# Treelike structures arising from continua and convergence groups

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**Abstract.**

In this paper, we develop the theory of a very general class of treelike structures based on a simple set of betweenness axioms. Within this framework, we explore connections between more familiar treelike objects, such as **R**-trees and dendrons. Our principal motive is provide tools for studying convergence actions on continua, and in particular, to investigate how connectedness properties of such continua are reflected in algebraic properties of the groups in question. The main applications we have in mind are to boundaries of hyperbolic and relatively hyperbolic groups and to limit sets of kleinian groups. One of the main results of the present paper constructs dendritic quotients of continua admitting certain kinds of convergence actions, giving us a basis for introducing the techniques of **R**-trees into studying such actions. This is a step in showing that the boundary of a one-ended hyperbolic group is locally connected. There are further applications to constructing canonical splitting of such groups, and to limit sets of geometrically finite kleinian groups, which are explored elsewhere. We proceed in a general manner, discussing other connections with Λ-trees, protrees, pseudotrees etc. along the way.

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## 0. Introduction

The term "treelike structure" of the title is intended to encompass the many different kinds of trees that have appeared in the literature, for example, simplicial trees, Λ-trees (in particular **R**-trees), dendrons, protrees, pseudotrees, tree algebras etc. One of the principal objects of study in this paper will be a very general kind of treelike structure which we shall call a "pretree". The axioms of a pretree appear in a paper by Ward [W2], and were rediscovered in a different context by Adeleke and Neumann [AN], who use the term "B-set" to refer to the same structure. Here, we aim to develop the theory in a somewhat different direction.

One reason for interest in pretrees is that most other treelike structures, including those referred to above, can be viewed as special cases of pretrees. A surprising amount can be done using the pretree axioms alone, thus unifying many arguments found distributed through the literature. Another reason, which will form the principal theme of this paper, will be its application to the topology of continua.

One of the main technical results arising out of this work is a condition on convergence

groups acting on continuum which implies that every global cut point is a parabolic fixed point [Bo6]. The principal motivation for this study was the conjecture of Bestvina and Mess that the boundary of a one-ended hyperbolic is locally connected. This was known to be equivalent to the non-existence of a global cut point [BeM,Bo2]. This was proven for "strongly rigid" groups in [Bo3] (see also [L]) and generalised to "strongly accessible" groups in [Bo4]. Swarup showed how to adapt these arguments to deal with the general case [Swa]. The result of [Bo6] places this in a more general dynamical setting, which also has applications to boundaries of relatively hyperbolic groups, and hence in particular to limit sets of geometrically finite kleinian groups. We discuss this further at the end of Section 6.

Recall that a "continuum" is a connected compact hausdorff topological space. By a "dendron" we shall mean a locally connected continuum in which every pair of points can be connected by a unique arc, where the term "arc" is interpreted here to mean a subset homeomorphic to a closed real interval. (This is a slightly more restrictive definition than that often used elsewhere.) Most of the continua in which we are primarily interested are separable (i.e. contain a countable dense subset). We shall use the term "dendrite" to mean a separable dendron. An equivalent definition of a dendrite describes it as a separable continuum in which every pair of distinct points are separated by a third point. (It is this latter criterion which is often used to define a dendron elsewhere.) The equivalence of these definitions for separable continua is well-known. In fact, it will follow from some of the results in this paper. We shall say that a dendrite (or dendron) is "non-trivial" if it is not just a single point.

One principal construction in this paper associates to any separable continuum a natural quotient which is a dendrite. This proceeds by analysing the cut-point structure of such a continuum using pretrees. A condition is given under which the quotient dendrite is guaranteed to be non-trivial. A particular application is to the boundary of a one-ended hyperbolic group containing a cut point (and is thus a step toward proving its non-existence). In this case, we show that the quotient dendrite is always non-trivial. In this way, we arrive at a convergence action of the group on the dendrite. In [Bo3], it is shown that such a group must split over a virtually cyclic subgroup. Another proof is given in [L]. The proof of this result involves constructing a non-trivial isometric action on an **R**-tree, and applying the result of Bestvina and Feighn [BeF2]. There is a well established theory of group actions on **R**-trees (see [Sha1,Sha2,Mor,Pa2]), which generalises the classical Bass-Serre theory on simplicial trees [Ser]. This leads us naturally to consider connections with convergence actions on dendrites (or more generally dendrons).

The main tool used will be that of a pretree, and we develop some of the general theory of these structures here. They have also been studied in some detail by Adeleke and Neumann [AN], though their motivation was somewhat different from ours, and so we shall be aiming in a rather different direction. Our reason for introducing them (as in the case of Ward's paper [W2]) was to capture the cut point structure of a continuum (or more general topological spaces). It is well known (see for example [HocY]) that set of cut points which separate a given pair of points in a continuum carries a natural linear order. If we want to consider the set of all cut points simultaneously, then the pretree axioms give a natural generalisation. A pretree thus consists of a set with a ternary "betweenness"

relation arising from the way cut points separate the space.

A pretree is a natural analogue of a protree as defined by Dunwoody [Du2,Du3] where instead of trying to capture the combinatorial structure of the edge set, we work with the vertex set. Thus, in a pretree, every finite set of points "looks like" a subset of some finite simplicial tree, or more formally, the betweenness relation on this finite set is compatible with that derived from an embedding of this set set in some finite tree. (The same is true of a protree where vertices are replaced by directed edges, and the betweenness relation is replaced by a binary relation describing the compatibility of orientations.) Indeed the above condition could serve, in principle, as the definition of a pretree, though, in fact we can get away with four simple explicit axioms (see Section 2).

As we stated at the beginning, most of the other kinds of treelike structure have natural pretree structures associated with them, and so can be viewed as special kinds of pretrees. We have already talked about protrees. Their connection pretrees is discussed in some detail at the end of Section 2. There is a also a fairly extensive literature about pseudotrees (see for example [Ni]), and pretrees seem a natural generalisation of this idea, as we outline in Section 3.

A another related concept to a pretree is that of a median algebra. Median algebras (under a variety of names) have been studied for some time — see [BaH] for a survey. A median algebra consists of a set together with a function which associates to any three points $x$, $y$ and $z$, a fourth point $\mathrm{med}(x, y, z)$, called the "median" of $x$, $y$ and $z$. The function med should satisfy certain axioms (see Section 2). An example of a median algebra would be a dendrite, where $\mathrm{med}(x, y, z)$ could be defined as the unique point of intersection of the arcs $[x, y]$, $[y, z]$ and $[z, x]$. More generally, in a pretree, a median of $x, y, z$ could be defined as a point lying between any pair of points from $\{x, y, z\}$. Such a point must always be a unique, but is not in general assumed to exist. A pretree in which every triple of points has a median will be termed a "median pretree". A median pretree is indeed a median algebra, though only very special median algebras arise in this way. For this reason, much of the general theory of median algebras will not be directly relevant to us here.

The notion of a median pretree is not a new one. They appear in many places in the literature under a great variety of names. (In [BaH] they are referred to as "tree algebras".) One of the first accounts seems to be that of Sholander [Sho]. Every $\Lambda$-tree, as defined by Morgan and Shalen [MorS], is an example of a median pretree.

There is a natural completion process which embeds every pretree in a median pretree (see Section 3). Another (different but related) way of constructing such an embedding is described in [AN]. We use the term "completion" for our construction since the resulting pretree also has a certain "completeness" property, namely that every arc (full linearly ordered subset) is an interval. We remark that Chiswell [Ch1] has shown that any median pretree (and hence any pretree) can be embedded in $\Lambda$-tree for some ordered abelian group $\Lambda$. We shall discuss connections between different kinds of trees more fully in at the end of Section 2.

Returning to the subject of continua, we mentioned earlier a process of associating to each separable continuum a natural quotient. For our purposes, it will be convenient to work mainly with pretrees rather than topological spaces. Viewing the construction

in these terms, we consider the cut point set of our continuum as a pretree, take the completion of this pretree, and then take an appropriate quotient of this completion. This quotient is also a topological quotient of the original continuum, and turns out to be a dendrite. We refer to it as the "dendritic quotient". There is a much more direct way of describing this quotient in purely topological terms, as we explain in a moment. First, we describe some applications.

Suppose $M$ is a separable continuum, and suppose $\Gamma$ is a group acting by homeomorphisms on $M$. Suppose $\Gamma$ acts minimally as a (discrete) convergence group without parabolics (in the sense of [GeM1]). Precise definitions in this context are given in Section 6. (See also [Tu1] or [Bo5].) Suppose that $M$ contains a cut point, and that $\Gamma$ does not split as an amalgamated free product or HNN extension over a finite subgroup, and does not contain an infinite torsion subgroup. Then, the dendritic quotient of $M$ is non-trivial. This is Theorem 6.1. In fact, we can allow parabolics in the hypotheses provided we assume that no cut point is also a parabolic fixed point.

Of particular interest is the case where $M = \partial \Gamma$ is the boundary of a (word) hyperbolic group $\Gamma$, as defined by Gromov [Gr]. The boundary is always a compact metrisable space, on which $\Gamma$ acts as a convergence group (see [Tu1,F,Bo5]). It is not hard to see that $\partial \Gamma$ is connected if and only if $\Gamma$ has only one end (see, for example, [GhH]). By Stallings's theorem [St], this is in turn equivalent to saying that $\Gamma$ is not finite or virtually cyclic, and does not split over a finite subgroup. In such a case, it was conjectured that the boundary must be locally connected. Bestvina and Mess [BeM] showed that if $\partial \Gamma$ is connected but not locally connected, then it must contain a global cut point. A converse was given in [Bo2].

Now any subgroup of a hyperbolic group is finite, virtually cyclic, or else contains a free subgroup on two generators (see [GhH]). In particular a hyperbolic group cannot contain an infinite torsion subgroup. Thus, from Theorem 6.1, we deduce that:

**Theorem 0.1 :** *If the boundary of a one-ended hyperbolic group contains global cut point, then it has an equivariant quotient which is a non-trivial dendrite.*

Now, the group $\Gamma$ also acts as a discrete convergence group on this dendrite. Thus, using the result from [Bo3] or [L] mentioned earlier, we obtain:

**Corollary 0.2 :** *If the boundary of a one-ended hyperbolic group contains global cut point, then it splits over a virtually cyclic subgroup.*

A somewhat different route through some of this material, also based on the notion of pretrees, has recently been suggested by Swenson [Swe].

We say that a one-ended hyperbolic group is "strongly rigid" if it does not split over any virtually cyclic subgroup. Thus a strongly rigid group is "rigid" in the sense that its outer automorphism group is finite [Pa1]. Thus, one consequence of our result is that a strongly rigid group has locally connected boundary. (In fact, one can build on this to show that every strongly accessible one-ended hyperbolic group has this property [Bo4]. This raises the question of whether every hyperbolic group, or indeed every finitely presented

group, is strongly accessible.)

In the case where the boundary is locally connected, one can obtain the "JSJ splitting" of Sela [Sel], by analysing the structure of local cut points [Bo2]. A "local cut point" can be defined as a point whose complement has more than one end. The JSJ splitting, in some sense, describes all possible splittings of the group over virtually cyclic subgroups. In particular, provided our group is not fuchsian (i.e. a virtual surface group), we see that the existence of a local cut point implies that the group is not strongly rigid. The converse is a fairly simple exercise. Putting this together with Corollary 0.2, we deduce that a one-ended non-fuchsian hyperbolic group is strongly rigid if and only if its boundary does not contain a (local or global) cut point. In particular, modulo fuchsian groups, this property is quasiisometry invariant.

As mentioned earlier, one can push this further using an idea of Swarup [Swa] to prove the cut point conjecture (or equivalently, the local connectedness conjecture) for all one-ended hyperbolic groups. This result can be placed in a more general dynamical setting [Bo6] (see Section 6).

The local connectedness of the boundary (or more specifically the non-existence of a global cut point) has further consequences. The Bestvina-Mess paper [BeM] shows, in fact, that if a one-ended hyperbolic group does not contain a global cut point in its boundary, then it must satisfy a kind of coarse uniform local connectedness property for spheres in its Cayley graph (a property referred to as "$\ddagger_M$" in the original paper). This in turn implies that the boundary is locally connected, and also that the group is semistable at infinity (see [Mih]). Now Dunwoody's accessibility theorem [Du1] tells us that any finitely presented group (in particular any hyperbolic group) can be decomposed into a finite number of finitely presented one-ended groups, joined together by finite subgroups (in the sense of a graph of groups — i.e. a finite sequence of amalgamated free products and HNN-extensions). It thus follows from the local connectedness theorem, that all hyperbolic groups are semistable at infinity. (In fact, it has been conjectured that all finitely presented groups are semistable at infinity.)

The result about splittings of convergence groups referred to ([Bo3,L]) relies on examining the connection between **R**-trees and dendrons. One direction of this correspondence is fairly well understood. Starting with an **R**-tree (or a more general topological tree) there is a natural "compactification" process which embeds it in a dendron. One can describe when an isometric action on an **R**-tree gives rise to a convergence action in this way. Going in the opposite direction is more subtle, and many questions remain open. However, one can construct an isometric action on a certain tree with a "monotone metric" which is the starting point of the result of [Bo3]. This will be discussed further in Section 7.

Returning to our general continuum, $M$, we give an explicit description of our dendritic quotient. We have already observed that, if $x, y \in M$, the set of cut points separating $x$ from $y$ forms a linearly ordered set. (See for example [HocY]. This also follows from the more general fact that the set of cut points is a pretree.) We define a relation $\sim$ on $M$ by saying that $x \not\sim y$ if this linear order does not contain any subset which is order-isomorphic to the rational numbers (i.e. a countable subset which is intrinsically dense). We see (using the pretree structure of the cut-point set) that $\sim$ is an equivalence relation. Moreover the equivalence classes are closed. Indeed, it gives an upper-semicontinuous decomposition of

$M$, so the quotient $M/\sim$ is hausdorff. More specifically, we see almost directly from the definition, that any pair of distinct points of the quotient will be separated by a third point. Thus, if $M$ is separable, then $M/\sim$ is a dendrite, by the second (apparently weaker) definition.

