duality, there is some $d \in H_1(Q)$ with $\mathbf{Z}_2$-intersection $\langle c, d \rangle = 1$.

Now the 2-chains b and $\lambda(c)$ both have the same boundary, $\lambda(a)$, in M, and so $e = b \cup \lambda(c)$ gives us a cycle representing an element of $H_2(M)$.

Recall that $\tilde{\phi} : K \longrightarrow Q$ is a homotopy equivalence, and so $d = \tilde{\phi}(h)$ for some $h \in H_1(K)$. Thus $\lambda(d) = \lambda(\tilde{\phi}(h)) = \phi(h)$, which is supported on $\phi(K) = \Phi$. Since b does not meet Φ, and since $(\lambda|Q)^{-1}\Phi = \tilde{\Phi} = \tilde{\phi}(K)$ and $\lambda|\tilde{\Phi}(K)$ is injective, we see that $\langle e, \lambda(d) \rangle = \langle b \cup \lambda(c), \lambda(d) \rangle = \langle \lambda(c), \lambda(d) \rangle = \langle c, d \rangle = 1$ where the first three angle brackets $\langle , \rangle$ denote the intersection number on $H_2(M) \times H_1(M)$.

But $H_2(M)$ is generated by the homology classes $h(e_1), \ldots, h(e_n)$ of its inner ends $e_1, \ldots, e_n$, and so the intersection form on $H_2(M) \times H_1(M)$ is trivial, giving a contradiction. $\diamondsuit$

We have shown that $f : \Sigma \longrightarrow M \setminus \Phi$ is injective on both H_1 and H_2. We can thus apply Lemma 2.6.2 to give us an embedded surface S in $M \setminus \Phi$, with $\text{genus}(S) \leq \text{genus}(\Sigma) = g$, with S in a small neighbourhood of $f(\Sigma)$ and non trivial in $H_2(M \setminus \Phi)$. (We remark that we could bypass Lemma 4.8, using the Thurston norm [G,P].) In particular, $\rho(\tau, S)$ and the ρ-diameter of S are both uniformly bounded. Note that S is contained in the component, Υ, of $\Pi(\Phi)$ containing e. Since $\partial\Upsilon = \Phi \cap \Pi(\Phi)$ is connected, and e is the only end of Υ, it follows that S must separate e from $\partial\Upsilon$. Since $\text{genus}(e) = g = \text{genus}(\Sigma)$, we have $\text{genus}(S) = g$ by Lemma 2.4.4.

This proves Proposition 4.1 in the compressible case.

The incompressible case calls for a few minor modifications. In this case, M, has two ends e and e'. We cannot assume that e' is topologically finite. Now Φ_α separates e and e', and in this case, we take incompressible surfaces, F and F', in $M \setminus \Phi_\alpha$ separating Φ_α from e and e' respectively. Let $W \supseteq \Phi_\alpha$ be the relatively compact open submanifold of M with $\partial W = F \sqcup F'$. In this case also, W is atoroidal (by essentially the same argument) and is therefore covered by a topologically finite manifold X. We find a compact manifold $P \subseteq X$, with $\Phi_\alpha \subseteq P \subseteq X$ and with $X \setminus P$ just a product. In this case P is homeomorphic to a product $\Sigma \times [0, 1]$, where Σ is a surface of genus g. (This follows from the fact that a manifold with fundamental group a surface group is just a product.) Now τ crosses one of the boundary components, $\partial_0 P$, of ∂P essentially. We proceed as before, with $\partial_0 P$ playing the role of ∂P. In this case, $\partial_0 P \longrightarrow M$ is a homotopy equivalence, and so most of the topological reasoning is simpler. In particular, Lemmas 4.7 and 4.8 are immediate.

This proves Proposition 4.1 in the incompressible case.

We now return to:

Proof of Lemma 4.2.

We want to prove that W is atoroidal. An alternative argument in a slightly different set-up is given in [CalG].

We start with some general lemmas.

Suppose that $\Omega \subseteq \mathbf{H}^3$ is a (perhaps disconnected) locally finite balanced polyhedron.

Lemma 4.9 : *Suppose $V \subseteq \mathbf{H}^3$ is a (connected) codimension-0 submanifold with $\Omega \cap \partial V = \emptyset$ and with ∂V incompressible in $\mathbf{H}^3 \setminus \Omega$. Then $\pi_1(V)$ is free.*

To begin the proof, we fix a basepoint, x, and set $d_x(y) = d(x, y)$. We can assume that all surfaces in $\mathbf{H}^3$ are in general position with respect to d_x, i.e. the restriction of d_x to the surface is a Morse function. Our aim is to show that we can isotope ∂V in $\mathbf{H}^3 \setminus \Omega$ so that $d_x|\partial V$ has no local maximum. We use:

Lemma 4.10 : *Suppose that $t > 0$ and that $F \subseteq \mathbf{H}^3 \setminus \Omega$ is a compact surface with $\partial F \subseteq \partial N(x, t)$, and which is π_1-injective in $\mathbf{H}^3$. Suppose that $F_0 \subseteq F \setminus \operatorname{int} N(x, t)$ is connected. Then there is a closed surface $J \subseteq F$ containing F_0, with $\partial J \subseteq \partial N(x, t)$ which is isotopic in $\mathbf{H}^3 \setminus \Omega$ relative to ∂J to a compact surface $J' \subseteq \partial N(x, t) \setminus \Omega$.*

Proof : Let us first consider the case where $F = F_0$. Write $R = \mathbf{H}^3 \setminus (\Omega \cup \operatorname{int} N(x, t))$ and given $u > t$, write $R(u) = R \cap \operatorname{int} N(x, u)$. These are manifolds with boundary $\partial R = \partial N(x, t) \setminus \Omega$. There is some $u_0 > t$ such that $F \subseteq R_0 = R(u_0)$.

Since Ω is balanced, the function d_x has no local maximum on Ω. From this, it follows that, as u increases from t to u_0, the topology of $R(u)$ changes a finite number of times by the addition of 0-handles or 1-handles (never 2-handles). Thus R_0 is homeomorphic to $\partial R_0 \times [0, 1)$ with a finite number of (open) 1-handles attached. In particular, it follows that $H_2(R_0, \partial R_0) = 0$, and that $\pi_1(R_0)$ is supported on the end of R_0. (In other words, if $H \subseteq R_0$ is compact, then $\pi_1(R_0 \setminus H) \longrightarrow \pi_1(R_0)$ surjects.)

From the first observation, we see that F cuts off a compact manifold $H \subseteq R_0$. By assumption, $\pi_1(F) \longrightarrow \pi_1(R_0)$ injects, so we can write $\pi_1(R_0) = \pi_1(R_0 \setminus H) *_{\pi_1(F)} \pi_1(H)$. But $\pi_1(R_0 \setminus H) \longrightarrow \pi_1(R)$ surjects, and so the splitting is trivial, i.e. $\pi_1(F) \longrightarrow \pi_1(H)$ surjects. Thus $F \hookrightarrow H$ induces a bijection on π_1, and so $(H, F) \cong (F \times [0, 1], F \times \{0\})$ by (see [H]). Thus, F is isotopic to a surface $J' \subseteq \partial R$ as required.

Now consider the general case. If F_0 were incompressible in R, then we could apply the above observation with $J = F_0$. If not, let $D_1 \subseteq R$ be a compressing disc for F_0 in R. Since F is incompressible in $\mathbf{H}^3 \setminus \Omega$, ∂D_1 bounds a disc $D'_1 \subseteq F$. Now the 2-sphere $D_1 \cup D'_1$ bounds a ball in $\mathbf{H}^3$ which does not meet Ω. This means we can isotope D'_1 to D_1 in $\mathbf{H}^3 \setminus \Omega$ fixing ∂D_1. Now surgering F_0 along D_1 and continuing in this way, we get a sequence of compressing discs $D_1, \ldots, D_p$, so that the surface $J = F_0 \cup D'_1 \cup \cdots \cup D'_p \subseteq F$ is isotopic to an incompressible surface in R. By the first paragraph, this is now isotopic to a compact surface $J' \subseteq \partial R = \partial N(x, t) \setminus \Omega$, as required. $\diamondsuit$

Lemma 4.11 : *Suppose that $S \subseteq \mathbf{H}^3 \setminus \Omega$ is a properly embedded π_1-injective surface. Then we can isotope S in $\mathbf{H}^3 \setminus \Omega$ so that $d_x|S$ has no local maximum.*

Proof : This is the same as requiring that no component of $S \setminus N(x, t)$ is relatively compact for any $t > 0$.

If not, let $t > 0$ be minimal so that $S \setminus N(x,t)$ has a relatively compact component. Its closure, F_0, satisfies $\partial F_0 = F_0 \cap N(x,t) = F_0 \cap \partial N(x,t)$ and meets the closure of the remainder of $S \setminus \mathrm{int}\, N(x,t)$ at a single point, $z \in \partial F_0 \subseteq \partial N(x,t)$. Let $F \subseteq F_0$ be the surface with any disc components of $S \setminus F$ adjoined. By Lemma 3.10, we can find some surface $J \subseteq F$, with $J \supseteq F_0$, which is isotopic to some $J' \subseteq \partial N(x,t) \setminus \Omega$ relative to ∂J. Since t is minimal, $J' \cap S = \emptyset$ (otherwise J' would cut off a compact component of $S \setminus \mathrm{int}\, N(x,t)$, sandwiched between J and J', contradicting the minimality of t). Now $z \in \partial J'$. and we can tilt J' slightly so that $d_x | J'$ has a unique maximum at z. By replacing J by this surface, we can eliminate the critical point z. We then proceed to the next smallest value of t for which $S \setminus N(x,t)$ has a relatively compact component. Since this set is discrete, we eventually isotope S so that $d_x | S$ has no maximum, as required. ◊

Proof of Lemma 4.9 : By Lemma 4.11, we can isotope V so that $d_x | \partial V$ has no local maximum. Now as t increases, the topology of the manifold $V \cap N(x,t)$ can change only by the addition of 0-handles or 1-handles. The result follows. ◊

We can now proceed with the proof of Lemma 4.2.