It's not hard to see that any subcontinuum (i.e. closed connected subset) of a dendrite is itself a dendrite (cf. Lemma 1.4). Moreover, the preimage in $M$ of any subcontinuum of $M/\sim$ will be a subcontinuum of $M$ (Proposition 5.22).

There are several ways one can see that a dendrite $D$ by the first definition (in terms of separation properties) satisfies the hypotheses of the first definition (in terms of arcs). For example, first show that $D$ is locally connected, then use the fact that a locally connected separable continuum is arc connected, and finally observe that $D$ cannot contain any topologically embedded circles. We shall not write out the details of this argument here. The result is a standard one in continuum theory. It will also follow, by a somewhat contorted route, directly from the results of this paper. For example, we shall see directly by a different argument, that for any separable continuum $M$, the quotient $M/\sim$ is a dendrite by the first definition. If $M$ already happens to be a dendrite by the second definition, then clearly the relation $\sim$ is just equality, so $M$ can be identified with its quotient.

The route taken in this paper is not intended to be the shortest to any particular objective. Instead we explore at some length various operations on treelike structures of the types we have outlined.

It seems that the idea of using treelike structures of some sort in relation to the cut point problem arose independently from a number of sources at various times — both before and after I began work on the present paper. In this connection, one should mention the efforts of Phil Bowers, Bill Grosso, Michah Sageev, Eric Swenson, and no doubt others. Although some progress was made in a number of directions, as far as I am aware, no published work emerged from these other attempts.

The structure of this paper in outline is as follows. In Section 1, we review some facts about dendrons and more general topological trees. In Section 2, we give the definitions of pretrees and median pretrees, and get the subject off the ground. We look at various additional hypotheses one can impose on pretrees, and give a characterisation of dendrites in these terms. We also look at connections with other types of treelike structures. In Section 3, we define what we call "flows" on a pretree. We define the completion of a pretree as the in terms of flows, and investigate some of its properties. In Section 4, we look at some natural quotients of a median pretree, in particular we construct "dense" median pretrees. In Section 5, we get involved in some general topology, and put together the various ingredients to describe our dendritic quotient. In Section 6, we consider convergence groups, and give a condition for the quotient to be non-trivial. In Section 7, we examine connections between isometric group actions on $\mathbf{R}$-trees and convergence actions on dendrons.

I would like to thank Martin Dunwoody, Damien Gaboriau and Frédéric Paulin for helpful conversations, particularly in relation to $\mathbf{R}$-trees. Shortly after starting work on this paper, I received a copy of the preprint of Samson Adeleke and Peter Neumann, which I found very informative, and has influenced this paper in many places. This paper was

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revised at the Universities of Zürich and Melbourne. I'm grateful to both these institutions for their hospitality, and in particular, to Viktor Schroeder and to Craig Hodgson and Walter Neumann for their respective invitations.

## 1. Trees

In this section we summarise some of the basic facts about treelike topological spaces. A much more detailed account will be given in Section 7, after we have developed the theory of pretrees. For the moment, we are principally interested in defining the terms "real tree" and "dendrite", and describing their basic properties. We begin fairly generally, and add additional hypotheses as we need them.

**Definition :** A *uniquely arc-connected* space, $T$, is a hausdorff topological space in which every pair of distinct points are joined by a unique arc.

(Here, the term "arc" should be interpreted as a subset homeomorphic to a closed real interval.) Equivalently, it can be defined as a path-connected hausdorff space which contains no topologically embedded circle.

We write $[x, y]$ for the arc joining $x$ to $y$, with the obvious convention that $[x, x] = \{x\}$. We write $[x, y) = (y, x]$ for $[x, y] \setminus \{y\}$ and $(x, y) = [x, y] \setminus \{x, y\}$. Note that if $z \in [x, y]$, then $[x, y] = [x, z] \cup [z, y]$ and $[x, z] \cap [z, x] = \{z\}$. Conversely if $[x, z] \cap [z, y] = \{z\}$, then $z \in [x, y]$. Note that every closed arc in $T$ is an interval, i.e. has the form $[x, y]$ for some $x, y \in T$.

Suppose $x, y, z \in T$. Since $T$ has no embedded circle, we see that $[x, y] \cap [x, z]$ is connected, and thus has the form $[x, m]$ for some $m \in T$. Now $[y, m] \cap [m, z] = \{m\}$ so $m \in [y, z]$. Thus $m$ lies in the intersection $[x, y] \cap [y, z] \cap [z, x]$. Indeed $m$ will be unique with this property. We see:

**Lemma 1.1 :** *Given $x, y, z \in T$ then $[x, y] \cap [y, z] \cap [z, x]$ consists of a single point $m \in T$.*
$\diamondsuit$

We refer to $m$ as the *median* of $x$, $y$ and $z$, and write $m = \mathrm{med}(x, y, z)$.

Note that the above argument shows that the the union of these three arcs is a "tripod" consisting of the three "legs" $[x, m]$, $[y, m]$ and $[z, m]$ which are joined at the point $m$. (It is possible that one or more of these three legs may consist of just a single point.)

More generally, given a finite subset $A \subseteq T$, write $\mathrm{hull}(A) = \bigcup\{[x, y] \mid x, y \in A\}$. An induction argument shows:

**Lemma 1.2 :** *If $A \subseteq T$ is finite, then $\mathrm{hull}(A)$ is a finite tree.*
$\diamondsuit$

Here, and in the rest of this paper, by a "finite tree" we really mean the topological realisation of a finite simplicial tree.

If $x, y, z \in T$ we shall say that $z$ lies *between* $x$ and $y$ if $z \in (x, y)$. We shall denote this by writing $xzy$. This defines a ternary "betweenness" relation on $T$. Note that we are dealing with "strict" betweenness — $xzy$ implies that $x$, $y$ and $z$ are all distinct. (This is in contrast to some other formulations elsewhere.) Using Lemma 1.2, its easily seen that this relation satisfies the axioms of a median pretree as laid out in Section 2.

We are not really interested in uniquely arc-connected spaces in this generality. For a start, we shall want to impose some local connectedness hypothesis. We shall call the resulting structure a "real tree". A convenient way of formulating this is as follows:

**Definition :** A *real tree* is a uniquely arc-connected space $T$ such that for every $x \in T$ and every neighbourhood $V$ of $x$, there is a neighbourhood $U$ of $x$, such that if $y \in U$ then $[x, y] \subseteq V$.

**Lemma 1.3 :** *If $T$ is a real tree then the map $[(x, y, z) \mapsto \mathrm{med}(x, y, z)] : T^3 \longrightarrow T$ is continuous.*

**Proof :** This follows easily from Lemma 1.2. Suppose the points $x', y', z'$ are close respectively to $x, y, z \in T$. Consider the tree hull$\{x, y, z, x', y', z'\}$. By local connectedness, the arcs $[x, x']$, $[y, y']$ and $[z, z']$ lie respectively in small neighbourhoods of $x$, $y$ and $z$. By considering the combinatorial possibilities for the finite tree, we see easily that the median cannot move very much. $\diamondsuit$

If we fix two points $a, b \in T$, we may define the projection map $\mathrm{proj} = \mathrm{proj}_{[a,b]} : T \longrightarrow [a, b]$ by $\mathrm{proj}(x) = \mathrm{med}(a, b, x)$. By Lemma 1.3, this map is continuous.

For subsets of real trees, connectness coincides with the obvious notion of convexity:

**Lemma 1.4 :** *A subset $Q$ of a real tree $T$ is connected if and only if $[x, y] \subseteq Q$ for all $x, y \in Q$.*

**Proof :** The "if" bit is obvious, so suppose $Q$ is connected. Suppose, for contradiction, $a, b \in Q$ and $c \in [a, b] \setminus Q$. Write $\pi = \mathrm{proj}_{[a,b]}$. Now $\pi^{-1}(c)$ cannot be open (otherwise we could write $T$ as a disjoint union of the two open sets $\pi^{-1}[a, c]$ and $\pi^{-1}[b, c)$). Thus there is some point $x \in \pi^{-1}(c)$ at which $\pi$ is not locally constant. By local connectedness, there is a neighbourhood $U$ of $x$ such that $y \in U$, then $[x, y] \cap [a, b] = \emptyset$. But we can choose $y \in U$ so that $\pi(y) \neq \pi(x)$. Considering the tree hull$\{a, b, x, y\}$ we see that $[\pi(x), \pi(y)] \subseteq [x, y]$ which gives $[x, y] \cap [a, b] \neq \emptyset$. $\diamondsuit$

Suppose that $V \subseteq T$. Applying Lemma 1.4, we see that $x, y \in V$ lie in the same connected component of $V$ if and only if $[x, y] \subseteq V$.

**Lemma 1.5 :** *Suppose $V \subseteq T$ is open. The connected components of $V$ are open.*

**Proof :** Suppose $Q \subseteq V$ is a component of $V$, and $x \in V$. There is a neighbourhood $U$ of $x$ such that if $y \in U$, then $[x, y] \subseteq V$. Thus $U \subseteq Q$, and so $Q$ is open. $\diamondsuit$

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Note that it follows that $T$ is locally connected in the usual topological sense: if $x$ lies in an open set $U$, then the component of $U$ containing $x$ is an open connected neighbourhood of $x$.

Given $x \in T$, the *degree* of $x$ is the cardinality of the set of components of $T \setminus F$. We write it as $\deg(x)$.

We say that $x$ is an *terminal point* of $T$ if $\deg(x) = 1$. We say that $x$ is a *node* if $\deg(x) \geq 3$. Note that $x$ is an terminal point if and only if it does not lie between any pair of points.

For some purposes, we shall want to add a hypothesis of separability. There are several notions of dense subset which we should first clarify.

Suppose $R \subseteq T$. We use the term *topologically dense* to mean the usual thing, namely that every open subset of $T$ contains a point of $R$. We say that $R$ is *weakly dense* if for any pair of distinct points $x, y \in T$, there is a point $z \in R$ such that $\mathrm{med}(x, y, z) \in (x, y)$. We say that $R$ is *strongly dense* (sometimes called "interval dense") if for any pair of distinct points $x, y \in T$, $(x, y) \cap R \neq \emptyset$. Clearly strongly dense implies topologically dense which in turn implies weakly dense (using local connectedness and continuity of projection respectively). Moreover, if $R$ is weakly dense, then the set $\{\mathrm{med}(x, y, z) \mid x, y, z \in R\}$ is strongly dense (see Lemma 2.11).

**Definition :** We say that a real tree $T$ is separable if it contains a countable dense subset.

¿From the above discussion, we see that it doesn't matter which of the three notions of density we take.

**Lemma 1.6 :** *If $T$ is separable, then $T$ has at most countably many nodes, each of which has at most countable degree.*

**Proof :** The latter statement follows from Lemma 1.5. To prove the first part, let $R \subseteq T$ be a countable topologically dense subset. If $w \in T$ is a node, we can choose $x, y, z \in R$ lying in distinct components of $T \setminus \{w\}$ (using Lemma 1.5). Thus $w = \mathrm{med}(x, y, z)$. There are thus at most countably many possibilities for $w$. $\diamond$

**Definition :** A *dendron* is a compact real tree. A *dendrite* is a separable dendron.

(This usage of "dendron" is not quite standard. The more usual definition is that it is a continuum in which every pair of point is separated by a third point. This is more general than our notion. If we add an assumption of separability, however, these definition become equivalent; so our notion of "dendrite" agrees with the usual one.)

Note that every compact metrisable space is separable. For dendrons, we have a converse:

**Lemma 1.7 :** *A dendrite is metrisable.*

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**Proof :** Let $T$ be a dendrite. By the metrisability theorem for separable spaces, it is enough to show that $T$ is regular and second countable (see [Ke, Theorem 4.17]). Since $T$ is compact hausdorff, it is certainly regular.

Given $x, y \in T$, let $H(x, y) = \mathrm{proj}_{[x,y]}^{-1}([x, y))$. Thus, by continuity of projection, $H(x, y)$ is an open subset of $T$. Note that if $z \in [x, y)$ then $H(x, y) = H(z, y)$. Suppose that $R \subseteq T$ is strongly dense, then we claim that $\{H(x, y) \mid x, y \in R\}$ forms a subbase for the topology. ¿From the previous observation, we can allow $x$ to vary over all of $T$.

To prove the claim, suppose $x \in T$ and that $U$ is an open neighbourhood of $x$. Let $V$ be an open connected neighbourhood of $x$, whose closure, $\bar{V}$ is contained in $U$. By Lemma 1.5, and compactness, all but a finite number, $W_1, W_2, \ldots, W_n$, of components of $T \setminus \bar{V}$ are contained in $U$. For each $i \in \{1, 2, \ldots, n\}$ choose some $w_i \in W_i$, and let $y_i \in [x, w_i] \cap V \cap R$. Thus $[x, y_i] \subseteq V$. Now if $z \in W_i$, then $[z, w_i] \cap [x, y_i] = \emptyset$, so we see that $y_i \in [x, z]$. In other words, $\mathrm{med}(x, y_i, z) = y_i$ so $z \notin H(x, y_i)$. This shows that $W_i \cap H(x, y_i) = \emptyset$. It follows that $\bigcap_{i=1}^{n} H(x, y_i) \subseteq U$. The result follows. $\diamond$

It's not hard to see that a dendrite is locally connected in the usual topological sense and thus a Peano continuum (i.e. a locally connected continuum). In fact, a dendrite may be defined as a separable Peano continuum which contains no topologically embedded circle (see [Ni], [Na] or [Ku] for a discussion). Dendrites have a long history. For example, they arise as cut loci on certain convex surfaces, and were considered by Poincaré [Po].