Suppose, for contradiction, that Δ is an embedded essential torus in W. Now Δ cannot be essential in M, and so it has a compressing disc in M. Since M is aspherical, it follows that Δ bounds either a solid torus or a ball with knotted hole in M. In the former case, the solid torus would have to contain $F = \partial W$, which easily leads to a contradiction. We can therefore assume that Δ bounds a ball with knotted hole. In particular, Δ is homotopically trivial in M and so lifts to $\mathbf{H}^3$. Let V be the component of the preimage of W in $\mathbf{H}^3$ that contains Δ. Note that Δ is also incompressible in V. Let Ω be the preimage of Φ_α in $\mathbf{H}^3$. Note that ∂V is incompressible in $\mathbf{H}^3 \setminus \Omega$. By Lemma 4.9, $\pi_1(V)$ is free. But $\pi_1(\Delta) \longrightarrow \pi_1(V)$ is injective, giving a contradiction.

This completes the proof.

5. Proof of tameness when there are no cusps.

Let M be a complete hyperbolic 3-manifold without cusps and with $\pi_1(M)$ finitely generated. Let e be an end of M. We show:

Proposition 5.1 : *e is topologically finite.*

We write core(M) for the convex core of M.

Definition : We say that e is *geometrically finite* if there is a neighbourhood, E, of e with $E \cap \mathrm{core}(M) = \emptyset$.
Otherwise, we say that e is *degenerate*.

If e is geometrically finite, then we can take E to be bounded by a convex surface that is a boundary component of core(M). In this case, topological finiteness follows easily.

The real interest is in the degenerate case. In other words, we have a sequence $x_i \in \mathrm{core}(M)$ with $x_i \to e$. (We remark that, using Ahlfors's finiteness theorem, one can see, in fact, that $\mathrm{core}(M)$ contains a neighbourhood of the end. However, we shall not use this, since we want an argument that is adaptable to the case of variable curvature where this is not known a-priori.)

For the purposes of the proof, we split the degenerate case into two subcases, depending on whether e is incompressible or compressible. We shall assume the incompressible case in dealing with the compressible case.

Before continuing, we make the following observation (cf. Section 1.5 of [Bon2]).

Lemma 5.2 : *There is some constant k_0 such that for any $b \in M$, any point of* $\mathrm{core}(M)$ *lies a distance k_0 from some geodesic loop based at b.*

Proof : Writing $M = \mathbf{H}^3/\Gamma$, we note that the convex hull of the Γ-orbit, B, corresponding to b, is the union of all geodesic 3-simplices with vertices in B. Any point in such a 3-simplex is a bounded distance from one of its edges. This edge projects to a loop based at b in M. $\diamondsuit$

To prove that e is topologically finite, after passing to the cover corresponding to e, we can suppose that $\pi_1(M)$ is supported on e. We are thus in the situation described in Section 4.

Let $\phi : K \longrightarrow M$ be the map described there, and write $\Phi = \phi(K)$. Let Υ be the component of $\Pi(\Phi)$ containing e. Given $x \in \Upsilon$ or $Q \subseteq \Upsilon$ we write $D(x) = \rho(x, \partial\Upsilon)$ and $D(Q) = \rho(Q, \partial\Upsilon)$, where ρ is the electric pseudometric described in Section 3.4. Thus $D : \Upsilon \longrightarrow [0, \infty)$ is a proper function, which we can think of as measuring "depth" in the end.

Let a be some fixed vertex of K. Since e is degenerate, we can find a sequence of points, $x_i \in \mathrm{core}(M)$ with $x_i \to e$. In other words, $D(x_i) \to \infty$. Moreover, we can find properly embedded rays τ_i based at x_i, with $D(\tau_i) \to \infty$. By Lemma 5.2, for each i, there is a loop, α_i, based at a, such that the corresponding geodesic loop in M based at $\phi(a)$ passes within a bounded distance of x_i (in the hyperbolic metric). We can extend $\phi : K \longrightarrow M$ to a map $\phi : K_{\alpha_i} \longrightarrow M$ as in Section 3.2. For all sufficiently large i, Proposition 4.1 gives us an embedded surface S_i in $\mathrm{int}\,\Upsilon$, of genus g, which separates e from $\partial\Upsilon$, and which is π_1-injective in $\Pi(\Phi)$ and hence also in Υ. Moreover, $\rho(\tau_i, S_i)$ and the ρ-diameter of S_i are uniformly bounded. In particular, $D(S_i) \to \infty$. In other words, the S_i go out the end.

Suppose first that e is incompressible. In this case, each S_i is incompressible in M, and so its inclusion into M is a homotopy equivalence. Thus, the S_i are all homotopic, and so, by Lemma 2.5.2, e is topologically finite.

We can thus assume that e is compressible. In this case, we show (cf. [Sou]):

Lemma 5.3 : *The S_i lie in finitely many homotopy classes in Υ.*

Proof : Since e is compressible none of the S_i are incompressible in M. Thus, by Dehn's lemma, there is some essential simple closed curve, $\beta_i \subseteq S_i$ which is homotopically trivial in M. Now S_i is incompressible in Υ, and so (by Lemma 3.2.1 applied to a locally CAT(-1) space), we can find a balanced map $\psi_i : S_i \longrightarrow \Upsilon$ homotopic to the inclusion, $S_i \hookrightarrow \Upsilon$, with $\psi_i(\beta_i)$ a closed (local) geodesic in the intrinsic path-metric on Υ. Since this cannot be geodesic in M, we must have $\psi_i(\beta_i) \cap \partial\Upsilon \neq \emptyset$. In particular, each $\psi_i(S_i)$ meets $\partial\Upsilon$.

Now, since Φ is compact and fat, the injectivity radius of Υ on $\partial\Upsilon$ is positive; that is, any loop in Υ that is non-trivial in Υ and meets $\partial\Upsilon$ has length at least some $\zeta > 0$. It follows that any essential loop in Υ of length less that ζ lies in a Margulis tube in M. Using Lemma 3.4.1, we conclude that the ρ-diameters of the sets $\psi_i(S_i)$ are bounded. Since the function D is proper, they all lie in some compact subset $R \subseteq \Upsilon$. Let $\eta > 0$ be the injectivity radius of Υ on R. If ψ_i and ψ_j agree everywhere to within a distance η, they must be homotopic in Υ, by linear homotopy along short geodesic segments. It now follows easily that there are only finitely many possibilities for the homotopy class, as claimed. $\Diamond$

Thus, passing to a subsequence, we can assume that the S_i are all homotopic in Υ and so e is topologically finite, again by Lemma 2.5.2.

This proves Proposition 5.1.

Now, as noted in Section 2.3, M has only finitely many ends, and it follows that M is topologically finite, proving Theorem 0.1 in the case without cusps.

6. Tameness with cusps.

6.1. Outline.

In this section, we explain how to adapt the earlier arguments to give a proof of topological finiteness in the general case (Theorem 0.1). We therefore have to deal with $\mathbf{Z}$-cusps and $\mathbf{Z} \oplus \mathbf{Z}$-cusps, but only the former cause significant complications. Much of the modification is fairly routine, though there are some more subtle points that call for more detailed comment. We work through each section in turn.

6.2. Topological constructions in the relative case.

Typically we will be dealing with a pair of spaces, (A, B), where A is connected, and $B \subseteq A$ is closed. We often loosely refer to B as the "peripheral structure" since it will be associated to cusps or boundary components. Our notation will frequently suppress explicit reference to B. We shall assume that A an B are reasonably nice spaces (in practice manifolds or simplicial complexes).

The fundamental group $\pi_1(A)$ carries a *peripheral structure* namely a set of conjugacy classes of subgroups — those supported on the connected components of B.

A standard trick for dealing with the situation when $B \neq \emptyset$ is to take the *double* $DA = D_B A$, i.e. take two disjoint copies of A and glue them along B. Note that $\pi_1(DA)$ is represented by a graph of groups with two vertex groups isomorphic to $\pi_1(A)$ connected by a set of edges, one for each (conjugacy class of) peripheral subgroup.

Suppose that $C \subseteq A$ is closed and meets every component of B. Then $DC \subseteq DA$ is connected. If $\pi_1(C) \longrightarrow \pi_1(A)$ is surjective then it is not hard to verify that $\pi_1(DC) \longrightarrow \pi_1(DA)$ is also surjective.

We now move on to consider more specific cases. Suppose that S is a compact surface, neither a disc nor an annulus, with boundary ∂S, which we view as its peripheral structure. (Here, the letter Σ will be reserved for surfaces without boundary, compact or otherwise.) The *type* of S is the pair (g, p) where g is the genus, and p the number of boundary components. A convenient measure of the complexity of S will be its first Betti number, $b(S) = \dim H_1(S) = 2g + (p-1)_+$ where $x_+ = \max(0, x)$. If $p > 0$ then $b(S)$ is also the genus of the closed surface DS. In fact, $b(DS) = 2b(S)$.

Suppose that F is a disjoint union of non-annular subsurfaces of a closed surface Σ, and that no boundary component of F is homotopically trivial in Σ. One can check that $b(F) \leq b(\Sigma)$. (Of course, the map $H_1(F) \longrightarrow H_1(\Sigma)$ need not be injective in general.)

Returning to S, we can form the product manifold, $P = S \times [0,1]$. We refer to $\partial^V P = \partial S \times [0,1]$ and $\partial^H P = S \times \{0,1\}$ as the *vertical* and *horizontal* boundaries respectively. (This ties in with the terminology used in [Mi,Bow2,Bow3] etc.) We think of $\partial^V P$ as the peripheral structure of P. (Thus, $DP \equiv DS \times [0,1]$.)