Note that is was shown independently by Bing and Moise that a metric Peano continuum admits a path-metric [Bi1,Moi] (see also [Bi2]). Now if we put a path-metric on a dendrite, every arc will necessarily be geodesic, and so in fact we get an **R**-tree. In other words, yet another definition of a dendrite is a compact separable **R**-tree (but without any preferred metric). For a more general discussion, see [MayO]. We shall discuss these matters further in Section 7.

## 2. Pretrees

In this section we define the notion of a "pretree". As we mentioned in the introduction, the axioms for a pretree, in complete generality, appear in a paper by Ward [W2], and were rediscovered in the article of Adeleke and Neumann (where they are referred to a "B-sets"). As we described in the introduction, a pretree is essentially set $T$ together with a ternary "betweenness" relation, such that every finite subset of $T$ "looks like" a subset of a finite simplicial tree — see Lemma 2.5. It is thus analogous to a protree as defined by Dunwoody [Du2,Du3].

Betweenness relations of various sorts have been considered by many authors. The idea of betweenness is fundamental in the axiomatisation of euclidean geometry (see for example [V]). Since then other kinds of betweenness relations have been studied in relation to treelike objects (see for example some of the references given in [BaH]). In particular, what we call "median pretrees" appear in [Sho], and have since appeared in a number of places under a variety of names. We shall say more about this at the end of this section.

Given $x, y, z$, we shall write $xyz$ for the ternary relation, and say that $y$ *lies between* $x$ and $z$.

We shall make frequent use of the logical symbols $\wedge$, $\vee$ and $\neg$ for "and", "or" and "not".

**Definition :** The set $T$ with such a ternary relation is a *pretree* if the following three axioms are satisfied:

(T0) $(\forall x, y)(\neg xyx)$,

(T1) $xzy \Leftrightarrow yzx$,

(T2) For all $x, y, z$, $\neg(xyz \wedge xzy)$,

(T3) If $xzy$ and $z \neq w$ then $(xzw \vee yzw)$.

We shall write T3$(xzy, w)$ for the third axiom.

Note that putting (T0) and (T1) together, we see that $xyz$ implies that $x$, $y$ and $z$ are all distinct.

Clearly any subset of a pretree is a pretree. We shall say that $x$ is a *terminal point* if $(\forall y, z \in T)(\neg yxz)$.

Given $x, y, z, w \in T$ we write $xyzw$ to mean $(xyz \wedge yzw \wedge xyw \wedge xzw)$.

We get ourselves off the ground with four elementary lemmas. These results can also be found in [AN].

**Lemma 2.1 :** $(xyz \wedge yzw) \Rightarrow xyzw$.

**Proof :** We have to show $xyw \wedge xzw$. By axiom (T2), $x \neq w$. Since $yzw$, we have $\neg ywz$, so $xyw$ (by T3$(xyz, w)$). Similarly, $xzw$. $\qquad\qquad\qquad\qquad\qquad\qquad \diamondsuit$

**Lemma 2.2 :** $(xyw \wedge yzw) \Rightarrow xyzw$.

**Proof :** By Lemma 2.1, it's enough to show $xyz$. But note $z \neq x$, so if $\neg xyz$, then $zyw$ by T3$(xyw, z)$, contradicting $yzw$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \diamondsuit$

**Lemma 2.3 :** $(xyw \wedge xzw) \Rightarrow (xyzw \vee xzyw)$.

**Proof :** By T3$(xzw, y)$, we can assume without loss of generality that $yzw$, and so $xyzw$ by Lemma 2.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \diamondsuit$

Now there are precisely four combinatorial possibilities (up to pretree isomorphism) for the betweenness relation of a set $F$ of four points embedded in a finite simplicial tree. They are illustrated in Figure 1, and labelled (A), (B), (C) and (D). The black dots represent the points of $F$. Clearly, we can assume that the tree is equal to the hull of $F$. Thus, (A) is "four in a row", (B) is a tripod with whose node is in $F$, (C) consists of a tripod whose node is not in $F$ and so one branch contains two points of $F$, and (D) has a node of degree 4 which is not in $F$. Case (D) is referred to as a "null" pretree — there are no relations — a concept that shall prove more useful than one might imagine. Note also

that case (D) is really the same as a tree with two nodes of degree 3, since the difference would not be noticed in $F$.

In fact there was no need to assume a-priori that $F$ admits an embedding in a simplicial tree — these are the only pretrees on four points. Expressed another way:

**Lemma 2.4 :** *Suppose $x, y, z, w$ are distinct points of in pretree. Then the betweenness relations on $\{x, y, z, w\}$ are compatible with one of the pictures (A), (B), (C) or (D) described above.*

**Proof :** If not (D), then without loss of generality, $xyz$. By T3$(xyz, w)$, we can also assume $xyw$. If not (C), then we must have another relation — without loss of generality one of $xzw$, $yzw$, $wxz$ or $wyz$. By the previous lemmas, either $xzw$ or $yzw$ implies $xyzw$. In this case, no other relations are possible, so we are in case (A). If $wxz$, then since $xyz$ we would have $yxw$ contradicting $xyw$, so this case is not possible. Finally, if $wyz$, we verify that not other relations are possible (by applying the previous lemmas to derive a contradiction), and so we are in case (B). $\diamond$

We can generalise Lemma 2.4 to any finite pretree $F$. We aim to embed $F$ in a finite tree, $\tau$, conisting of a set, $V$, of vertices, and a finite set $E$ of edges. Let $N = V \setminus F$. We can suppose that each vertex of $N$ has degree at least 3. Moreover we can suppose that each edge has at least one endpoint in $F$ (otherwise we could contract it to a point without altering the betweenness relations on $F$). Note that if $F$ is known to arise in this way then it must have at least two terminal points (assuming it has at least two points).

Recall that pretree $F$ is *null* if $(\forall x, y, z \in F)(\neg xyz)$.

**Lemma 2.5 :** *If $F$ is a finite pretree, then it can be embedded in a finite tree $\tau$ (in the manner described above) so that the pretree structure is induced by that on $\tau$.*

**Proof :** By induction on the cardinality of $F$, we can suppose that the lemma is true for any proper subset of $F$. We assume that $F$ has at least 3 points.

We first claim that there is some point $p \in F$ which is terminal. For suppose not. Choose any $x \in F$. Now $F \setminus \{x\}$ can be embedded in a tree and so has at least two terminal points, say $y$ and $z$. But $y$ is not terminal in $F$, so there is some $w \in F$ with $xyw$. If $w = z$ then $xyz$. Also if $w \neq z$, then since $y$ is terminal in $F \setminus \{x\}$, by T3$(xyw, z)$, we deduce again that $xyz$. But reversing the roles of $y$ and $z$, we obtain $xzy$. This contradiction shows that there must indeed be some terminal point, say $p \in F$.

We now embed $F' = F \setminus \{p\}$ in a finite tree $\tau'$ with vertex set set $N \sqcup F'$ and edge set $E$ as described above. We shall put an orientation on some of the edges $E$ as follows.

Suppose $e \in E$ has both its endpoints $x$ and $y$ in $F'$. We orient $e$ from $x$ to $y$ if $xyp$ and from $y$ to $x$ if $yxp$. Otherwise we leave $e$ unoriented.

Suppose $v \in N$, and let $F_v \subseteq F'$ be the set of adjacent vertices. Thus $F_v$ is null. If $F_v \cup \{p\}$ is also null, then we orient all edges incident on $v$ towards $v$. If not, then there exist $x, y \in F_v$ with $xyp$. We see that $zxp$ for all $z \in F_v$ (using T3$(yxp, z)$). Thus $x$ is determined uniquely. In this case, we orient the edge joining $v$ to $x$ towards $x$, and we direct all other edges incident on $v$ towards $v$.

This process accounts for every edge in $E$. By construction, each vertex in $N$ has at most one incident edge pointing away, and all other incident edges pointing towards it. Also (for example by applying Lemma 2.4), we see that if $x \in F'$, then all but at most one edge incident on $x$ points towards $x$. (The remaining edge might be unoriented, or might point in either direction.) It now follows that there is either a unique vertex $v \in N \sqcup F'$ towards which all edges of $E$ are oriented, or else a unique edge $e \in E$ with is unoriented and towards which all other edges of $E$ are oriented.

We can now extend $\tau'$ to a finite tree containing all of $F$ as follows. In the first case, we join $p$ to $v$ by a new edge. In the second case we split the edge $e$ in two by inserting a new vertex $w$ in its interior, and then join $p$ to $w$ by new edge. In this way we obtain a new tree $\tau$.

We need to verify that the relations on $F$ are precisely those induced from $\tau$. By construction, we need only worry about relations involving the point $p$. To begin with, we should note that $p$ is terminal in both $F$ and $\tau$, so we only have relations of the form $xyp$. Now by the construction of $\tau$, we see that if $x, y \in F'$, then $y$ lies between $x$ and $p$ in $\tau$ if and only if the interval $[x, y] \subseteq \tau'$ consists of a sequence of edges of $E$ which are all oriented towards $y$. We thus need to verify that this can only happen if $xyp$. ¿From the construction of the orientations on $E$, this easily reduced to the assertion that if $x, y, z$ are distinct points of $F'$ with $xzy$, then $xyp$ if and only if $xzp$ and $zyp$. This can in turn be seen for example by applying Lemma 2.4 to the set $\{x, y, z, p\}$. $\diamond$

This lemma is useful in working with pretrees. Often we find ourselves dealing with only a finite number of points at a given moment, and so this allows us to assume for the purposes of argument that we are living in a finite tree.

Given distinct points $x, y$ in a pretree $T$, we shall write $(x, y) = \{z \in T \mid xzy\}$, $[x, y) = (y, x] = (x, y) \cup \{x\}$ and $[x, y] = (x, y) \cup \{x, y\}$. We refer to these sets as *open*, *half-open* and *closed intervals* respectively, and collectively as *intervals*. Note that without reference to the points $x$ and $y$ the distinction between open, half-open and closed might be ambiguous. We shall use the convention that $(x, x) = [x, x) = \emptyset$ and $[x, x] = \{x\}$. We say that two points distinct points, $x, y \in T$ are *adjacent* if $(x, y) = \emptyset$.

**Lemma 2.6 :**    If $x_0, x_1, \ldots, x_n$ is a finite sequence of points of $T$, then $[x_0, x_n] \subseteq \bigcup_{i=1}^{n} [x_{i-1}, x_i]$.

**Proof :**  This can be proved by induction over $n$. The case where $n = 2$ is a simple consequence of axiom (T3). $\diamond$

Note that a pretree can alternatively be defined in terms of closed intervals. In other words, we have a set $T$, and to each pair of elements $x, y \in T$, we associate subset $[x, y] \subseteq T$. The pretree axioms then translate as $[x, y] = [y, x]$, $[y, z] \subseteq [x, y] \cup [x, z]$, and $((y \in [x, z]) \wedge (z \in [x, y])) \Rightarrow (y = z)$.

We shall need some further definitions:

**Definitions :**
A subset $A$ of $T$ is *full* if $[x, y] \subseteq A$ for all $x, y \in A$.

A subset $A$ is *linear* if for all distinct $x, y, z \in T$, we have $xyz \lor yzx \lor zxy$.

An *arc* is a nonempty full linear subset.

If $A$ is a linear subset of $T$, a *direction* on $A$ is a linear (i.e. total) order $<$ on $A$ such that $xyz \Leftrightarrow ((x < y < z) \lor (z < y < x))$. We refer to $(A, <)$ as a *directed* linear set.

Note that if $(A, <)$ is a directed linear set, then so also is $(A, >)$ where $x > y \Leftrightarrow y < x$. In fact, by applying Lemma 2.5, it is easily verified that:

**Lemma 2.7 :** *A linear set with at least two elements admits precisely two directions.* ◇

Note that every interval is an arc, though the converse need not hold in general. Given an interval, say $[a, b]$, we can define a direction on $[a, b]$ by writing $x < y$ if and only if $x \in [a, y)$, or equivalently $y \in (x, b]$.

Note that the points $a$ and $b$ are precisely the terminal points of $[a, b]$. Thus, if $[a, b] = [a', b']$ then $\{a, b\} = \{a', b'\}$. (This fails in general for half-open or open intervals, but see Lemma 2.9 and the subsequent discussion.)

**Definition :** Given $x, y, z \in T$, we shall say that $c \in T$ is a *median* of $x, y, z$ if $c \in [x, y] \cap [y, z] \cap [z, x]$.

Applying Lemma 2.5, we see that if a median exists, then it must be unique. In this case we write $c = \mathrm{med}(x, y, z)$.

**Definition :** A *median pretree* is a pretree in which every set of three points has median.

Thus in median pretree, we have a map $\mathrm{med} : T^3 \longrightarrow T$. Applying Lemma 2.5, we see that this map must satisfy the axioms of a median algebra, namely

(M1) $\mathrm{med}(x, x, y) = x$,

(M2) $\mathrm{med}(x, y, z) = \mathrm{med}(x, z, y) = \mathrm{med}(y, z, x)$, and

(M3) $\mathrm{med}(\mathrm{med}(x, y, z), u, v) = \mathrm{med}(x, \mathrm{med}(y, u, v), \mathrm{med}(z, u, v))$.