We can define a "relative compression body" by the following construction. Take a (possibly empty) collection, $P_i = S_i \times [0,1]$, of product manifolds, for $i = 1, \ldots, n$, and take an embedded disc, $D_i \subseteq (\operatorname{int} S_i) \times \{1\}$. Now take a handlebody, H, of genus g_0 (possibly 0), and a set, $D'_1, \ldots, D'_n$, of disjoint embedded discs in ∂H. We identify D_i with D'_i to give us a connected manifold, P. Let $\partial^V P = \bigcup_{i=1}^n \partial^V P_i$, and let $\partial^H P = \partial P \setminus \operatorname{int} \partial^V P$. Now, $\partial^H P$ consists of a set of *inner* boundaries $S_i \times \{0\}$, and an *outer* boundary component $S = \partial^H_0 P$. If S_i has type (g_i, p_i), then S has type (g, p), where $g = g_0 + g_1 + \cdots + g_n$ and $p = p_1 + \cdots + p_n$. We refer to $(g_0, g_1, \ldots, g_n, p_1, \ldots, p_n)$ as the *type* of P. A manifold of this sort, with peripheral structure, $\partial^V P$, is referred to as a *relative compression body*. (This ties in with the inductive definition of compression body described in Section 2.2.)

We shall refer to a group, G, if the from $\pi_1(P)$, with its peripheral structure as a *relative compression group*, of *genus* g, and *complexity* $b(G) = b(\partial^H_0 P) = 2g + (p-1)_+$, where p is the number of conjugacy classes of peripheral subgroups. (Note that $b(G)$ is not the same as $\dim H_1(P)$.) We see that $\pi_1(P) = F_{g_0} * (*_i \pi_1(S_i))$, with p_i peripheral subgroups conjugate into $\pi_1(S_i)$.

Note that DP is a compression body of type $(2g_0, g_1, g_1, g_2, g_2, \ldots, g_m, g_m, g'_{m+1}, g'_{m+2}, \ldots, g'_n)$, where we assume that the $p_i = 0$ if and only if $i \leq m$, and set $g'_i = 2g_i + p_i - 1$ for $i > m$. From the canonical structure of $\pi_1(DP)$ as a free product of surface groups and the free group F_{2g_0}, and counting the number of peripheral subgroups in each of the surface group factors, we see that this type, and hence $g_0, (g_1, p_1), \ldots, (g_n, p_n)$ are determined by the structure of $\pi_1(P)$. In other words, the type of a relative compression group is well defined (up to consistently permuting the g_i and p_i). Moreover, the factors $\pi_1(S_i)$ are well defined up to conjugacy in $\pi_1(P)$. We refer to these as the *surface factors*.

We have the following analogue of Lemmas 2.2.1 and 2.2.2.

Lemma 6.2.1 : *Suppose that P is a compact manifold and that $F \subseteq \partial P$ is a disjoint union of closed annuli, pairwise non-homotopic in ∂P. Suppose that $\pi_1(P)$ is a relative* *compression group, with peripheral structure coming from F. Suppose that each of the surface factors of $\pi_1(P)$ is conjugate into the fundamental group of a component of $\partial P \setminus F$. Then P is a relative compression body with $\partial^V P = F$.*

Lemma 6.2.2 : *Suppose that P is a compact manifold and that $F \subseteq \partial P$ is a disjoint union of closed annuli. Suppose that S is a component of $\partial P \setminus \operatorname{int} F$, and that $\pi_1(S) \longrightarrow \pi_1(P)$ is surjective. Then P is a relative compression body with vertical boundary $\partial^V P = F$ and outer boundary $\partial^H_0 P = S$.*

These follow by similar arguments, decomposing along compression discs. We use the fact that if Q is a compact manifold and $S \subseteq \partial Q$ is a closed subsurface with $\pi_1(S) \longrightarrow \pi_1(Q)$ an isomorphism, then $Q \cong S \times [0,1]$ with $S \equiv S \times \{0\}$ (see Theorem 10.2 of [H]).

Let R be a (non-compact) 3-manifold with boundary. As usual we assume our manifolds to be aspherical. The discussion of Section 2.3 goes through much as before, with R in place of M. Typically we use relative second homology, $H_2(P, \partial^V P)$ or $H_2(R, \partial R)$, but the usual (non-relative) first homology, $H_1(P)$ or $H_1(R)$.

The following strengthening of the Scott core theorem is due to McCullough [Mc].

Theorem 6.2.3 : [Mc] *Suppose R is a 3-manifold with $\pi_1(R)$ finitely generated, and that $F \subseteq \partial R$ is a compact subsurface. Then there is some compact submanifold, $P \subseteq R$, with $P \hookrightarrow R$ a homotopy equivalence, and with $P \cap \partial R = F$.* ♢

Suppose that ∂R is a union of bi-infinite cylinders, and that there are no proper essential discs or annuli. Let $F \subseteq \partial R$ be union of compact cores of a finite set of these cylinders, and let P be as given by Theorem 6.2.3, so that $F \subseteq \partial P$. Now the Euler characteristic of ∂P is twice the Euler characteristic of P, which is determined by $\pi_1(M)$. Thus, there is a bound on the number of non-toroidal components of ∂P and hence the number of components of F (since no two of them can be homotopic). We can deduce that ∂R, in fact, has only finitely many components, and so we can retrospectively assume that F contains a core from each such component. (This is a standard topological proof of Sullivan's cusp finiteness theorem [Su], see [FeM].) In this situation, we will write $\partial^V P = F$.

We say that an embedded surface S in a 3-manifold, R, is *proper* if $\partial S = S \cap \partial R$. Suppose now that ∂R is incompressible and anannular (that is, any proper disc or annulus in R can be homotoped into ∂R). Suppose that ∂R is a union of bi-infinite cylinders. Let e be an isolated end of R. It has a one-ended neighbourhood, E, with $E \cap \partial D$ consisting of semi-infinite cylinders, $S^1 \times [0, \infty)$. We write $\partial^V E = E \cap \partial R$, and write $\partial^H E$ for the relative boundary of E in M. Thus, $\partial^H R$ is a proper surface in R. We write $p(e)$ for the number of components of $\partial^V E$. (This is independent of E.) As before, we say that E is *full* if $\pi_1(R \setminus E) \longrightarrow \pi_1(R)$ is surjective.

The earlier discussion now goes through. The end e has associated with it an end group, $G(e)$, which is a relative compression group. This determines the *type* of the end e. In particular, we see that it has associated to it a genus, g, and a *complexity* $b(e) = b(G(e))$. Note that $b(e) = 2g + (p(e) - 1)_+$.

The above can be arrived at by similar arguments, or assisted by the observation that, if $p > 0$, then e determines an end, De of DR. Moreover, if E is a full neighbourhood of e

in R, then DE is a full neighbourhood of De in DR.

The following result could be elaborated upon (taking account of genus) but will suffice for our purposes. It is the analogue of Corollary 2.4.5, and can be proven similarly or by doubling.

Lemma 6.2.4 : *Suppose that $P \subseteq R$ is a closed subset carrying $\pi_1(R)$ and separating the end e from any other end. Suppose that $S \subseteq M \setminus P$ is a proper surface separating P from e. Then $b(S) \geq b(e)$. Moreover, if $b(S) = b(e)$, then S is π_1-injective in $M \setminus P$.* $\diamondsuit$

The relative version Waldhausen's cobordism theorem states that if two proper embedded subsurfaces, $S_1, S_2 \subseteq R$ are homotopic relative to ∂R, then they bound a product region $S_i \times [0,1]$.

We now move on to Section 2.6. We say that a map, $f : S \longrightarrow R$ is *proper* if $f^{-1}\partial R = \partial S$. For the analogue of Theorem 2.6.1, homotopies are interpreted relative to ∂R. It is discussed in [Bon1]. Here is a generalisation of Theorem 2.6.2:

Theorem 6.2.5 : *Suppose that $f : S \longrightarrow R$ is a proper map inducing injections $H_1(S) \longrightarrow H_1(R)$, $H_1(\partial S) \longrightarrow H_1(\partial R)$ and $H_2(S, \partial S) \longrightarrow H_2(R, \partial R)$. Given any open neighbourhood, U, of $f(S)$, there is a proper embedding $S \hookrightarrow U$ of another surface, S', with $b(S') \leq b(S)$, and with $H_2(S', \partial S') \longrightarrow H_2(R, \partial R)$ injective.*

Proof : We follow the proof of Theorem 2.6.2. At the top of the tower we have maps $h : S \longrightarrow N$ and $\lambda : N \longrightarrow U$. The map $H_1(S) \longrightarrow H_1(N)$ is an isomorphism, and so $b(\partial N) = \dim H_1(\partial N) \leq 2 \dim H_1(N) = 2b(S)$. Let $\partial^V N = \lambda^{-1}(\partial R)$ and let $\partial^H N$ be the closure of $\partial N \setminus \partial^V N$. Note that $\partial^V N$ is a disjoint union of essential annuli. We decompose $\partial^H N$ as $A \sqcup B$, so that $h(S)$ homologically separates A and B (in terms of relative homology, $H_2(S, \partial S)$ and $H_2(N, \partial^V N)$). Now the boundary components of $\partial^H N$ are all essential in the closed surface ∂N. From an earlier observation, we see that $b(A) + b(B) = b(A \cup B) \leq b(\partial N) \leq 2b(S)$. We can thus assume that $b(A) \leq b(S)$.

We can now continue as before. On going down the tower, we have the additional possibility of having to carry out surgery on a double arc connecting boundary components (in addition to double curves). However, as before, this does not change the Euler characteristic, and so the first Betti number also remains unchanged. At the bottom of the tower, we arrive at our desired surface. $\diamondsuit$

Moving on to Section 2.7, we have the following generalisation of Lemma 2.7.1, which is proven in [Bow3].

Lemma 6.2.6 : *Suppose that R is a 3-manifold and $S \subseteq \partial R$ is a compact subsurface and that no component of ∂S is homotopically trivial in S. Suppose that S can be homotoped into a closed subset, $P \subseteq R$ with $P \cap S = \emptyset$. Then S is π_1-injective in $R \setminus P$.* $\diamondsuit$

(The homotopy referred to need not be relative to ∂R, though this case would be sufficient for our application here.)

We have the following slight strengthening of Theorem 2.8.1.

Theorem 6.2.7 : *Let N be an (aspherical) topologically finite 3-manifold (without boundary), such that each π_1-injective torus in N can be homotoped out an end of N. We suppose that N has at least one non-toroidal end (i.e. genus at least 2). Let X be a cover of N with $\pi_1(X)$ finitely generated. Then X is topologically finite.*

Proof : The argument is again due to Thurston. By Thurston's hyperbolisation theorem [O,K], N is homeomorphic to $\mathbf{H}^3/\Gamma$ where $\Gamma = \pi_1(N)$ is a geometrically finite kleinian group acting on $\mathbf{H}^3$ with no $\mathbf{Z}$-cusps. Each non-toroidal boundary component of the compactification of N can be identified with a component of $\Omega(\Gamma)/\Gamma$ where $\Omega(\Gamma)$ is the discontinuity domain. Let $Y \subseteq \mathbf{H}^3$ be the closed 1-neighbourhood of the convex hull of the limit set, $\Lambda(\Gamma)$, of Γ, and let $\mathcal{B}$ be a (possibly empty) strictly invariant collection of horoballs (coming from the $\mathbf{Z} \oplus \mathbf{Z}$-cusps). We write B for the interior of $\bigcup \mathcal{B}$. Thus, $(Y \setminus B)/\Gamma$ is compact, and $Y \setminus B \subseteq N(\partial Y \setminus B, r)$ for some $r \geq 0$. Let $G \leq \Gamma$ correspond to $\pi_1(X) \subseteq \pi_1(N)$, and let Z be the 1-neighbourhood of the convex hull of $\Lambda(G)$. Thus $Z \setminus B \subseteq N(\partial Z \setminus B, r)$. By Ahlfors's Finiteness Theorem [Ah], $(\partial Z(G) \setminus B)/G$ is compact, and so $(Z \setminus B)/G$ is compact. It follows that G is geometrically finite. (Note that if we only remove those horoballs with non-trivial G-stabilisers, it is easily seen that the above quotient remains compact, and this is one of the standard characterisations of geometrical finiteness for kleinian groups.) In particular, $\operatorname{int} Z/G \cong \mathbf{H}^3/G \cong X$ is topologically finite. $\diamondsuit$

There is an addendum to the above result. Suppose that $\bar{N}$ is a manifold compactification of N and that $\hat{N}$ is a manifold with $N \subseteq \hat{N} \subseteq \bar{N}$. Lifting, we see from the above argument, that we can embed the cover, $\hat{X}$ of $\hat{N}$ in the compactification of $\bar{X}$. In our application of this, each boundary component of $\hat{N}$ is a cylinder which lifts to a cylinder in $\hat{X}$. In fact, this addendum can be bypassed, but only at the cost of introducing another, more complicated, argument, as we discuss at the relevant time (after Lemma 6.4.1).

6.3. Polyhedra with cusps.

The main novelty here is that we allow our hyperbolic simplices to have ideal points. We can extend continuously over "blow-ups" around our ideal points. Rather than attempt to give a formal unified account, we deal separately with manifolds and complexes.

Let M be a complete hyperbolic manifold. The *non-cuspidal part*, $\Psi(M)$, of M is obtained by removing the interiors of all Margulis cusps from M. The thick part, $\Theta(M) \subseteq \Psi(M)$, is then obtained by removing Margulis tubes from $\Psi(M)$. In what follows, we shall deem a certain set of $\mathbf{Z}$-cusps of M be be *essential*, and construct a manifold, $R(M)$ by removing only the essential cusps. Thus $\Psi(M) \subseteq R(M) \subseteq M$. (All the $\mathbf{Z} \oplus \mathbf{Z}$-cusps are considered inessential.) Note that each component of $\partial R(M)$ is a bi-infinite cylinder. We can write $M \setminus \operatorname{int} R(M)$ as $\partial R(M) \times [0, \infty)$, where $\partial R(M) \equiv \partial R(M) \times \{0\}$, and each $\{x\} \times [0, \infty)$ is a geodesic ray. We can adjoin a copy of $\partial R(M)$ to M as $\partial R(M) \times \{\infty\}$. The resulting manifold, denoted $M \cup \partial^V M$, is homeomorphic to R. It has boundary $\partial^V M \cong \partial R$, which we view as the peripheral structure.

We can do something similar with simplicial complexes. Suppose that $\bar{K}$ is a locally finite simplicial complex, and that $A \subseteq V(\bar{K})$ is a set of vertices. Let $K = \bar{K} \setminus A$. We define the blow-up of, $K \cup \partial^V K$, by adjoining a copy of the link of a, for each vertex $a \in A$.

(Note that $K \cup \partial^V K$ can also be triangulated as a simplicial complex.)

This construction can be given a geometric interpretation. Suppose that K is built out of hyperbolic simplices in such a way that each "missing" vertex in A corresponds to an ideal point. Then the points of $\partial^V K$ can be thought of as obtained by adjoining an ideal point to each geodesic ray.

We can now generalise the notion of a polyhedron in M. This is a simplicial complex, $\Phi \subseteq M$, where the simplices are allowed to have ideal vertices in the essential cusps of M. We see that the inclusion $\Phi \hookrightarrow M$ extends naturally to an injective map $\Phi \cup \partial^V \Phi \hookrightarrow M \cup \partial^V M$.

More generally, we have an obvious notion of polyhedral map, $\phi : K \longrightarrow M$, where again, ideal vertices go to essential cusps. This has a natural extension, $\phi : K \cup \partial^V K \longrightarrow M \cup \partial^V M$. Moreover, $\Phi = \phi(K)$ is a polyhedral complex, with $\Phi \cup \partial^V \Phi = \phi(K \cup \partial^V K)$.

The discussion of fat and balanced polyhedra and maps in Sections 3.1 and 3.2 goes through with little change. When realising maps of finite complexes (cf. Lemma 3.1.1) we can talk about proper homotopy classes of proper maps, $K \longrightarrow M$, where the ends of K (corresponding to missing vertices) go out essential cusps of M. This is equivalent to considering relative homotopy classes of maps $K \cup \partial^V K, \partial^V K \longrightarrow M \cup \partial^V M, \partial^V M$.

There is again little change to Section 3.3.

Moving on to Section 3.4, we let d_R be the induced path-metric on $R(M)$, and let ρ_R be the path-pseudometric obtained from d_R by deeming each Margulis tube and each inessential Margulis cusp to have diameter 0.

Now, Σ, becomes a topologically finite surface which can be compactified to a surface $\Sigma \cup \partial^V \Sigma$ with boundary $\partial^V \Sigma$. The metric, σ, on S will be a finite-area singular hyperbolic metric. In Lemma 3.4.1, we get instead, a bound either on the ρ_R-diameter of $f(\Sigma) \cap R(M)$, or on $\rho_R(x, f(\beta))$.

6.4. Wrapping with cusps.

Let M be a complete hyperbolic 3-manifold with $\pi_1(M)$ finitely generated. We suppose, as before, that M has a preferred set of $\mathbf{Z}$-cusps deemed "essential". Let e be an end of $M \cup \partial^V M$. We let $\partial^V_e M$ be the set of components of $\partial^V M$ that meet each neighbourhood of e. It is a consequence of Theorem 6.2.3 and the subsequent discussion that we can choose such a neighbourhood so that it intersects each such component in a bi-infinite cylinder. The subgroup, $G(e)$ of $\pi_1(M)$ supported on e is a relative compression group (see Section 6.2). Let $\bar{N}(e)$ be the the associated cover of $M \cup \partial^V M$, and write $N(e)$ for its interior. The cusps associated to the end e (namely bounded by $\partial^V_e M$) lift to $N(e)$. We view these cusps as the essential cusps of $N(e)$. In other words, we can identify $\partial^V N(e)$ with $\partial^V_e(M)$. The manifold $N(e) \cup \partial^V N(e)$ has an "outer" end, e_0, which is a lift of e, together with a (possibly empty) set of inner ends.

In general these ends may contain other $\mathbf{Z}$-cusps (traditionally termed "accidental cusps"). Note, however, that any such $\mathbf{Z}$-cusp in e_0 must have already existed (as an inessential cusp) in M.

To do the construction properly, we therefore begin by deeming all $\mathbf{Z}$-cusps in M to be essential. We can take any end, e, of $M \cup \partial^V M$, and construct $N(e) \cup \partial^V N(e)$. Then the outer end has no $\mathbf{Z}$-cusps. Replacing M with $N(e)$. We are therefore reduced to the following situation:

$(*)$ M is a complete hyperbolic 3-manifold with an associated set of "essential" $\mathbf{Z}$-cusps. The manifold $M \cup \partial^V M$ has an "outer" end e, which meets each component of $\partial^V M$ in an unbounded set. The fundamental group, $\pi_1(M)$, is supported on (any neighbourhood of) e. The end e contains no $\mathbf{Z}$-cusps.