A survey of median algebras is given in [BaH]. Recently median algebras have been studied from a more geometric viewpoint by Roller [R]. In particular, "discrete" median algebras turn out to have close connections with non-positively curved cubed complexes. Note that median algebras are much more general than median pretrees (for example the cartesian product of two median algebras is naturally a median algebra). We shall not have much to say about this general theory here. Note that various kinds of betweenness relations and their connections with median algebras are to be found in [Sho].

In a median pretree, we can define *projection* to any closed interval $\mathrm{proj}_{[a,b]} : T \longrightarrow [a, b]$ by $\mathrm{proj}_{[a,b]}(x) = \mathrm{med}(a, b, x)$.

There are various other conditions we shall want to put on pretrees, for example density, completeness and separability. We shall consider these in turn.

**Definition :** A pretree $T$ is *dense* if for all distinct $x, y \in T$ there exists $z \in T$ with $xzy$.

In other words, no two points in $T$ are adjacent.

**Lemma 2.8 :** Suppose $T$ is a dense pretree, and $x, y, z \in T$. If $[x, y) \subseteq [x, z]$ then $y \in [x, z]$.

**Proof :** Suppose the conclusion fails. In particular, $y \neq z$, so we can find $w$ with $ywz$. By T3$(yzw, x)$, we have $zwx \vee ywx$. But since $[x, y) \subseteq [x, z]$, $ywx$ implies $zwx$. Thus $zwx$, and so $w = \mathrm{med}(x, y, z)$. Now choose $w'$ so that $ww'z$. Thus $yw'z$ so applying the same argument we get $w' = \mathrm{med}(x, y, z)$ contradicting the uniqueness of medians. $\diamond$

**Lemma 2.9 :** Suppose $T$ is a dense pretree, and $x, y, z \in T$. If $[x, y) = [x, z)$ then $y = z$.

**Proof :** We have $[x, y) \subseteq [x, z]$ so by Lemma 2.9, we see that $y \in [x, z]$ so $[x, y] \subseteq [x, z]$. Similarly, we also have $z \in [x, y]$ and so $y = z$. $\diamond$

Note that in a dense pretree, $x$ is the unique terminal point of $[x, y)$ so in this case we might generalise Lemma 2.9 to say that if $[x, y) = [x', y')$, then $x = x'$ and $y = y'$. An open interval $(x, y)$ has no terminal points. In this case (for example by splitting it into two half-open intervals), we see that if $(x, y) = (x', y')$, then $\{x, y\} = \{x', y'\}$. In particular, note that in a dense pretree, one can speak unambiguously about open, half-open and closed intervals.

By a *cut* of an interval $[a, b]$ we mean a partition of $[a, b]$ into two non-empty subarcs, $[a, b] = A \sqcup B$. Clearly no proper subarc of $[a, b]$ can contain both $a$ and $b$. Thus, without loss of generality we can suppose that $a \in A$ and $b \in B$.

**Definition :** A pretree is *complete* if every arc is an interval.

There is an alternative way of formulating competeness by saying that every directed arc has a "supremum" as we shall describe in Section 3.

**Lemma 2.10 :** Suppose $T$ is complete and dense. Suppose $a, b \in T$ and that $[a, b] = A \sqcup B$ is a cut with $a \in A$ and $b \in B$. Then there is some $c \in [a, b]$ such that either $A = [a, c]$ and $B = (c, b]$ or else $A = [a, c)$ and $B = [c, b]$.

**Proof :** The case where either $A$ or $B$ is a singleton is trivial, so we shall assume that this is not the case. By the completeness hypothesis, there are points $x, y \in T$ such that $A$ is equal to $[a, x]$ or $[a, x)$ and $B$ is equal to $[b, y]$ or $[b, y)$. Lemma 2.8 tells us that $x, y \in [a, b]$, and by our initial assumption we see in fact that $x, y \in (a, b)$. Now if $x \neq y$, then by Lemma 2.3, we have either $axyb$ or $ayxb$. By choosing any $z \in (x, y) \subseteq [a, b]$, we get the contradiction $z \notin A \cup B$ in the former case, and the contradiction $z \in A \cap B$ in the latter case. We conclude that $x = y$. Now $x$ must lie in either $A$ or $B$, say $A$. Then $A = [a, x]$ and $B = [b, x)$ as required. $\diamond$

**Lemma 2.11 :** *A complete dense pretree is median.*

**Proof :** Suppose, for contradiction, that the three points $x, y, z \in T$ do not have a median. Certainly $x$, $y$ and $z$ are all distinct. Let $A = [x, y] \cap [x, z]$, $B = [y, z] \cap [y, x]$ and $C = [z, x] \cap [z, y]$. Thus $A$, $B$ and $C$ are all arcs, and thus intervals.

Now if $w \in (x, y)$, then (using T3$(xwy, z)$) we have $w \in [x, z]$ or $w \in [y, z]$. Thus $[x, y] = A \cup B$. Also any point of $A \cap B$ would have to be a median of $x, y, z$, and so $A \cap B = \emptyset$. Thus $[x, y] = A \sqcup B$ is a cut. By Lemma 2.11 one of $A$ and $B$ is a half-open interval, while the other is a closed interval. (Recall that this is a well-defined distinction for dense pretrees.) But the same argument applies to the partitions $[y, z] = B \sqcup C$ and $[z, x] = C \sqcup A$, so we arrive at a contradiction. $\diamondsuit$

We now want to consider separability of dense median pretrees.

Given distinct points $x$ and $y$ in a median pretree $T$, we write $H(x, y) = \mathrm{proj}_{[x,y]}^{-1}([x, y)) = \blacksquare$ $\{z \in T \mid \mathrm{med}(x, y, z) \neq y\}$. We write $P(x, y) = H(x, y) \cap H(y, x)$. Generalisting definitions in Section 1, we define:

**Definition :** A subset $R \subseteq T$ is *weakly dense* if for any pair of distinct points $x, y \in T$, $R \cap P(x, y) \neq \emptyset$.

A subset $R \subseteq T$ is *strongly dense* if for any pair of distinct points $x, y \in T$, $R \cap (x, y) \neq \emptyset$.

Note that saying that $T$ is dense is the same as saying that it is strongly dense in itself. In fact, the existence of any strongly dense subset implies immediately that $T$ must be dense.

**Lemma 2.12 :** *Suppose that $T$ is a median pretree and that $R \subseteq T$ is weakly dense in $T$. Let $R' = \{\mathrm{med}(x, y, z) \mid x, y, z \in R\}$. Then $R'$ is strongly dense.*

**Proof :** Given distinct $x, y \in T$, choose any $a \in R \cap P(x, y)$ and let $z = \mathrm{med}(x, y, a) \in (x, y)$. We claim that $z \in R'$. To see this, choose $b \in R \cap P(x, z)$ and $c \in R \cap P(y, z)$. Now applying Lemma 2.5 to the set $\{x, y, z, a, b, c\}$ we see that $z = \mathrm{med}(a, b, c)$. $\diamondsuit$

**Definition :** A median pretree is *separable* if it contains a countable dense subset.

By Lemma 2.12, we can interpret "dense" to mean either weakly dense or strongly dense, with the same result. A separable pretree is necessarily dense.

Suppose that $T$ is separable and that $a, b \in T$ are distinct. Let the total order $<$ be a direction on $[a, b]$ (see Lemma 2.7). If $R \subseteq T$ is strongly dense, then $R \cap [a, b]$ is dense with respect to this order. If $R$ is countable, then $R \cap (a, b)$ has the order type of of the rational numbers. If $T$ is also complete, then Lemma 2.10 tells us that $[a, b]$ has the Dedekind cut property (i.e. "no gaps"). We conclude:

**Lemma 2.13 :** *Suppose $T$ is a complete separable pretree, and $a, b \in T$ are distinct. Let $<$ be a direction on $[a, b]$. Then $([a, b], <)$ is order-isomorphic to the real interval $[0, 1]$ with the standard order.* $\diamondsuit$

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We now give a characterisation of dendrites as compact topological spaces with compatible pretree structure.

**Theorem 2.14 :**  *Suppose that $T$ is a compact separable topological space. Suppose $T$ also carries a complete dense pretree structure such that $H(x, y)$ is open for any pair of distinct points $x, y \in T$. Then $T$ is a dendrite. Moreover if $x, y \in T$, then the interval $[x, y]$ (as defined by the pretree structure) is the unique arc joining $x$ to $y$.*

We shall need some more preliminary discussion before proving this fact.

It's clear that $T$ must be hausdorff, since given distinct $x, y \in T$, choose any $z \in (x, y)$; then $x \in H(x, z)$, $y \in H(y, z)$ and $H(x, z) \cap H(x, y) = \emptyset$. By Lemma 2.11, we know that $T$ is a median pretree. Also $T$ is separable as a pretree; since if $R \subseteq T$ is topologically dense, and $x, y \in T$ are distinct, then $P(x, y) = H(x, y) \cap H(y, x)$ is open, so $R \cap P(x, y) = \emptyset$, and so it follows that $R$ is weakly dense.

Now, suppose $a, b \in T$ are distinct. Then the "closed" interval $[a.b]$ is closed in the topological sense. To see this, suppose $x \in T \setminus [a, b]$. Let $c = \mathrm{med}(a, b, x)$ and choose $y \in (c, x)$. Then $x \in H(x, y)$ and $H(x, y) \cap [a, b] = \emptyset$. Thus $[a, b]$ is closed, and is therefore compact.

Let $<$ be a direction on $[a, b]$. We can assume that $a < b$, so the order is given by $x < y \Leftrightarrow x \in [a, y)$. We know by Lemma 2.13, that $[a, b]$ is order isomorphic to the real closed interval $[0, 1]$. If $x, y \in [a, b]$ then from the characterisation of a direction, we see that $(x, y) = \{z \in [a, b] \mid x < y < z\}$, so this agrees with the usual notion of open interval for totally ordered sets. Note that $(x, y) = [a, b] \cap P(x, y)$, and so is topologically open in $[a, b]$ with its given compact topology coming from $T$. But now $[a, b]$ can be given another topology arising from some (in fact any) order isomorphism with the interval $[0, 1]$. In this "order topology", $[a, b]$ is thus homeomorphic to $[0, 1]$. Since open intervals form a base for the order topology, we see that the given topology is finer than the order topology. In fact, these topologies must agree. This follows from the observation:

**Lemma 2.15 :**  *Let $\mathcal{T}_0$ be the standard order topology on $[0, 1]$. If $\mathcal{T}$ is a compact topology on $[0, 1]$ finer than $\mathcal{T}_0$, then in fact $\mathcal{T} = \mathcal{T}_0$.*

**Proof :** The identity map from $([0, 1], \mathcal{T})$ to $([0, 1], \mathcal{T}_0)$ is a continuous bijection from a compact space to a hausdorff space, and hence a homeomorphism. $\diamond$

In summary, we have shown:

**Lemma 2.16 :**  *Suppose $a, b \in T$ are distinct. Then $[a, b]$ (in the subspace topology) is homeomorphic to the closed real interval $[0, 1]$. Moreover, pulling back the standard order by this homeomorphism gives us a direction on $[a, b]$.* $\diamond$

In particular, this shows that $T$ is topologically arc-connected. Our next objective is a form of local connectedness. We first show that the median map is continuous at certain points:

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**Lemma 2.17 :** *Suppose $x, z \in T$ are distinct. Suppose $V$ is a neighbourhood of $x$. Then there exist open sets $U \ni x$ and $O \ni z$ such that if $y \in U$ and $w \in O$, then $\mathrm{med}(x, y, w) \in V$.*

**Proof :** Using Lemma 2.16, we can find some $t \in (x, z]$ with $[x, t] \subseteq V$. Let $U = H(x, t)$ and $O = H(z, t)$. If $y \in U$ and $w \in O$, then (using Lemma 2.5) we see that $\mathrm{med}(x, y, w) = \mathrm{med}(x, y, z) \in [x, t) \subseteq V$ as required. $\diamondsuit$

We deduce the following formulation of local connectedness. (Recall that any pair of points $x, y \in T$ are joined by a preferred topological arc namely $[x, y]$ as derived from the pretree structure, applying Lemma 2.16. However, we have not yet shown that it is the only topological arc between these points.)

**Lemma 2.18 :** *Suppose $x \in T$ and that $V$ is a neighbourhood of $x$. Then there is a neighbourhood $U$ of $x$ such that if $y \in U$ then $[x, y] \subseteq V$.*

**Proof :** We can suppose that $V$ is open so that $T \setminus V$ is compact. Given $z \in T \setminus V$, choose open sets $U(z)$ about $x$ and $O(z)$ about $z$ satisfying the conclusion of Lemma 2.17. Let $\{O(z_i) \mid 1 \le i \le n\}$ be a finite subcover of $T \setminus V$, and let $U = \bigcap_{i=1}^{n} U(z_i)$. We claim that if $y \in U$, then $[x, y] \subseteq V$. For if not, choose $z \in [x, y] \setminus V$. Now $z \in O(z_i)$ for some $i$, and $y \in U(z_i)$. Thus $\mathrm{med}(x, y, z) \in V$. But $z \in [x, y]$ so $\mathrm{med}(x, y, z) \in z \notin V$. This proves the claim. $\diamondsuit$

It remains to show that there is only one topological arc joining two given points. Since shall refer to this again later, we shall formulate this result for general pretrees (without any compactness assumption).

**Lemma 2.19 :** *Suppose that $S$ has the structure of both a pretree and a hausdorff topological space. Suppose that for all $x, y \in S$, the pretree interval, $[x, y]$, is homeomorphic to a closed real interval with endpoints at $x$ and $y$. In addition, suppose that $S$ is locally arc-connected in the sense that if $x \in S$, and $V$ is a neighbourhood of $x$, then there is a neighbourhood $U$ of $x$ such that for all $y \in U$, we have $[x, y] \subseteq V$. Then, $S$ is topologically a real tree. Moreover the induced pretree structure agrees with the original.*

**Proof :** Suppose $x, y \in S$ are distinct, and that $h : [0, 1] \longrightarrow S$ is a continuous injective map with $h(0) = x$ and $h(1) = y$. Then, we claim that $h([0, 1]) = [x, y]$.

Since $h$ is a homeomorphism onto its range, we know that $h([0, 1])$ is an arc joining $x$ to $y$. By Lemma 2.16, so is $[x, y]$. Clearly if one were a subset of the other, they would have to be identical. Thus, suppose for contradiction, that there exists $z \in [x, y] \setminus h([0, 1])$. Note that $S \setminus \{z\}$ is open. ¿From the local connectedness hypothesis, we see that if $t \in [0, 1]$, there is some open interval $J(t) \ni t$ such that if $u \in J(t)$, then $z \notin [h(t), h(u)]$. By the Heine-Borel theorem, we can find points $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for each $i = 1, \ldots, n$, we have $t_i \in J(t_{i-1})$. Let $x_i = h(t_i)$, then $z \notin [x_i, x_{i-1}]$. But $x = x_0$ and $y = x_n$, so $z \in [x, y] = [x_0, x_n]$ contradicting Lemma 2.6.

We thus see that $T$ is uniquely arc-connected in the topological sense. The local connectedness hypothesis now reduces to that used in the definition of a real tree, and so

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the result follows. ◇

**Proof of Theorem 2.14 :** By Lemmas 2.16, 2.18 and 2.19. ◇

Before we finish this section, we describe how examples of pretrees occur in "nature". One of the principal motivations for introducing the idea was to capture the cut-point structure of a continuum. This will be discussed in some detail in Section 5. Pretrees can also be viewed as a kind of generalisation of pseudotrees as discussed in Section 3. Other connections with partial orders and their automorphism groups are discussed in [AN] and [Tr].

The special case of a median pretree (under a variety of different names) has been studied in various contexts for some time. They can be found in Sholander's paper [Sho], and have appeared in various guises, for example see [He] and [Ch1].

More recently, a connection has been made between median pretrees and and $\Lambda$-trees as defined in [MorS]. The case of $\mathbf{R}$-trees will be discussed in some detail in Section 7, and $\mathbf{Z}$-trees are mentioned in Section 6. We recall briefly the definition. Let $\Lambda$ be an ordered abelian group. A $\Lambda$-*tree*, $T$, is a $\Lambda$-metric space such that given any $x, y \in T$, there is a unique segment $[x, y] \subseteq T$ joining $x$ to $y$, where a "segment" is a subset of $T$ $\Lambda$-isometric to an interval in $\Lambda$. In addition, $T$ must satisfy the two axioms: $(\forall x, y, z \in T)(\exists w \in T)([x, y] \cap [x, z] = [x, w])$ and $[x, y] \cap [x, z] = \{x\} \Rightarrow [x, y] \cup [x, z] = [y, z]$. It is easily verified, and has been observed elsewhere, that a $\Lambda$-tree is also a median pretree, where the betweenness relation is defined by interpreting a segment $[x, y]$ as a pretree interval. Conversely, it has been shown by Chiswell [Ch1], that every median pretree can be embedded in a $\Lambda$-tree for some suitable $\Lambda$. Moreover, a countable median pretree can be embedded in an $\mathbf{R}$-tree. Now, the completion process (Theorem 3.19) embeds any pretree in a median pretree. Another construction which achieves this is given in [AN]. It follows that every pretree can be embedded in a $\Lambda$-tree. A modification of the completion process (along the lines of that given in [AN]) shows that every countable pretree embeds in a countable median pretree, and hence in an $\mathbf{R}$-tree. For further details, see the remarks after Lemma 3.29.

Returning to general pretrees, another natural connection, which we shall discuss in more detail here, can be made with protrees, as defined by Dunwoody [Du2]. It turns out that any protree can be viewed as a special kind of pretree (with some additional structure). Also the vertex set of a protree has a natural structure as a median pretree.

Recall the definition of a *protree*. It is a partially ordered set $(E, <)$, together with an order-reversing involution $[x \mapsto x^*] : E \longrightarrow E$ satisfying the "nesting condition", namely that given any $x, y \in E$, precisely one of the following six statements is true: $x = y$, $x = y^*$, $x < y$, $x < y^*$, $x > y$ or $x > y^*$. Note that every subset of a protree, which is closed under the involution, is itself a protree. We should imagine $x$ as directed edge, and $x^*$ as the same edge directed in the opposite sense. We can interpet $x < y$ intuitively to mean simultaneously that $x$ "points towards" $y$, and $y$ "points away from" $x$. This can be made precise if, for example, $E$ were the set of directed edges of a simplicial tree, in particular, of a finite (simplicial) tree. In fact, every finite protree arises in this way. This can be viewed as an analogue of Lemma 2.5. In fact, it is a corollary of Lemma 2.5, as we

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shall see.

Suppose $E$ is a protree. Let $T$ be the (set-theoretic) quotient of $E$ under the involution. Thus an element of $T$ is formally a pair $\{x, x^*\}$, which can be thought of intuitively as an undirected edge of the protree. We define a betweenness relation on $T$ as follows. Given $X, Y, Z \in T$, we write $XYZ$ to mean that $(\exists x \in X)(\exists y \in Y)(\exists z \in Z)(x < y < z)$.

**Proposition 2.20 :**  *With this structure, $T$, is a pretree.*

**Proof :**

(T1): If $x < y < z$, then $z^* < y^* < x^*$, and so $ZYX$.

(T2): Suppose $XYZ \wedge XZY$. Now, $x < y < z$ and $x' < z' < y'$, where $x' \in X$, $y' \in Y$ and $z' \in Z$. Now since $x < y$ and $x' < y'$, the uniqueness part of the nesting condition tells us that $x = x'$ and $y = y'$. Similarly, we find that $z = z'$. But then $y < z$ and $z < y$.

(T3): Suppose $XYZ$ and $W \in T \setminus \{Y\}$. We have $x < y < z$. Now by the nesting condition (repalacing $w$ by $w^*$ if necessary) we see that there is some $w \in W$ with $(y < w) \vee (w < y)$. Thus $XYW \vee WYZ$ as required. $\diamondsuit$

Of course, a lot of information is lost in passing in this way from a protree to a pretree. Perhaps a more useful way is to associate to every undirected edge of the protree a pair of points, which can be thought of as corresponding to the "endpoints" of the edge. A more formal way to approach this construction is to first define the "binary subdivision" of a protree.

Suppose that $E$ is a protree. Let $E' = E \times \{-, +\}$. We think of $(x, -)$ and $(x, +)$ as subdividing the directed edge $x$ into two halves, with $(x, -)$ at the tail end of $x$ and $(x, +)$ at the head end. With this in mind, we define a protree structure on $E'$ as follows. Given $x \in E$, we set $(x, -)^* = (x^*, +)$ and $(x, +)^* = (x^*, -)$. Given $x, y \in E$ and $\sigma, \tau \in \{-, +\}$, we write $(x, \sigma) < (y, \tau)$ if $(x < y) \vee ((x = y) \wedge (\sigma = -) \wedge (\tau = +))$. In particular, $(x, -) < (x, +)$ for all $x \in E$. It is now a simple exercise to verify that $E'$ with this structure is indeed a protree.

Now, given any protree, $E$, we can define a pretree, $\Sigma = \Sigma(E)$, by first taking the binary subdivision, $E'$, and then taking the associated pretree (of undirected edges) as previously defined. An element of $\Sigma$ thus consists of an unordered pair $\beta(x) = \{(x, -), (x^*, +)\}$ for some $x \in E$. Note that $\Sigma$ comes with a natural involution which sends $\beta(x)$ to $\beta(x^*)$. This involution can be thought of intuitively as swapping the endpoints of an undirected edge, $\{x, x^*\}$, of $E$.

Note that the last construction is invertible in the sense that we can recover the protree, $E$, from the pretree $\Sigma$ together with the involution. Thus a protree can be viewed as a particular kind of pretree together with an involution. It's not difficult to write down a set of axioms required of such an involution in order for it to give rise to a protree, though we shall not bother with that here.

One consequence to note from this construction is that if $E$ is finite, then so is $\Sigma$. Thus, applying Lemma 2.5, we see that $\Sigma$ can be embedded in a finite tree $\tau$. Moreover, we can assume that for every undirected edge $\{x, x^*\}$ of $E$, the corresponding pair of elements $\{\beta(x), \beta(x^*)\} \subseteq \Sigma$, are the endpoints of some (undirected) edge, $e(x)$, of $\tau$. We can thus

associate to $x$ the directed edge $\vec{e}(x)$ of $\tau$, with tail at $\beta(x)$, and head at $\beta(x^*)$. In this way, if $x, y \in E$, we have $x < y$ if and only if $\vec{e}(x)$ points towards $e(y)$ and $\vec{e}(y)$ points away form $e(x)$. If we like, we can now contract to a point each component of $\tau$ minus the union of the interiors of all the edges of the form $e(x)$. In this way, we can identify $E$ as precisely the set of directed edges of a finite tree. This can be viewed as an analogue of Lemma 2.5. It can also be seen as an instance of a more general construction which embeds every countable protree in an **R**-tree [Du2]. A still more general connection between protrees and $\Lambda$-trees is explored in [Ch2].

As already mentioned, there is another way in which a protree gives rise to a pretree. An *orientation*, $a$, on $E$, is a transversal to the involution. In other words, for any $x \in E$, precisely one of the statements $x \in a$, $x^* \in a$ holds. A *vertex* of $E$ is an orientation, $a$, with the property that if $y \in a$ and $x < y$ then $x \in a$. Note that this condition forbids the possibility of having $x, y \in a$ with $x^* < y$, in other words, the edges $x$ and $y$ pointing away from each other. We can thus intuitively imagine all the directed edges in $a$ pointing towards some phantom vertex, which we can abstractly identify with $a$.

We write $V = V(E)$ for the set of vertices of $E$. Thus $V \subseteq \mathcal{P}(E)$ where $\mathcal{P} = \mathcal{P}(E)$ is the power set of $E$. We shall describe a pretree structure on $V$.

Note that $\mathcal{P}$ is a distributive lattice with respect to the operations of union and intersection. There is a standard way of defining a median in such a lattice, namely given $a, b, c \in \mathcal{P}$, set

$$\mathrm{med}(a, b, c) = (a \cup b) \cap (b \cup c) \cap (c \cup a) = (a \cap b) \cup (b \cap c) \cup (c \cap a).$$

One can verify (though we won't need to here) that $\mathcal{P}$ is a median algebra. Now, given $a, b \in \mathcal{P}$ one can define the "median algebra interval", $[a, b]_{\mathcal{P}} \subseteq \mathcal{P}$ as $[a, b]_{\mathcal{P}} = \{c \in \mathcal{P} \mid \mathrm{med}(a, b, c) = c\}$. Thus, in this case, $c \in [a, b]_{\mathcal{P}}$ if and only if $c \subseteq a \cup b$. Note that $\mathrm{med}(a, b, c) \in [a, b]_{\mathcal{P}} \cap [b, c]_{\mathcal{P}} \cap [c, a]_{\mathcal{P}}$. So far, these are basic constructions for median algebras (see [BaH]). Note that $\mathcal{P}$ is not in any sense a "treelike" object. However the situation changes, when we restrict to $V \subseteq \mathcal{P}$.

Now, it's a simple exercise to verify that $V$ is closed under the operation of taking medians. We claim, in fact, that $V$ is a median pretree, where we define the betweenness relation, by interpreting $[a, b] = \{c \in V \mid c \subseteq a \cup b\}$ as a pretree interval.

We begin with the following lemma:

**Lemma 2.21 :** *Suppose $a, b, c, d \in V$ and $d \subseteq a \cup b \cup c$. Then at least two of the three statements $d \subseteq a \cup b$, $d \subseteq b \cup c$ and $d \subseteq c \cup a$ hold.*

**Proof :** Suppose not. Then, without loss of generality, we can find $x \in d \setminus (a \cup b)$ and $y \in d \setminus (a \cup c)$. Thus, $x^*, y^* \in a$. Since $x, y \in d$, we have $\neg(x^* \leq y)$, and since $x^*, y^* \in a$, we have $\neg(x \leq y^*)$. Thus, by the nesting condition, we have $(x < y) \vee (y < x)$. Now, $d \subseteq a \cup b \cup c$, so $x \in c$ and $y \in b$. If $x < y$ we obtain the contradiction $x \in b$, whereas if $y < x$ we obtain the contradiction $y \in c$. $\diamond$

**Proposition 2.22 :** *With the structure defined above, $V$ is a median pretree.*

**Proof :**

(T1): Clearly $[a, b] = [b, a]$.

(T2): Suppose $b \in [a, c]$ and $c \in [a, b]$. Then $b \subseteq a \cup c$ and $c \subseteq a \cup b$. If $x \in b \setminus c$, then $x^* \in c \subseteq a \cup b$, so $x^* \in a$. But $x \in b \subseteq a \cup c$, contradicting $x^* \in a \cap c$. We conclude that $b \subseteq c$. Similarly $c \subseteq b$. Thus $b = c$.

(T3): Suppose $d \in [b, c]$ and $a \in V$. Now $d \subseteq b \cup c \subseteq a \cup b \cup c$. Thus, by Lemma 2.21, we see that $d \subseteq a \cup b$ or $d \subseteq a \cup c$. Thus $d \in [a, b] \cup [a, c]$. This shows that for all $a, b, c \in V$, we have $[b, c] \subseteq [a, b] \cup [a, c]$.

This shows that $V$ is a pretree, and the existence of medians has already been observed.