Now $\pi_1(M)$ is a relative compression group, and $M \cup \partial^V M$ has a (possibly empty) set of "inner" ends $e_1, \ldots, e_n$. Such an inner end may contain $\mathbf{Z}$-cusps, or may be a $\mathbf{Z} \oplus \mathbf{Z}$-cusp. If e is itself a $\mathbf{Z} \oplus \mathbf{Z}$-cusp, then it is clearly topologically finite, so we can ignore this case.

We will again give separate attention to the cases where e is compressible or incompressible. In the former case, we will assume all the inner ends to be topologically finite. In the latter case, we will not be able to assume that, but much of the reasoning is simpler anyway. The following discussion deals mostly with the compressible case.

Let K_0 be a compact 2-complex with a balanced polyhedral homotopy equivalence, $\Phi : K_0 \longrightarrow M$. Each essential cusp of M corresponds to a closed curve in K which we take to lie in the 1-skeleton. We can form another compact complex by coning over each such curve. Removing the cone points we then obtain a complex, K, with "missing vertices". We can now extend ϕ to a proper polyhedral homotopy equivalence $\phi : K \longrightarrow M$, where the ends of K get sent to the essential cusps. This extends further to a continuous map $\phi : K \cup \partial^V K \longrightarrow M \cup \partial^V M$, which is again a homotopy equivalence. (Note that $\partial^V K$ is a union circles.) Let $\Phi = \phi(K)$. We similarly define K_α, and let $\Phi_\alpha = \phi(K_\alpha)$. Now $\Phi_\alpha \cup \partial^V \Phi_\alpha$ separates the ends of $M \cup \partial^V M$, and we let $\Upsilon \cup \partial^V \Upsilon$ be the closure of the complementary component containing e. (Here $\Upsilon \subseteq M$ and $\partial^V \Upsilon \subseteq \partial^V M$.) We write $\partial^H \Upsilon = (\Upsilon \cup \partial^V \Upsilon) \cap (\Phi \cup \partial^V \Phi)$. This is the relative boundary of $\Upsilon \cup \partial^V \Upsilon$ in $M \cup \partial^V M$.

In this case, we will assume that our ray τ lies in $\Upsilon \cap R(M)$, based in $\Phi_\alpha \cap R(M)$, and going out the end. We will assume that $\rho_R(\partial^H \Upsilon \cap R(M), \tau)$ is sufficiently large.

We now define $W \cup \partial^V W \subseteq M \cup \partial^V M$ with relative boundary $\partial^H W$, similarly as we did with $W \subseteq M$ in the case where $\partial^V M = \emptyset$. This manifold contains all the inner ends of $M \cup \partial^V M$. We have the following version of the statement that Δ is atoroidal (cf. Lemma 4.2).

Lemma 6.4.1 : *Any π_1-injective torus in W can be homotoped out an end of N.*

Proof : Let Δ be an embedded π_1-injective torus. If Δ is π_1-injective in M, then it bounds a $\mathbf{Z} \oplus \mathbf{Z}$-cusp. This cannot contain $\partial^H W$ and so Δ bounds some cusp in W. We can therefore assume that Δ is not π_1-injective in M, and the argument proceeds as with Lemma 4.2. $\diamondsuit$

As before, we can now take a cover, X, of W so that $\phi : K \longrightarrow W$ lifts to a homotopy equivalence, $\tilde{\phi} \longrightarrow X$. By Lemma 6.2.7, X is topologically finite. In other words we can embed it in a compact manifold $\bar{X}$. By the addendum to Lemma 6.2.7, we can lift $\partial^V W$ to a subset $\partial^V X \subseteq \partial \bar{X}$, so that the closure of $\partial^V X$ is a disjoint union of annuli. (The addendum can be bypassed using the observation that a properly embedded essential semi-infinite cylinder, $S^1 \times [0, \infty)$ cannot be knotted in a 3-manifold. However, this requires a bit of work to prove.) By Lemma 6.2.1, $\bar{X}$ is a relative compression body with respect to this peripheral structure. We write $\partial^H X \subseteq X$ for the outer boundary. Thus, $X \cup \partial^V X \cup \partial^H X$ is a relative compression body with the inner boundary components removed. Pushing the surface $\partial^H X$ slightly inside this manifold, while keeping its boundary components in $\partial^V X$, we a surface, denoted $\partial^H P$, in $X \cup \partial^V X$. This is the outer boundary of another relative compression body, $P \cup \partial^V P \cup \partial^H P$, with $P \subseteq X$ and $\partial^V P \subseteq \partial^V X$.

We now proceed to construct a locally CAT(-1) space Z pretty much as before. It has a vertical boundary, $\partial^V Z$, coming from $\partial^V X$. There is a natural map $\lambda : Z \longrightarrow M$, which extends to a relative homotopy equivalence $Z \cup \partial^V Z \longrightarrow M \cup \partial^V M$.

Next, we construct maps $h, f : \Sigma \longrightarrow \Upsilon$ as before, by homotoping $\partial^H P$ in $Z \cup \partial^V Z$. By construction, the maps are just products in some neighbourhood of the essential cusps (i.e. they send rays to geodesic rays). In fact, we can take these cusps to be uniform. By construction, we can then assume that the 1-skeleton of the triangulation does not enter the cusps. Also by construction, Σ has the same type as the outer boundary component of the relative compression body. In particular (from the definition of the complexity of an end) we have $b(\Sigma) = b(e)$. Now $f(\Sigma \cup \partial^V \Sigma)$ is a properly embedded surface that bounds a manifold $Q \cup \partial^V Q \subseteq Z \cup \partial^V Z$. We write $\partial^H Q = f(\Sigma \cup \partial^V \Sigma)$ for the relative boundary. Again $Q \cup \partial^V Q \cup \partial^H Q$ is a relative compression body with the inner boundary components removed. The map $\lambda : Q \cup \partial^V Q, \partial^V Q \longrightarrow M \cup \partial^V M, \partial^V M$ is a relative homotopy equivalence.

The analogue of Lemma 4.7 now follows by applying the analogous statement to Lemma 4.5, where we need to take care that the common image of the curves α and β is not homotopic into $\mathbf{Z} \oplus \mathbf{Z}$-cusp of M.

We now have:

Lemma 6.4.2 : *$f : \Sigma \longrightarrow \Upsilon$ induces an injection $H_2(\Sigma \cup \partial^V \Sigma, \partial^V \Sigma) \longrightarrow H_2(\Upsilon \cup \partial^V \Upsilon, \partial^V \Upsilon)$.*

Proof : The case where M is a handlebody (with $\partial^V M = \emptyset$) has already been accounted for (Lemma 4.7).

In all other cases, f induces an injection to $H_2(M \cup \partial^V M, \partial^V M)$ as with Lemma 4.7, and the result follows. ◊

Note that, by construction, the map $H_1(\partial^V \Sigma) \longrightarrow H_1(\partial^V \Upsilon)$ is injective.

Lemma 6.4.3 : *$f : \Sigma \longrightarrow \Upsilon$ induces an injection $H_1(\Sigma) \longrightarrow H_1(\Upsilon)$.*

Proof : This is most conveniently seen via the doubling trick. We get doubled maps $D\tilde{\phi} : DK \longrightarrow DQ$ and $D\lambda : DQ \longrightarrow DM$, which are both homotopy equivalences. We argument proceeds as before to show that $H_1(D\Sigma) \longrightarrow H_1(D\Upsilon)$ is injective. But $H_1(\Sigma) \longrightarrow H_1(D\Sigma)$ is injective, and the result follows. ◊

Using Theorem 6.2.5, we can find a proper embedded surface $F \subseteq \Upsilon \cup \partial^V \Upsilon$, which is non-trivial $H_2(\Upsilon \cup \partial^V \Upsilon, \partial^V \Upsilon)$, and with $b(S) \leq b(e)$. In particular, we can assume that $S \cap R(M)$ is in a small neighbourhood $f(\Sigma) \cap R(M)$. Thus, $S \cap R(M)$ has bounded ρ_R-diameter, and is a bounded ρ-distance from $\tau \subseteq \Upsilon$. (Since we have a bound on complexity, the bounded diameter lemma applies.) After pushing contractible components of $S \setminus R(M)$ into M, we we can assume that each component of $S \setminus R(M)$ is an annulus (and so $S \cap R(M)$ is homeomorphic to S). Indeed, we can modify S so that it is a product in a neighbourhood of the cusps (i.e. is ruled by geodesic rays going out the cusps). Thus, $S \subseteq \Upsilon$. Applying Lemma 6.2.4, we see that $b(S) = b(e)$.

In summary, we have found an embedded π_1-injective surface, $S \subseteq \Upsilon$, of bounded complexity, with $S \cap R(M)$ of bounded ρ_R-diameter, and with $\rho(\tau, S)$ bounded.

The argument in the case where e is compressible requires the same modification as in the case with no cusps.

6.5. Proof of tameness when there are cusps.

We begin with a fairly general and straightforward observation. Suppose M is a complete manifold with $\pi_1(M)$ finitely generated, and with an associated set of essential $\mathbf{Z}$-cusps. Let e be an end $M \cup \partial^V M$. This end may be subdivided by inessential $\mathbf{Z}$-cusps in e. (That is, several end of M minus the inessential cusps might lie in a single end of M.) But if each of the subdivided ends is topologically finite, then so is e.

To prove that an end of $M \cup \partial^V M$ is topologically finite, we proceed by induction on the complexity, $b(e)$, of e. By the discussion at the beginning of Section 6.4, we can reduce to the case satisfying $(*)$. In this case, one easily checks that either e is incompressible, or else all of the inner ends have strictly lower complexity and hence can be assumed topologically finite. (They may be subdivided by inessential $\mathbf{Z}$-cusps, but that can only lower complexity further.) We start the induction with the case of tori (which are necessarily $\mathbf{Z} \oplus \mathbf{Z}$-cusps) or 3-holed spheres (which are necessarily geometrically finite).