$\diamondsuit$

We remark that this construction can also be interpreted in terms of flows in pretrees, and ties in with the previous construction. Given a protree, $E$, we construct the pretree, $\Sigma = \Sigma(E)$, as above, and then embed $\Sigma$ in the complete pretree $\Phi(\Sigma)$ as constructed in Section 3. Now $V$ can be identified as a certain subset of $\Phi(\Sigma) \setminus \Sigma$. In this way, we obtain a natural pretree structure on the disjoint union $\Sigma \sqcup V$. We shall not work through the details of this construction here.

## 3. Flows

In this section, we describe what we call "flows" on a pretree. For a finite pretree, a flow can be imagined by first embedding it in a finite tree, and then putting orientations on each of the edges in such a way that they all point towards a single vertex (which may or may not lie in our original pretree). This was essentially the structure that arose in the proof of Lemma 2.5. (Note that the vertex to which all the edges point should be allowed to have degree 2, so it might correspond to an unoriented edge in that proof.) This gives rise to a binary relation given by the property that two points should be connected by a sequence of positively oriented edges. A flow on a general pretree is thus a binary relation which is consistent with the above picture for all finite subsets.

We note that there is close connection between flows on pretrees and pseudotrees as we discuss after Proposition 3.4.

One purpose in introducing flows is that enables us to embed any pretree, $T$, in a complete median pretree, $\Phi$. This embedding will have the property that for any distinct pair of points $x, y \in \Phi \setminus T$, there is some $z \in T$ with $xzy$. Also for any pair of distinct points $x, y \in T$, there is some $z \in \Phi$ with $xzy$. The pretree $\Phi$ arises as the set of flows on $T$ given an appropriate pretree structure.

In some respects, $\Phi$ is larger than really necessary. For example, even if $T$ is already a complete median pretree, then $\Phi \setminus T \neq \emptyset$ (unless $T$ is a singleton or empty). However, once we have constructed $\Phi$ there are various ways one could trim it down to give a more sensible notion of completion, if desired. These will be discussed later.

Note that a means of embedding a pretree in a median pretree is given in [AN] (or in their terminology embedding a "B-set" in a "B-set of positive type"). Their construction is different, and does not guarantee completeness.

In outline, what we propose to do is to define the set, $P$, of flows on a pretree $T$, and put a pretree structure on $T \sqcup P$ extending that on $T$. In this structure, every point of $P$ is terminal. In particular, $P$ is null, so it's not very interesting in itself. However, we can use the pretree $T \sqcup P$ to define a new pretree structure on $P$ which admits a natural embedding of $T$. Since we shall eventually want to revert to using our original notation for the betweenness relation, we shall be obliged to rename $P$ as $\Phi$. We can then regard $T$ as a subset of $\Phi$ without a clash of notation.

As already mentioned, a flow, $p$, on $T$ is a binary relation which we shall denote by $xy.p$ for $x, y \in T$. Thus, in some sense, we imagine $p$ as flowing from $x$ to $y$.

**Definition :** The binary relation $p$ is a *flow* on $T$ if it satisfies the following axioms for all $x, y, z \in T$:

(F1) $\neg(xy.p \wedge yx.p)$,

(F2) $xzy \Rightarrow (xz.p \vee yz.p)$, and

(F3) $(xy.p \wedge z \neq y) \Rightarrow (xyz \vee zy.p)$.

Note that (F1) implies that if $xy.p$, then $x \neq y$.

We have already observed that any subset $S$ of a pretree $T$ is a a pretree. Also we can restrict a flow, $p$, to $S$ to give a flow on $S$ which we shall denote by $p|S$.

A simple example of a flow is obtained by choosing any $a \in T$, and defining $\bar{a} = \text{flow}(a) = \text{flow}_T(a)$ by

$$xy.\bar{a} \Leftrightarrow y \in (x, a]$$

(i.e $xy.\bar{a} \Leftrightarrow xya \vee (y = a \wedge x \neq a)$). It is easily verified that $\bar{a}$ is indeed a flow. In fact it has a simple characterisation which will be useful later:

**Lemma 3.1 :** *Suppose $p$ is a flow on $T$. Then $p = \bar{a}$ if and only if $(\forall x \in T \setminus \{a\})(xa.p)$.*

**Proof :** The "only if" bit follows immediately from the definition of $\bar{a}$. For the "if" bit, suppose $(\forall x \in T \setminus \{a\})(xa.p)$. We want to show that $xy.p \Leftrightarrow xy.\bar{a}$. We can assume that $y \neq a$. By axiom (F1) we have $\neg ay.p$. If $xy.p$, then by (F3), $xya$, so $xy.\bar{a}$. Conversely, if $xy.\bar{a}$ then $xya$, so by (F2), $xy.p$. Thus $p = \bar{a}$ as required. $\diamond$

There is a sense in which all flows on $T$ arise in this way, but we might first have to add an extra point. Thus, suppose $p$ is a flow on $T$. We put a pretree structure on $T \sqcup \{p\}$ which extends that on $T$ by declaring that $p$ is terminal (i.e. $(\forall x, y \in T)(\neg xpy)$), and if $x, y \in T$, we set $xyp \Leftrightarrow pyx \Leftrightarrow xy.p$. The fact that this is indeed a pretree is almost immediate: axiom (T1) is contained in the construction, axiom (T2) follows from (F1) given that $p$ is terminal, and axiom (T3) follows from (F2) and (F3) again given that $p$ is terminal. Also, directly from the construction, we see that $p$ is just the restriction to $T$ of flow towards $p$ in $T \sqcup \{p\}$. We have shown:

**Lemma 3.2 :** *If $p$ is a flow on $T$, then there is a natural pretree structure on $T \sqcup \{p\}$ (as defined above) such that $p = \text{flow}(p)|T$.* $\diamondsuit$

In fact, we can generalise this result to take account of all possible flows simultaneously. Let $P$ be the set of all flows on $T$. We put a pretree structure on $T \sqcup P$ as follows. We assume that the induced structure on $T$ agrees with the given structure. The following three cases account for all other possibilities:

(C1) If $p \in P$ and $u, v \in T \sqcup P$, then $\neg upv$.

(C2) If $p \in P$ and $x, y \in T$, then $xyp \Leftrightarrow xy.p$.

(C3) If $p, q \in P$, $p \neq q$ and $x \in T$, then $pxq \Leftrightarrow (\forall y \in T \setminus \{x\})(yx.p \vee yx.q)$.

Note that for any $p \in P$, the structure induced on the subset $T \sqcup \{p\}$ agrees with that defined as for Lemma 3.2. In case (C3), it is important to insist that $p \neq q$, otherwise if $x \in T$ and $p = \text{flow}(x)$ then we would have $pxp$ which is not allowed in a pretree.

**Lemma 3.3 :** *With the betweenness relation defined above, $T \sqcup P$ is a pretree.*

**Proof :** The proof is fairly mechanical. We verify the axioms in turn:

(T1): This is implicit in the construction.

(T2): This follows from (C1) and axiom (T1) for $T$.

(T3): There are essentially four cases to consider. Throughout, we shall assume that $x, y, z \in T$ and $p, q, r \in P$.

Case(a), T3$(xyp, z)$: Now $xy.p$ so by (F3), we have $xyz \vee zy.p$ and so $xyz \vee zyp$ as required.

Case(b), T3$(xyp, q)$: Again, we have $xy.p$ and want to show $xy.q \vee pyq$. Suppose the conclusion fails. Then $\neg xy.q \wedge \neg pyq$. The latter states that $\neg(\forall z \in T \setminus \{y\})(zy.p \vee zy.q)$ so $(\exists z \in T \setminus \{y\})(\neg zy.p \wedge zy.q)$. But $xy.p \wedge \neg zy.p$ implies $xyz$ by (F3). Thus, by (F2), we have $xy.q \vee zy.q$. But we already know that $\neg xy.q \wedge \neg zy.q$. This contradiction shows that we must indeed have $xy.q \vee pyq$, so $xyq \vee pyq$ as required.

Case(c), T3$(pxq, y)$: We have $pxq$ so $(\forall z \in T)(zx.p \vee zx.q)$. In particular, $yx.p \vee yx.q$ so $yxp \vee yxq$ as required.

Case(d), T3$(pxq, r)$: We have $pxq$ and we want $pxr \vee qxr$. Suppose, for contradiction, that $\neg pxr \wedge \neg qxr$. Then $(\exists y \in T \setminus \{x\})(\neg yx.p \wedge \neg yx.r)$ and $(\exists z \in T \setminus \{x\})(\neg zx.q \wedge \neg zx.r)$. ¿From $pxq$, we have $yx.p \vee yx.q$, and so since $\neg yx.p$, we deduce $yx.q$. Now since $yx.q \wedge \neg zx.q$ we have $yxz$ by (F3). Thus, by (F2), we obtain $yx.r \vee zx.r$, contradicting $\neg yx.r \wedge \neg zx.r$. We thus conclude that $pxr \vee qxr$ as required. $\diamondsuit$

¿From the construction, we see that the structure on $T \sqcup P$ induces the original pretree structure on $T$. In addition it satisfies the following:

(P1) Every point of $P$ is terminal,

(P2) Every flow on $T$ has the form $\text{flow}(p)|T$ for some $p \in P$, and

(P3) If $p, q \in P$ and $\text{flow}(p)|T = \text{flow}(q)|T$, then $p = q$.

Although we shall not need it, we make the observation that $T \sqcup P$ is characterised by properties (P1)–(P3):

**Proposition 3.4 :**    *Suppose $T \sqcup P'$ is another pretree satisfying (P1)–(P3) (with $P'$ replacing $P$). Then there is a pretree isomorphism from $T \sqcup P$ to $T \sqcup P'$ fixing $T$.*

**Proof :** By (P2) and (P3), there is a natural bijective correspondence between $P'$ and the set of flows on $T$, so we may identify $P'$ with $P$. Given $u, v, w \in T \sqcup P \equiv T \sqcup P'$ we denote the pretree relation coming from $T \sqcup P'$ by $|uvw|$. We thus want to show that this is the same as that already defined. By hypothesis, we have $|xyz| \Leftrightarrow xyz$ for $x, y, z \in T$. Also, from the nature of our identification, we have $|xyp| \Leftrightarrow xy.p \Leftrightarrow xyp$ for $x, y \in T$ and $p \in P$. We thus need to verify that if $x \in T$ and $p, q \in P$, then $|pxq| \Leftrightarrow pxq$.

Now if $|pxq|$, then for all $y \in T \setminus \{x\}$ we have $|yxp| \vee |yxq|$ (by (T3)), so $yx.p \vee yx.q$. Thus, by definition, $pxq$. Conversely, suppose $\neg|pxq|$. By (T3), we have $|yxp| \Leftrightarrow |yxq|$, and so $yx.p \Leftrightarrow yx.q$, for $y \in T$. Now, if $pxq$, then for all $y \in T \setminus \{x\}$ we have $yx.p \vee yx.q$ and so $yx.p \wedge yx.q$. Thus, by Lemma 3.1, we see that $p = \bar{x} = q$ contradicting $pxq$. We thus deduce $\neg pxq$ as required. $\diamondsuit$

**Remark.**

We shall digress for a moment to describe a connection with pseudotrees, about which there is fairly extensive literature (see [Ni], and the references therein). Recall that a *pseudotree* is a partially ordered set, $(X, <)$, with the property that the set of predecessors, $\{y \in X \mid y \leq x\}$, of any element $x \in X$, is totally ordered. The *infinum*, $\inf\{x, y\}$ of two elements $x$ and $y$, if it exists, is the maximum of the set of common predecessors of $x$ and $y$. If every pair of elements of $X$ have an infinum, then $X$ is a meet-semilattice, where the meet operation is defined by $x \wedge y = \inf\{x, y\}$.

Now, suppose that $T$ is a pretree with a flow $p$. Given $x, y \in T$, write $x < y \Leftrightarrow yx.p$. We can easily verify that $(T, <)$ is a pseudotree. (This is probably most easily done using Lemma 3.2, which reduces us to considering flow towards a point, and Lemma 2.5, which reduces us to finite trees.) Now, if $(T, <)$ has a minimum $a \in T$, we see that $p = \text{flow}(a)$. If also $T$ is a median pretree, then $(T, <)$ is a meet-semilattice with $x \wedge y = \text{med}(a, x, y)$.

Conversely, suppose we are given a pseudotree $(T, <)$. We can put a compatible pretree structure on $T$, though this will not in general be unique. The simplest pretree structure to describe would be to set $xzy$ if and only if precisely one of the relations $z < x$ and $z < y$ holds. The pretree axioms are readily verified. However, probably a more natural pretree structure is obtained by adding to this all relations of the form $xzy$ where $z = x \wedge y$. In either case, the binary relation, $p$, defined by $yx.p \Leftrightarrow y < x$ gives a flow on $T$.

The advantage of the second definition is that if $(T, <)$ happens to be a meet-semilattice, and has a (globally) minimal element, $a$, then $T$ is a median pretree, with $\text{med}(x, y, z) = \max\{x \wedge y, y \wedge z, z \wedge x\}$ (which exists by the pseudotree axioms). Moreover, the flow defined by the partial order is equal to $\text{flow}(a)$. We see that there is a natural bijective correspondence between median pretrees with a preferred point, and pseudotrees which are meet-semilattices with minima.

Note that the construction of Lemma 3.2 has a simple interpretation in terms of pseudotrees: if the pseudotree does not have a minimum, then we simply adjoin one.

It seems that much of the theory of pseudotrees can be carried out in the more general context of pretrees. For our purposes, the latter will be more natural, and we shall not

make any further direct use of pseudotrees.

Before we continue with our discussion of the set of flows, we first need to describe particular flows which arise from directed arcs.

Recall that a directed arc $(A, <)$ is a nonempty full linear subset with a total order $<$ such that for $x, y, z \in A$, $xyz \Leftrightarrow ((x < y < z) \vee (z < y < x))$. If $A$ happens to be an interval, we shall use the convention of writing it in the form $[a, b]$ or $[a, b)$ etc. where we assume that $a < b$.

A *final segment* of a directed arc $A$ is a nonempty subset, $B \subseteq A$, such that if $x \in B$ and $x \leq y \in A$, then $y \in B$. Clearly $B$ is itself a directed arc. We say that two directed arcs, $A$ and $A'$, are *cofinal* if they have a common final segment. In such a case, $A \cap A'$ will be such a segment (as we shall see later). Clearly cofinality is an equivalence relation on the set of directed arcs.

We say that $a \in A$ is a *maximum* of $A$ if $\{a\}$ is a final segment of $A$. Obviously a maximum, if it exists, must be unique. We write $a = \max(A)$. We shall say that $A$ is *endless* if it has no maximum.

**Lemma 3.5 :** *Suppose that $A$ is an endless directed arc. Then given any $x, y \in T$, the following statements are equivalent:*

*(1) $(\forall z \in A)(\exists w \in A)(xyw \wedge z < w)$.*

*(2) $(\exists z \in A)(\forall w > z)(xyw)$*

**Proof :**

$(1) \Rightarrow (2)$: Suppose (2) fails. Then $(\forall z \in A)(\exists w \in A)(\neg xyw \wedge z < w)$. Applying (1) and $\neg(2)$ alternately, we can find points $a < b < c$ in $A$ with $\neg xya \wedge xyb \wedge \neg xyc$. By $T3(xyb, a)$ and $T3(xyb, c)$, we see that $a, c \in [x, y]$. Now, since $abc$, we have $b \in [x, y]$, so $\neg xyb$. This contradiction shows that (2) must hold.

$(2) \Rightarrow (1)$: Let $z_0$ be the point given by condition (2). If $z$ is any other point of $A$, then choose $w \in A$ greater then both $z$ and $z_0$. We have $xyw$. $\diamond$

Given a directed arc, $A$, we shall define a flow, $\text{flow}(A) = \text{flow}_T(A)$ on $T$, which we denote by $xy.A$ for $x, y \in T$ as follows. If $a = \max(A)$ we set $\text{flow}(A) = \text{flow}(a)$ as defined earlier. If $A$ is endless, we write $xy.A$ if either (hence both) of the statements of Lemma 3.5 is are satisfied. We shall verify that this does indeed define a flow. First we make a couple of observations.

**Lemma 3.6 :** *If $A$ and $B$ are cofinal, then $\text{flow}(A) = \text{flow}(B)$.*

**Proof :** If $a = \max(A)$, then $a = \max(B)$ so $\text{flow}(A) = \text{flow}(a) = \text{flow}(B)$. So suppose $A$ and $B$ are both endless. We can assume that $B \subseteq A$ is a final segment. We use condition (1) to see that $xy.A \Rightarrow xy.B$, and condition (2) to see that $xy.B \Rightarrow xy.A$. $\diamond$

**Lemma 3.7 :**    Suppose that $A$ is a directed arc, and $x, y \in T$.

(1) If $x, y \in A$, then $xy.A \Leftrightarrow x < y$.

(2) If $x, y \notin A$, then $xy.A \Leftrightarrow (\exists z \in A)(xyz) \Leftrightarrow (\forall z \in A)(xyz)$.

**Proof :**

(1) If $a = \max(A)$, then $x < y \Leftrightarrow y \in (x, a] \Leftrightarrow xy.A$. So suppose $A$ is endless. If $x < y$ then $(\forall z > y)(xyz)$ and so $xy.A$. If $xy.A$ then $(\exists z)(xyz \wedge y < z)$ so $x < y$.

(2) Note that if $w, z \in A$, then since $\neg wyz$, we have $xyz \Leftrightarrow xyw$ (by $T3(xyz, w)$ and $T3(xyw, z)$. The statement now follows easily.    $\diamondsuit$

**Lemma 3.8 :**    *For any directed arc, $A$, flow$(A)$ is a flow on $T$.*

**Proof :** We have already observed that for any point, $a$, flow$(a)$ is a flow. We can thus assume that $A$ in endless. We need to verify the axioms (F1)–(F3) for any set of four points $x, y, z, w$. Now using Lemma 3.6 and the fact that $A$ is endless, we can suppose that each of these points lies outside $A$. This allows us to apply the condition given by Lemma 2.7(2). We thus choose any $a \in A$, and so for any pair $u, v \in \{x, y, z, w\}$ we have $uv.A \Leftrightarrow uva$. So again, the result follows from the fact that flow$(a)$ is a flow.    $\diamondsuit$

**Lemma 3.9 :**    *Suppose $A$ is directed arc and $x \in A$. If $y \in T$, then $xy.A \Leftrightarrow (y \in A \wedge x < y)$.*

**Proof :** If $a = \max(A)$, then $A = $ flow$(a)$, so $xy.A \Leftrightarrow y \in (x, a] \Leftrightarrow (y \in A \wedge x < y)$ as required. So suppose $A$ is endless. In this case, if $xy.A$, then $(\exists z \in A)(xyz)$ so $y \in A$, and by Lemma 3.7(1), $x < y$. The converse follows directly from Lemma 3.7(1).    $\diamondsuit$

¿From this, we may deduce the following partial converse of Lemma 3.6:

**Lemma 3.10 :**    *Suppose $A$ and $B$ are directed arcs with $A \cap B \neq \emptyset$ and flow$(A) = $ flow$(B)$ then $A$ and $B$ are cofinal.*

**Proof :** Choose $x \in A \cap B$. Let $C = \{y \in T \mid xy.A\} = \{y \in T \mid xy.B\}$. By Lemma 3.9, if $C$ is empty, then $x = \max(A) = \max(B)$. Otherwise, $C$ is final segment of both $A$ and $B$.    $\diamondsuit$

The following observation will prove useful:

**Lemma 3.11 :**    *Suppose $S \subseteq T$ is a full subset, and $(A, <)$ is a directed arc in $T$ with $A \cap S \neq \emptyset$. Then $(A \cap S, <)$ is a directed arc. Moreover, if $A \cap S$ is endless, then flow$_T(A)|S = $ flow$_S(A \cap S)$.*

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**Proof :** Let $B = A \cap S$. The fact that this is a directed arc follows directly from the definitions.

Suppose $B$ is endless, and $x, y \in S$. We want to verify that $xy.A \Leftrightarrow xy.B$. Now, since $B$ is endless, we can suppose, using Lemma 3.6, that $x, y \notin A$. Choose any $a \in A \cap S$. Then, by Lemma 3.7(2), we see that $xy.A \Leftrightarrow xya \Leftrightarrow xy.B$. $\diamond$

It will be convenient to have a slightly different formulation of the notion of completeness from that given in Section 2. We begin by making some observations about directed arcs.

Recall the definition of a directed arc $(A, <)$ given after Proposition 3.4. The set of final segments of $(A, <)$ are totally ordered by set inclusion. If $x \in A$ then $\{y \in A \mid x \leq y\}$ is a final segment. Also, if $B \subseteq A$ is a subarc, and contains a final segment of $A$, then $B$ is itself a final segment. (To see this, suppose $C \subseteq B$ is a final segment of $A$. Choose any $z \in C$. Then $\{w \in A \mid z \leq w\} \subseteq B$. Now, if $x \in B$, and $y \geq x$, then either $y \geq z$ so $y \in B$, or $y \leq z$ so $y \in [x, z]$ and again $y \in B$, since $B$ is full.) This justifies the assertion made earlier that if directed arcs $A$ and $A'$ are cofinal, then $A \cap A'$ is a common final segment.

Consider now an interval $[x, b)$. We consider $[x, b)$ as a directed arc so that $x$ is a minimum. If $y \in [x, b)$, then $[y, b)$ is a final segment. Note, more generally, that if $x, y, b$ are distinct points of $T$, then $[x, b)$ and $[y, b)$ are cofinal if and only if $[x, b) \cap [y, b) \neq \emptyset$ (since if $z \in [x, b) \cap [y, b)$ then $[z, b)$ is a common final segment).

Suppose now, that $A$ is any directed arc, $x \in A$, $b \in T$ and $[x, b)$ is cofinal with $A$. Then $[x, b)$ is a final segment of $A$. (To see this, choose $y \in A$ such that $[y, b)$ is a final segment of both $A$ and $[x, b)$. Now $[x, b) \subseteq [x, y] \cup [y, b) \subseteq A$. Thus $[x, a)$ is subarc of $A$ cofinal with $A$, and so, as observed above, $[x, b)$ is a final segment of $A$.)

**Lemma 3.12 :** *Suppose $A$ is an endless directed arc. If $x, y \in A$ and $b \in T$, then $[x, b)$ is a final segment of $A$ if and only if $[y, b)$ is.*

**Proof :** Without loss of generality, $x < y$.

Suppose $[x, b)$ is a final segment. Now $y \in [x, b)$ so $[y, b)$ is a final segment of $[x, b)$ and so also of $A$.

Conversely, suppose that $[y, b)$ is a final segment. Since $A$ is endless, we can choose some $z \in (y, b)$. Now $x < y < z$ so $xyz$. Also $y \neq b$ and $\neg byz$, so $xyb$. Thus $y \in [x, b)$. In particular, the intervals $[x, b)$ and $[y, b)$ intersect and are thus cofinal. It follows from the discussion immediately before the lemma that $[x, b)$ is a final segment. $\diamond$

**Definition :** Given a directed arc $(A, <)$, and $b \in T$, we say that $b$ is a *supremum* of $A$ if either $b = \max(A)$ or if $A$ is endless, and $[x, b)$ is a final segment of $A$ for some (and hence all) $x \in A$.

Note that if $a$ is a minimum of $A$, then $A = [a, b]$ or $A = [a, b)$ depending on whether or not $b \in A$. More generally, if both $(A, <)$ and $(A, >)$ have suprema, then $A$ must be an interval. We see:

**Lemma 3.13 :** *A pretree is complete if and only if every directed arc has a supremum.*
$$\diamondsuit$$

Note that in general, a directed arc might have more than one supremum. However in a dense pretree or a median pretree, a supremum, if it exists, will be unique.

We now return to our pretree $T \sqcup P$ satisfying the conditions (P1)–(P3) described earlier.

By (P1), we see that if $p, q \in P$, then $(p, q) \subseteq T$. If $(p, q) = \emptyset$, then given $x, y \in T$, we have $\neg pyx$, so $xyp \Leftrightarrow xyq$, and so, by (P3) we see that $p = q$. In other words, no pair of points of $P$ are adjacent in in $T \sqcup P$.

To each $x \in T$, we associate the unique point $\bar{x} \in P$ such that $\mathrm{flow}(x) = \mathrm{flow}(\bar{x})|T$. Note that $x$ and $\bar{x}$ are adjacent, i.e. $(x, \bar{x}) = \emptyset$.

**Lemma 3.14 :** *If $x \in T$, and $a \in T$, then $[a, \bar{x}) = [a, x]$.*

**Proof :** We have already noted that $(x, \bar{x}) = \emptyset$, so $[x, \bar{x}) = \{x\} = [x, x]$. So suppose that $a \neq x$. Now, $ax.\bar{x}$, so $ax\bar{x}$ and so $x \in (a, x)$. Thus $[a, \bar{x}) = [a, x] \cup [x, \bar{x}) = [a, x] \cup \{x\} = [a, x]$. $\diamondsuit$

**Lemma 3.15 :** *If $x, y \in T$ and $x \neq y$, then $(\bar{x}, \bar{y}) = [x, y]$.*

**Proof :** Now $(\bar{x}, \bar{y}) \neq \emptyset$, so choose any $a \in (\bar{x}, \bar{y})$. We see that $x \in [a, x] = [a, \bar{x}) \subseteq (\bar{x}, \bar{y})$. Thus, $(\bar{x}, \bar{y}) = (\bar{x}, x] \cup [x, \bar{y}) = \{x\} \cup [x, y] = [x, y]$. $\diamondsuit$

**Lemma 3.16 :** *Suppose that $A \subseteq T$ is an endless directed arc with minimum $a \in A$. Then there is some $m \in P$, such that $A = [a, m)$.*

**Proof :** By (P2) there is some $m \in P$ such that $\mathrm{flow}(A) = \mathrm{flow}(m)|T$. Now, using Lemmas 3.6, 3.11 and 3.14 respectively, we see that $\mathrm{flow}(A) = \mathrm{flow}(m)|T = \mathrm{flow}([a, m])|T = \mathrm{flow}([a, m] \cap T) = \mathrm{flow}([a, m))$. Thus, by Lemma 3.10, $A$ and $[a, m)$ are cofinal. Since $a$ is the minimum of $A$ we see that $A = [a, m)$. $\diamondsuit$

Now if $p, q, r \in P$, by Lemma 2.7, we have $[p, q] \subseteq [p, r] \cup [r, q]$. So, intersecting with $T$, we get $(p, q) \subseteq (p, r) \cup (r, q)$. Similarly, we have $(q, r) \subseteq (p, q) \cup (p, r)$ and so $(p, r) \subseteq (p, q) \Rightarrow (q, r) \subseteq (p, q)$. We deduce:

**Lemma 3.17 :** *If $p, q, r \in P$, then the following are equivalent:*
*(1) $(p, q) = (p, r) \cup (r, q)$,*
*(2) $(p, r) \subseteq (p, q)$,*
*(3) $(q, r) \subseteq (p, q)$.* $\diamondsuit$

Note that if $(p, q) = (p, r) \cup (r, q)$, then any point of $(p, r) \cap (r, q)$ must be a median of $p, q, r$. We see that, in such a case, $(p, r) \cap (r, q)$ can contain at most one point. In fact, if $x \in (p, r)$ and $y \in (r, q)$ then either $pxyq$, or $x = y = \mathrm{med}(p, q, r)$.