We now obtain a sequence, S_i, of properly embedded surfaces in $M \cup \partial^V M$ going out the end, of bounded complexity, $b(S)$.

We need to check:

Lemma 6.5.1 : The S_i lie in finitely many homotopy classes in $\Upsilon \cap R(M)$ (relative to $\Upsilon \cap \partial R(M)$).

Proof : The argument proceeds much as with Lemma 5.3. We assume that e is compressible. Since $b(S_i)$ is bounded, we can assume that each S_i is the image of an embedding of a fixed surface, S. We can find balanced maps $\psi_i : S \longrightarrow \Upsilon \cup \partial^V \Upsilon$ in the same homotopy classes as the respective embeddings, so that $\psi_i(\beta_i)$ is geodesic in Υ, where β_i is a simple closed curve in S with $\psi_i(\beta_i)$ homotopically trivial in M. Thus, $\psi_i(\beta_i) \cap \partial\Upsilon \neq \emptyset$. If the intersections $\psi_i(\beta_i) \cap \partial\Upsilon$ all lie in a compact set, we are done, by essentially the same argument. However, a-priori $\psi_i(\beta_i)$ might meet $\partial\Upsilon$ only far out the cusps. This is prevented by the observation that, in the induced metric on $\operatorname{int} S_i$, β_i is also geodesic and (homotopic to) a simple closed curve. Since S_i is singular hyperbolic, it is easily verified that there is a bound on how far β_i can go out a cusp. $\diamondsuit$

Thus, after passing to a subsequence, we can assume that the S_i are all homotopic in $\Upsilon \cap R(M)$, and it follows by the relative version of Waldhausen's cobordism theorem, that e is topologically finite.

Finally, using the relative Scott theorem [Mc] it follows that $R(M)$ has only finitely many ends and so $R(M)$ hence M is toplogically finite. In fact, the argument has shown directly that the non-cuspidal part, $\Psi(M)$, is M is topologically finite, in the sense that we can embed $\Psi(M)$ in a compact manifold M', so that the interiors of $\Phi(M)$ and M' coincide, and such that the boundary of $\Psi(M)$ is a subsurface of $\partial M'$.

6.6. Orbifolds.

We finally note that the tameness theorem extends to orbifolds in the following sense. Any complete hyperbolic orbifold, O, with orbifold fundamental group Γ is the quotient of hyperbolic 3-space by a properly discontinuous action of Γ. If Γ is finitely generated, then Selberg's Lemma tells us that it is virtually torsion free, and so O is the quotient of a hyperbolic manifold, M, by the action of a finite group, G. Tameness tells us that M is homeomorphic to the interior of a compact manifold M', and Theorem 8.5 of [MeS] tells us that we can take this homeomorphism to be equivariant with respect to an action of G on M'. It follows that O is topologically finite, in the sense that it is the interior of a compact orbifold (with boundary), M'/G.

7. Variable curvature.

In this section, we describe how the main ideas go through, with some modification, to the case of pinched negative curvature. Our goal is to prove the following (earlier stated as Theorem 0.4).

Theorem 7.1 : *Let M be a complete riemannian 3-manifold of pinched negative curvature with $\pi_1(M)$ finitely generated. Then M is diffeomorphic to the interior of a compact manifold with boundary.*

By "pinched negative curvature" we mean that all sectional curvatures lie between two negative constants. Note that we can write $M = \Xi/\Gamma$ where Ξ is a pinched Hadamard manifold (i.e. simply connected), and $\Gamma \cong \pi_1(M)$. Thus Ξ plays the role of $\mathbf{H}^3$ in what follows.

We remark that we also get tameness of orbifolds, as discussed in Section 6.6. However, in this more general situation, we do not have Selberg's Lemma, so we need to make the assumption that the orbifold fundamental group is virtually torsion free. In retrospect, one can show that a topologically finite negatively curved orbifold is finitely covered by a manifold, using Thurston's orbifold theorem (see [BoiLP] or [CoHK]). Thus, one can view tameness as equivalent to being virtually torsion free. I don't know whether is automatically implied by finite generation. In what follows we will deal only with (orientable) manifolds.

Many aspects of the discussion go through with little change from constant curvature. For example, M has a thick-thin decomposition, where the thin part consists of Margulis tubes and cusps, which (at least in dimension 3) are smooth submanifolds. (The boundaries need not be intrinsically euclidean, but that does not matter here.)

The issue of convex hulls and convex cores is a bit more subtle, since they can no longer be constructed out of simplices. However, the construction of Anderson [An] as elaborated on in [Bow1] gives us what we need here. In particular, we have the following.

If $Q \subseteq \Xi$ is closed, we write $\mathrm{join}(Q)$ for the union of all geodesic segments connecting pairs of points of Q, and $\mathrm{hull}(Q)$ for the closed convex hull of Q (i.e. the closure of the convex hull).

Lemma 7.2 : *There is some $r > 0$, depending only on the pinching constants, such that for any closed $Q \subseteq \Xi$, we have $\mathrm{hull}(Q) \subseteq N(\mathrm{join}(Q), r)$.*

From this, we get immediately the generalisation of Lemma 5.2:

Lemma 7.3 : *Suppose M is a complete riemannian 3-manifold of pinched negative curvature, and $b \in M$. Then any point of $\mathrm{core}(M)$ lies a distance at most r from some geodesic loop based at b, where $r > 0$ depends only on the pinching constants.* $\diamondsuit$

(In fact, this holds in any dimension, if we allow r to depend on dimension as well.) Here $\mathrm{core}(M)$ denotes the closed convex core.

We can give the same definition of geometrically finite end of $M \cup \partial^V M$, namely one which has a neighbourhood that does not meed $\mathrm{core}(M)$. Such ends are also easily seen to be topologically finite (see [Bow1]). Moreover (again in this dimension) cusps are also easily seen to be topologically finite. We are therefore reduced to understanding the case of non-geometrically finite ends.

The main complication arises in adapting the notion of a polyhedral complex, or polyhedral map. For a 1-complex, the same definition makes sense. However, to extend over 2-simplices, we would need to use ruled surfaces, or something similar, instead of totally geodesic ones. It seems intuitively clear that the theory should go through much as before, with these notions. However, there are some complicated technical issues, so we suggest an alternative means which avoids the worst of these. (In this discussion, we only need a negative upper curvature bound.)

We say that a locally finite embedded graph $\Phi \subseteq M$ is *polygonal* if each edge is a geodesic segment. We have the same notion of *balanced* as before: the tangent vectors to Φ at any vertex do not lie in any open hemisphere. Let $\hat{\Pi}(\Phi)$ be the completion of the universal cover, of $M \setminus \Phi$. We write $\mathrm{int}\,\hat{\Pi}(\Phi) \subseteq \hat{\Pi}(\Phi)$ for the universal cover of $M \setminus \Phi$.

Lemma 7.4 : *If $\Phi \subseteq M$ is a balanced polygon, then $\hat{\Pi}(M)$ is (globally) CAT(-1)*

Proof : (Sketch) Suppose $x \in \Phi$. We can find some small $t > 0$ so that $\Phi \cap N(x,t)$ just consists of geodesic segments emerging from x. (We can identify $\partial N(x,t)$ with the unit tangent space, $\Delta_x(M)$ at x via the exponential map.) We claim that $\partial N(x,t) \setminus \Phi \hookrightarrow M \setminus \Phi$ is π_1-injective. For otherwise, by Dehn's lemma, there would be an embedded disc $D \subseteq M \setminus (\Phi \cup \mathrm{int}\, N(x,t))$ with $\partial D = D \cap N(x,t)$ an essential curve in $\partial N(x,t) \setminus \Phi$. By gluing in a small disc, we can extend this to a sphere in M meeting Φ only in the point x. This sphere bounds a ball, $B \subseteq M$. By considering the point of $\Phi \cap B$ furthest from x, after lifting to $\mathbf{H}^3$, we contradict the fact that Φ is balanced.

This means that the universal cover of $N(x,t) \setminus \Phi$ is embedded in $\mathrm{int}\,\hat{\Pi}(\Phi)$. Since this holds true for any $x \in \Phi$, it is not hard to see that $\hat{\Pi}(\Phi)$ is simply connected. This reduces us to showing that $\hat{\Pi}(\Phi)$ is locally CAT(-1). Again from the above discussion this is now a fairly simple exercise using ruled surfaces. $\diamondsuit$

We have the complication that $\hat{\Pi}(\Phi)$ is not usually locally compact. However, since it is a complete, we still have a classification of isometries of $\hat{\Pi}(\Phi)$ into elliptic, parabolic and loxodromic. If Φ is finite, then in $\pi_1(M \setminus \Phi)$ the only elliptics are those that fix an edge of Φ, and the only parabolics are those that already existed in M.

Given a subset $Q \subseteq M \setminus \Phi$, we write $\hat{\Pi}_Q(\Phi)$ for the quotient of $\hat{\Pi}(\Phi)$ corresponding to the image of $\pi_1(Q)$ in $\pi_1(M)$. This is locally CAT(-1). We get a natural 1-lipschitz map from $\hat{\Pi}_Q(M)$ to M. The preimage Φ is a locally finite graph, and we can construct a collar neighbourhood of it in $\hat{\Pi}_Q(M)$, homeomorphic to a surface times an open interval. For this reason, the topological 3-manifold arguments used in Sections 4 and 5 go through much as before.

Suppose now that K is a locally finite simplicial 2-complex, with 1-skeleton K^1. We can define a polyhedral map as a proper map $\phi : K \longrightarrow M$ such that each edge gets mapped to a geodesic. We generally regard the maps on 2-simplices to be defined only up to homotopy relative to their boundaries. (We could always take these to be ruled surfaces, but with our modified constructions, their geometry becomes irrelevant.) We say that ϕ is *balanced* if $\phi|K^1$ is balanced in the obvious sense, reintepreting the definition in constant curvature. Note that $\phi(K^1)$ is a balanced polygonal set in M.

The wrapping construction only requires slight modification. We have balanced maps $\phi : K \longrightarrow M$ and $\phi : K_\alpha \longrightarrow M$. This time, we write $\Phi = \phi(K^1)$ and $\Phi_\alpha = \phi(K^1_\alpha)$. We let $F \subseteq M \setminus \Phi_\alpha$ be a surface of minimal genus (or complexity) separating e from Φ_α. It follows (using the fact that $H_1(\Phi_\alpha) \longrightarrow H_1(M)$ is surjective) that F is incompressible in $M \setminus \Phi_\alpha$. We define $W \supseteq \Phi_\alpha$ with $\partial W = F$ as before. Since ∂W is incompressible, we can arrange that $\phi(K_\alpha) \subseteq W$.

We define $\Upsilon_\alpha = \hat{\Pi}_F(\Phi_\alpha)$. This is locally CAT$(-1)$, and $F \hookrightarrow \Upsilon_\alpha$ is a homotopy equivalence. We now construct Z as before (using our modified definition of Υ_α). Thus, $Z \longrightarrow M \setminus \Phi$ is a covering space. We end up with maps $K \hookrightarrow K_\alpha \longrightarrow P \hookrightarrow X \hookrightarrow Z \longrightarrow M$ which are all homotopy equivalences, and the argument now proceeds with little modification.

8. Ahlfors's finiteness theorem.

In this section, we consider the relationship between tameness and Ahlfors's Finiteness Theorem, and describe a version of the latter applicable to pinched negative curvature (Theorem 8.8). Such connections have also been observed by Agol. Ahlfors's theorem [Ah] is probably the central finiteness result in the classical theory of finitely generated kleinian groups.

A *kleinian group* is a group, Γ, acting properly discontinuously on $\mathbf{H}^3$. Thus $\mathbf{H}^3/\Gamma$ is a hyperbolic orbifold. It is a manifold if and only if Γ is torsion-free. As observed in Section 6, Selberg's Lemma tells us that any finitely generated kleinian group is virtually torsion-free. It follows that we can reduce to the torsion-free case in constant curvature.

While tameness gives a new perspective on Ahlfors's theorem in constant curvature, it does not give an independent argument, since the result has already been used in Theorem 2.8.1. The classical proof uses deformation theory, which also features strongly in the proof of hyperbolisation [O,Ka], so there would seem to be little point in trying to circumvent this. The main interest therefore lies in its adaptability to (pinched) variable curvature, which we discuss at the end of this paper. We explain how tameness gives us back Ahlfors's Finiteness Theorem (Lemma 8.3 and 8.4 together). Reinterpreting for variable curvature, we will get a new result, namely Theorem 8.8. First, we confine our discussion to constant curvature.

Let $M = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold, and let $\Omega = \Omega(\Gamma) \subseteq \partial\mathbf{H}^3$ be the discontinuity domain, and write $F_\infty = F_\infty(\Gamma) = \Omega/\Gamma$. This is a complex surface. We say that such a surface is *analytically finite* if it is a finite union of finite type Riemann surfaces, i.e. compact surfaces with finite subsets removed.

Ahlfors's Finiteness Theorem [Ah] states:

Theorem 8.1 : *If Γ is a finitely generated group, then $F_\infty(\Gamma)$ is analytically finite.*

It turns out that each puncture of F_∞ is associated to a $\mathbf{Z}$-cusp of M. Related to this, is Sullivan's finiteness theorem [Su]:

Theorem 8.2 : *M has finitely many cusps.*

In fact, this can be given a purely topological proof. The fact that M has finitely many $\mathbf{Z} \oplus \mathbf{Z}$ cusps is a direct consequence of Scott's Theorem (Theorem 2.3.1)) — M has finitely many ends. The fact that there are finitely many $\mathbf{Z}$-cusp is a consequence of McCullough's theorem (Theorem 6.2.3) as observed in Section 6.1 (see also [FeM]).

In fact, part of Theorem 8.1 can also be viewed in this way. Note that by Theorem 6.1.3 [Mc], any compact subsurface, $F \subseteq F_\infty$ can be assumed to lie in the boundary of a Scott core, P, of $M \cup F_\infty$. The complexity of ∂P is bounded in terms of the Euler characteristic of P, and hence in terms of $\pi_1(M) \cong \Gamma$. This tells us that the union of those components of F_∞ that are not topological discs or annuli is topologically finite. It does not rule out the possibility of there being infinitely many discs or annuli. (Though in the latter case, they would certainly have to lie in finitely many homotopy classes in M.) Moreover, this particular argument says nothing about the analytic type of F_∞.

One can give a more geometric interpretation of Ahlfors's Finiteness Theorem. Let F_t be the boundary of the t-neighbourhood of the convex core of M, for $t \geq 0$. For $t > 0$, this is a properly embedded C^1-surface in M.

Lemma 8.3 : *Let $M = \mathbf{H}^3/\Gamma$ be a complete hyperbolic 3-manifold. The following are equivalent:*

(1) F_∞ is analytically finite,

(2) for some $t > 0$, F_t is analytically finite,

(3) for all $t > 0$, F_t is analytically finite,

(4) for some $t > 0$, F_t has finite area,

(5) for all $t > 0$, F_t has finite area,

(6) $M \cup \partial^V M$ has finitely many geometrically finite ends, and each geometrically infinite end has a neighbourhood whose intersection with M is contained in the convex core of M.

In (6), $\partial^V M$ is the union of the ideal boundaries of the $\mathbf{Z}$-cusps as defined in Section 6. (All $\mathbf{Z}$-cusps are deemed essential here.) Recall that $M \cup \partial^V M$ is homeomorphic to $R(M)$, that is M with its $\mathbf{Z}$-cusps removed.

Proof : The equivalence of (1)–(5) is well known, and arises from the fact that for all $t > 0$, the "nearest point" projection $F_\infty \longrightarrow F_t$ is a quasiconformal homeomorphism, and if $0 < t < u < \infty$, the projection $F_u \longrightarrow F_t$ is bilipschitz. It is easy to see that the ends of any F_t are horocyclic constant curvature cusps — and so do not pose any difficulties.

To relate this to (6), we note that the closure of F_t in $M \cup \partial^V M$ can be naturally identified with the compactification $F_t \cup \partial^V F_t$ of F_t. This cuts off a closed neighbourhood, E_t, of a geometrically finite end in $M \cup \partial^V M$. We can naturally compactify E_t to a product manifold, $(F_t \cup \partial^V F_t) \times [0, \infty]$, where we identify $\operatorname{int} F_t \times \{\infty\}$ with F_∞. Moreover, any geometrically finite end has this form. Condition (6) tells us that a finite set of such ends account for all of F_∞, and the equivalence of (6) with (5) follows easily. ◊

In Lemma 8.3 we did not need that $\pi_1(M)$ be finitely generated. In relation to Ahlfors's Finiteness Theorem, we need:

Lemma 8.4 : *Suppose that $M = \mathbf{H}^3/\Gamma$ is a complete hyperbolic 3-manifold with $\pi_1(M)$ finitely generated. Let e be a degenerate end of M. Then there is a neighbourhood, E, of e in $M \cup \partial^V M$ with $E \cap M \subseteq \operatorname{core}(M)$.*

This depends on the following observations:

Lemma 8.5 : *Let $\Psi \subseteq \mathbf{H}^3$ be a locally finite balanced polyhedron and let $L \subseteq \partial \mathbf{H}^3$ be the set of accumulation points of Ψ in $\partial \mathbf{H}^3$. Then $\Psi \subseteq \operatorname{hull}(L)$.*

Proof : If not, there is a closed half-space $H \subseteq \mathbf{H}^3$ with $H \cap \operatorname{hull}(L) = \emptyset$ and with $\Psi \cap \operatorname{int} H \neq \emptyset$. It follows that $H \cap \Psi$ is compact. If $x \in \Psi \cap \operatorname{int} H$, with $d(x, \partial H)$ maximal, then we see easily that Ψ cannot be balanced at x. ◊

Corollary 8.6 : *Suppose that $\Phi \subseteq M$ is a finite balanced polyhedron in M (allowing ideal vertices in $\partial^V M$). Then $\Phi \subseteq \operatorname{core}(M)$.*

Proof : Lift to $\mathbf{H}^3$ and apply Lemma 8.5. ◊

Lemma 8.7 : *Suppose that $Q \cup \partial^V Q \subseteq M \cup \partial^V M$. If $\partial Q \subseteq \partial^V M \cup \mathrm{core}(M)$, then $Q \subseteq \mathrm{core}(M)$.*

(Note that $\partial Q \setminus \partial^V Q$ is the relative boundary of Q in M.)

Proof : The closure of any component of $M \setminus \mathrm{core}(M)$ in $M \cup \partial^V M$ is non-compact and it is easily seen that Q cannot contain such a component. ♢

Proof of Lemma 8.4 : By Section 6.4, we have a sequence of separating surfaces, $S_i \subseteq M \cup \partial^V M$ going out the end, e. We can assume them to be products inside a (fixed) set of **Z**-cusps (i.e. geometric cones of the cusp points). Now S_i lies in small neighbourhood of $f_i(\Sigma)$, which in turn lies close to $h_i(\Sigma)$, where $f_i, h_i : \Sigma \longrightarrow \Upsilon \subseteq M$ are proper maps and h_i is balanced in Υ. Now $h_i(\Sigma)$ goes out the end, e (and is a product in a neighbourhood of the cusp), and so $h_i(\Sigma) \cap \partial\Upsilon = \emptyset$, for all sufficiently large i. Then $h_i(\Sigma)$ is balanced in M. By Corollary 8.6, we see that $h_i(\Sigma) \subseteq \mathrm{core}(M)$. We can therefore assume that $f_i(\Sigma)$ and hence $S_i \cap M$ also lie in core(M).

Now each point of Υ must be separated from e by the surface S_i. Thus $E = \bigcup_i E_i$ is neighbourhood of e in $M \cup \partial^V M$, where E_i is the region between S_0 and S_i. Now $\partial E_i \subseteq \partial^V M \cup \mathrm{core}(M)$ and so by Lemma 8.7, $E_i \subseteq \mathrm{core}(M)$. Thus $E \subseteq \mathrm{core}(M)$ as required. ♢

Now it is easy to see that Lemmas 8.3 and 8.4 together imply Ahlfors's Finiteness Theorem (Theorem 8.1). Of course we have used this theorem (applied to a different group) in order to prove this. The point of the exercise is that the argument is adaptable to variable curvature.

Let M be a complete manifold of pinched negative curvature, and write $F_t = N(\mathrm{core}(M), t)$ as before. This is C^1-surface, and the projection $F_u \longrightarrow F_t$ for $u > t$ is C^1-lipschitz and distance decreasing. In particular, area(F_t) (if finite) is a non-decreasing function of t.

We can now prove the variable curvature version of the Ahlfors Finiteness Theorem (in the torsion-free case) stated as Theorem 0.5 in the introduction:

Theorem 8.8 : *Let M be a complete riemannian 3-manifold of pinched negative curvature, and with $\pi_1(M)$ finitely generated. Let $t > 0$, and let F_t be the boundary of the t-neighbourhood of the convex core of M. Then F_t has finite area.*

It follows (or can be seen from the argument) that the quotient, F_∞, of the discontinuity domain is a topologically finite surface, though it is not clear how to formulate analytical finiteness directly in these terms.

Proof of Theorem 8.8 : Most of the proof of Theorem 8.1 generalises unchanged. (In discussion the balanced maps $f_i, h_i : \Sigma \longrightarrow \Upsilon$, we can extend over 2-simplices using ruled surfaces. In particular, any degenerate end has a neighbourhood in core(M). Moreover, McCullough's theorem tells us that $M \cup \partial^V M$ has finitely many ends. What remains is to explain why this implies that area(F_t) is finite.

Now in [Bow1] it was shown that if M is geometrically finite then the volume of $N(\mathrm{core}(M), t)$ is finite for all t. In particular, for any $u > t$, $\mathrm{vol}(C(t,u))$ is finite, where $C(t,u) = N(\mathrm{core}(M), u) \setminus \mathrm{int}\, N(\mathrm{core}(M), t)$. In our situation, M need not be geometrically finite. Nevertheless, $C(t,u)$ lies in the union of the geometrically finite ends. Thus the same argument as given in [Bow1] shows in fact that $\mathrm{vol}(C(t,u))$ is finite. But $\mathrm{vol}(C(t,u)) = \int_t^u \mathrm{area}\, F_v \, dv \geq (u-t)\, \mathrm{area}\, F_t$, and so $\mathrm{area}\, F_t < \infty$ as required. $\diamondsuit$

References.

[Ag] I.Agol, *Tameness and hyperbolic 3-manifolds* : preprint, Chicago (2004).

[Ah] L.V.Ahlfors, *Finitely generated Kleinian groups* : Amer. J. Math. **86** (1964) 413–429, **87** (1965) 759.

[An] M.Anderson, *The Dirichlet problem at infinity for manifolds of negative curvature* : J. Differential Geom. **18** (1983) 701–721.

[BoiLP] M.Boileau, B.Leeb, J.Porti, *Geometrization of 3-dimensional orbifolds* : Ann. of Math. **162** (2005) 195–290.

[Bon1] F.Bonahon, *Structures géométriques sur les variétés de dimension 3 et applications* : Thèse, Orsay (1985).

[Bon2] F.Bonahon, *Bouts des variétés hyperboliques de dimension 3* : Ann. of Math. **124** (1986) 71–158.

[Bow1] B.H.Bowditch, *Geometrical finiteness with variable negative curvature* : Duke Math. J. **77** (1995) 229–274.

[Bow2] B.H.Bowditch, *Geometric models for hyperbolic 3-manifolds* : preprint, Southampton (2005).

[Bow3] B.H.Bowditch, *End invariants of hyperbolic 3-manfolds* : preprint, Southampton (2005).

[BridH] M.R.Bridson, A.Haefliger, *Metric spaces of non-positive curvature* : Grundlehren der Math. Wiss. No. 319, Springer (1999).

[BrinJS] M.Brin, K.Johannson, P.Scott, *Totally peripheral 3-manifolds* : Pacific J. Math. **118** (1985) 37–51.

[BrinT] M.G.Brin, T.L.Thickstun, *3-manifolds which are end 1-movable* : Memoirs Amer. Math. Soc. No. 81 (1989).

[BrocB] J.F.Brock, K.W.Bromberg, *On the density of geometrically finite Kleinian groups* : Acta Math. **192** (2004) 33–93.

[BrocCM] J.F.Brock, R.D.Canary, Y.N.Minsky, *Classification of Kleinian surface groups II: The ending lamination conjecture* : preprint, Brown/Michigan/Yale (2004).

[Brom] K.Bromberg, *Projective structures with degenerate holonomy and the Bers density conjecture* : Ann. of Math. **166** (2007) 77-93.

[CalG] D.Calegari, D.Gabai, *Shrinkwrapping and the taming of hyperbolic 3-manifolds* : J. Amer. Math. Soc. **19** (2006) 385–446.

[Can1] R.D.Canary, *Ends of hyperbolic 3-manifolds* : J. Amer. Math. Soc. **6** (1993) 1–35.

[Can2] R.D.Canary, *A covering theorem for hyperbolic 3-manifolds and its applications* : Topology **35** (1996) 751–778.

[CanM] R.D.Canary, Y.N.Minsky, *On limits of tame hyperbolic 3-manifolds* : J. Differential Geom. **43** (1996) 1–41.

[Ch] S.Choi, *The PL-methods for hyperbolic 3-manifolds to prove tameness* : preprint, Daejeon (2006).

[CoHK] D.Cooper, C.D.Hodgson, S.Kerckhoff, *Three-dimensional orbifolds and cone manifolds* : MSJ Mem. No. 5 (2000).

[FeM] M.Feighn, D.McCullough, *Finiteness conditions for 3-manifolds with boundary* : Amer. J. Math. **109** (1987) 1155–1169, and **112** (1990) 41–45.

[FrHS] M.Freedman, J.Hass, P.Scott, *Least area incompressible surfaces in 3-manifolds* : Invent. Math. **71** (1983) 609–642.

[G] D.Gabai, *Foliations and the topology of 3-manifolds* : J. Differential Geom. **18** (1983) 445-503.

[H] J.Hempel, *3-manifolds* : Ann. of Math. Studies, No. 86, Princeton University Press (1976).

[JR] W.Jaco, J.H.Rubinstein, *PL minimal surfaces in 3-manifolds* : J. Differential Geom. **27** (1998) 493–524.

[Ka] M.Kapovich, *Hyperbolic manifolds and discrete groups* : Progress in Mathematics No. 183, Birkhäuser (2001).

[KlS] G.Kleineidam, J.Souto, *Algebraic convergence of function groups* : Comment Math. Helv. **77** (2002) 244–269.

[Ma] A.Marden, *The geometry of finitely generated kleinian groups* : Ann. of Math. **99** (1974) 383–462.

[Mc] D.McCullough, *Compact submanifolds of 3-manifolds with boundary* : Quart. J. Math. **37** (1986) 299-307.

[MeS] W.H.Meeks III, P.Scott, *Finite group actions on 3-manifolds* : Invent. Math. **86** (1986) 287–346.

[Mi] Y.N.Minsky, *The classification of Kleinian surface groups I : Models and bounds* : preprint, Stony Brook (2002).

[My] R.Myers, *End reductions, fundamental groups, and covering spaces of irreducible open 3-manifolds* : Geom. Topol. **9** (2005) 971–990.

[NS] H.Namazi, J.Souto, *Non-realizability and ending laminations* : in preparation.

[O] J.-P.Otal, *Thurston's hyperbolization of Haken manifolds* : Surveys in differential geometry, Vol. III, 77–194, International Press (1998).

[P] L.Person, *A piecewise linear proof that the singular norm is the Thurston norm* : Topology Appl. **51** (1993) 269–289.

[Sc1] P.Scott, *Finitely generated 3-manifold groups are finitely presented* : J. London Math. Soc. **6** (1973) 437-440.

[Sc2] P.Scott, *Compact submanifolds of 3-manifolds* : J. London Math. Soc. **7** (1973) 246-250.

[Som] T.Soma, *Existence of ruled wrapping in hyperbolic 3-manifolds* : Geom. Topol. **10** (2006) 1173–1184.

[Sou] J.Souto, *A note on the tameness of hyperbolic 3-manifolds* : Topology **44** (2005) 459–474.

[Su] D.Sullivan, *A finiteness theorem for cusps* : Acta Math. **147** (1981) 289–299.

[Th1] W.P.Thurston, *The geometry and topology of 3-manifolds* : notes, Princeton (1979).

[Th2] W.P.Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry* : Bull. Amer. Math. Soc. **9** (1982) 357–381.

[Tu] T.W.Tucker, *Non-compact 3-manifolds and the missing-boundary problem* : Topology **13** (1974) 267–274.

[Wa] F.Waldhausen, *On irreducible 3-manifolds which are sufficiently large* : Ann. of Math. **87** (1968) 56–88.

[Wh] J.H.C.Whitehead, *A certain open manifold whose group is unity* : Quart. J. Math. **6** (1935) 268-279.