Note also that if $(p,r) \cap (r,q) = \emptyset$ then it necessarily follows that $(p,q) = (p,r) \cup (r,q)$. We write $\bar{T} = \{\bar{x} \mid x \in T\} \subseteq P$.

We are now ready to define a new pretree structure on $P$ as follows. We shall denote the new betweenness relation by $\langle pqr \rangle$ for $p,q,r \in P$.

We write $\langle prq \rangle$ if $p,q,r$ are distinct points of $P$, and either $(\exists x \in T)(r = \bar{x} \wedge pxq)$ or $(r \notin \bar{T}) \wedge ((p,r) \cap (r,q) = \emptyset)$.

Applying Lemma 3.15, we see that if $x,y,z \in T$, then $\langle \bar{x}\bar{y}\bar{z} \rangle \Leftrightarrow xyz$, so that $\bar{T}$ is an isomorphic copy of $T$.

**Lemma 3.18 :**  *Suppose $p,q,r \in P$ are distinct. Then:*

*(1) $(p,r) \cap (r,q) = \emptyset \Rightarrow \langle prq \rangle$.*

*(2) $\langle prq \rangle \Rightarrow (p,q) = (p,r) \cup (r,q)$.*

**Proof :** If $(p,r) \cap (r,q) = \emptyset$, then it follows from Lemma 3.14, that $r \notin \bar{T}$, so $\langle prq \rangle$ by definition. Suppose $\langle prq \rangle$. Then if $r = \bar{x}$ for some $x \in T$, then by definition, $x \in (p,q)$, so $(p,q) = (p,x] \cup [x,q) = (p,r) \cup (r,q)$ by Lemma 3.14. If $r \notin \bar{T}$, then $(p,r) \cap (r,q) = \emptyset$ so from the earlier discussion we have again that $(p,q) = (p,r) \cup (r,q)$ as required. $\diamondsuit$

When referring to the new structure, $(P, \langle \rangle)$, we shall speak of "$\langle \rangle$-arcs" and "$\langle \rangle$-medians" etc. We shall use the notation $\langle [p,q] \rangle$ or $\langle [p,q) \rangle$ etc. for $\langle \rangle$-intervals in $(P, \langle \rangle)$.

We are now ready to prove the main result of this section:

**Theorem 3.19 :**  $(P, \langle \rangle)$ *is a complete median pretree.*

**Proof :** We first need to verify the axioms of a pretree.

(T0): This holds by hypothesis.

(T1): This is immediate from the construction.

(T2): Suppose $\langle prq \rangle \wedge \langle pqr \rangle$. Now, if $q = \bar{x}$ and $r = \bar{y}$ for $x,y \in T$, we get $px\bar{y} \wedge py\bar{x}$ so $pxy \wedge pyx$, contradicting (T2) for $T \sqcup P$. So suppose $r \notin \bar{T}$. ¿From $\langle pqr \rangle$, and Lemma 3.18, we get $(q,r) \subseteq (p,r)$, and from $\langle prq \rangle$ we get $(p,r) \cap (q,r) = \emptyset$. Thus $(q,r) = \emptyset$ and so $q = r$, contradicting the fact that $p,q,r$ must be distinct.

(T3): Suppose $\langle prq \rangle$ and $s \neq r$. If $r = \bar{x}$, for $x \in T$, then $pxq$ so $pxs \vee qxs$, and so $\langle prs \rangle \vee \langle qrs \rangle$ as required. Thus, suppose $r \notin \bar{T}$, so that $(p,r) \cap (q,r) = \emptyset$. If $\neg \langle prs \rangle \wedge \neg \langle qrs \rangle$ then we can find $x \in (p,r) \cap (r,s)$ and $y \in (q,r) \cap (r,s)$. We must have $x \neq y$. Also $rxs \wedge ryx$, so without loss of generality, $rxys$. But $ryq$, so $rxyq$. Thus $x \in (q,r)$ contradicting $(p,r) \cap (q,r) = \emptyset$. We must therefore have $\langle prs \rangle \vee \langle qrs \rangle$ as required.

This shows that $(P, \langle \rangle)$ is a pretree. The next job is prove the existence of medians. To this end, suppose $p,q,r \in P$. We can suppose that these are all distinct.

Now if $(p,r) \cap (r,q) = \emptyset$, we would have $\langle prq \rangle$ so $r$ would be $\langle \rangle$-median of $p,q,r$. We can thus suppose that $(p,r) \cap (r,q) \neq \emptyset$. Similarly, we suppose $(r,q) \cap (q,p) \neq \emptyset$ and $(q,p) \cap (p,r) \neq \emptyset$.

Suppose $x \in (q, p) \cap (p, r) \cap (r, q)$. Now if $\bar{x} = r$, then, since $pxq$, we would have $\langle prq \rangle$, and so $r$ would be a $\langle \rangle$-median. We can thus suppose that $\bar{x} \neq r$, and similarly, that $\bar{x} \neq p$ and $\bar{x} \neq q$. It then follows that $\bar{x}$ is a $\langle \rangle$-median of $p, q, r$. We can thus assume that $(q, p) \cap (p, r) \cap (r, q) = \emptyset$.

We now define a flow on $T$, denoted by $xy.s$ for $x, y \in T$, by the condition that

$$xy.s \Leftrightarrow (xyq \wedge xyp) \vee (xyp \wedge xyr) \vee (xyr \wedge xyq).$$

In other words, at least two of the conditions $xyp$, $xyq$ and $xyr$ should hold. We first need to verify the axioms of a flow. We take $x, y, z$ to be arbitrary elements of $T$.

(F1): If $xy.s \wedge yx.s$, then, without loss of generality, $xyp \wedge yxp$ which is impossible.

(F2): Suppose $xzy$. By (T3) for $T \sqcup P$, we have $(xzp \vee yzp) \wedge (xzq \vee yzq) \wedge (xzr \vee yzr)$, which clearly implies $xz.s \vee yz.s$.

(F3): Suppose $xy.s$ and $z \neq y$. Without loss of generality, we have $xyp \wedge xyq$. If $\neg xyz$, then $zyp \wedge zyq$ so $zy.s$.

This shows that what we have defined is indeed a flow, and so, by (P2), there is some $s \in P$ such that $xys \Leftrightarrow xy.s$ for $x, y \in T$. We claim that $s$ is a $\langle \rangle$-median.

First, we show that $(p, s) \subseteq (p, q)$. To see this, suppose $pxs$. Thus for all $y \in T \setminus \{x\}$ we have (by (T3) for $T \sqcup P$) $yxp \vee yxs$ which implies $xyp \vee xyq$. But directly from the way in which the structure on $T \sqcup P$ was defined, namely (C3), (or indirectly from properties (P1)–(P3)), we see that $pxq$ as required.

Now we can permute $p$, $q$ and $r$ as we please, so we see that $(p, s) \subseteq (q, p) \cap (p, r)$, $(q, s) \subseteq (r, q) \cap (q, p)$ and $(r, s) \subseteq (p, r) \cap (r, q)$. Since $(q, p) \cap (p, r) \cap (r, p) = \emptyset$, it follows that $(p, s)$, $(q, s)$ and $(r, s)$ are mutually disjoint. It now follows from earlier assumptions that $s$ must be distinct from $p$, $q$ and $r$. We thus have $\langle qsp \rangle \wedge \langle psr \rangle \wedge \langle rsq \rangle$. In other words, $s$ is a $\langle \rangle$-median of $p, q, r$.

In summary, we have so far shown that $(P, \langle \rangle)$ is a median pretree. It remains to show that it is complete. By Lemma 3.13, this is equivalent to showing that every directed arc has a supremum. Before we start, we make the following observation.

Suppose $p, q, r, s \in P$ and $\langle pqrs \rangle$. Then we claim that $(p, q) \cap (r, s) = \emptyset$. To see this, suppose $x \in (p, q) \cap (r, s)$. Now, from $\langle pqr \rangle$, we have $(p, q) \subseteq (p, r)$, so $x \in (p, r) \cap (r, s)$. Since $\langle prs \rangle$, we must have $r = \bar{x}$. But we can similarly deduce that $q = \bar{x}$, so we arrive at the contradiction that $q = r$. This proves the claim.

We are now ready for the final task, of showing that a directed arc as a supremum.

Suppose then that $(B, \prec)$ is a directed $\langle \rangle$-arc. We can suppose that it is $\prec$-endless. We can also assume, for convenience, that $B$ has a minimum of the form $\bar{a}$ for some $a \in T$. (Note that $B$ must meet $\bar{T}$, since if $p, q \in T$ are any distinct pair of points, we can choose $a \in (p, q)$. Thus $paq$, so $\langle p\bar{a}q \rangle$ and so $\bar{a} \in B$. We can now replace $B$ by the final segment $\{p \in B \mid \bar{a} \preceq p\}$.) Note that the order on $B$ can now be described by $p \prec q \Leftrightarrow \langle \bar{a}pq \rangle \vee (p = \bar{a} \wedge q \neq \bar{a})$. By Lemma 3.14, for any $p \in P$, we have $(\bar{a}, p) = [a, q]$. Note that if $p \preceq q$ then $[a, p] \subseteq [a, q]$.

Now let $A = \bigcup_{p \in B} [a, p]$. We can think of each interval $[a, p]$ as a directed arc with minimum $a$. Also this sequence is a nested, so we see that $A$ is an arc, which we may order so that $a$ is a minimum. We denote this order by $<$. Thus if $x, y \in A$, then $x < y \Leftrightarrow \bar{a}xy$.

Note that if $p, q \in B$ then $(p, q) \subseteq A$. (We can assume that $p \neq \bar{a}$ and that $p \preceq q$. Thus $\langle \bar{a}pq \rangle$, so $(p, q) \subseteq (\bar{a}, q) \subseteq A$.)

Suppose that $A$ has a maximum, say $b$. Thus $A = [a, b]$. Now, from the definition of $A$, there is some $p \in B$ such that $b \in [a, p)$. Thus $A \subseteq [a, p)$ and so $A = [a, p)$. Since $B$ is endless, we can find $q, r, s \in B$, with $p \prec q \prec r \prec s$. Thus $\langle \bar{a}qrs \rangle$, so $(\bar{a}, q) \cap (r, s) = \emptyset$. But $A = (\bar{a}, p) = (\bar{a}, q)$, and so $(r, s) = \emptyset$, giving us the contradiction $r = s$.

It follows that $A$ must be endless. Thus, by Lemma 3.16, there is some $m \in P$ such that $A = [a, m)$.

First we observe that $m \notin B$. Clearly, $m \neq \bar{a}$, since we know that $A$ is endless, and $[a, \bar{a}) = \{a\}$. For if $m \in B$, we could choose $p, q \in B$ with $m \prec p \prec q$. Now $\langle \bar{a}mpq \rangle$, so $(\bar{a}, m) \cap (p, q) = \emptyset$. But $(p, q) \subseteq A = (\bar{a}, m)$, so $(p, q) = \emptyset$, and we get the contradiction $p = q$. Thus $m \notin B$ as claimed.

We now claim that $B$ is equal to the $\langle \rangle$-interval $\langle [\bar{a}, m) \rangle$. This shows that $m$ is a $\langle \rangle$-supremum of $B$, and so, we can deduce that $(P, \langle \rangle)$ is complete.

First we show that $\langle [\bar{a}, m) \rangle \subseteq B$. Suppose $p \in \langle [\bar{a}, m) \rangle$. We can assume that $p \neq \bar{a}$, so $\langle \bar{a}pm \rangle$.

Consider, first, the case where $p = \bar{x}$ for some $x \in T$. Then $\bar{a}xm$, so $x \in (\bar{a}, m) = A$. ¿From the definition of $A$, there is some $q \in B$ such that $x \in (\bar{a}, q)$. Thus $\bar{a}xq$ so $\langle \bar{a}pq \rangle$. Since $\bar{a}, q \in B$ and $B$ is full, we see that $p \in B$ as required.

Now consider the case where $p \notin \bar{T}$. Since $\langle \bar{a}pm \rangle$ we have $(\bar{a}, p) \cap (p, m) = \emptyset$. Also, $(a, m) = (\bar{a}, p) \cup (p, m)$. We see that if $x \in (\bar{a}, p)$ and $y \in (p, m)$, then $\bar{a}xym$. Now choose any $y \in (p, m)$. Now $y \in (a, m) = A$, so there is some $q \in B$ such that $y \in (\bar{a}, q)$, i.e. $\bar{a}yq$. Suppose that $x$ is some point of $(\bar{a}, p)$. Then $\bar{a}xy$, so $\bar{a}xyq$. Thus $xyq \wedge xym$. It follows that $\neg qxm$ (since if $qxm$ then $yxq \vee yxm$). Also, $x \in (\bar{a}, p)$ so $x \notin (p, m)$, i.e. $\neg psm$. It follows that $x \notin (p, q)$ (since if $pxq$ then $pxm \vee qxm$). In other words, we have shown that if $x \in (\bar{a}, p)$ then $x \notin (p, q)$. Thus $(\bar{a}, p) \cap (p, q) = \emptyset$ so $\langle \bar{a}pq \rangle$. As before, we see that $p \in B$ as required.

Finally, we want to show that $B \subseteq \langle [\bar{a}, m) \rangle$. Thus suppose $p \in B$, and that $p \neq \bar{a}$. We want to deduce that $\langle \bar{a}pm \rangle$.

Suppose first that $p = \bar{x}$ for some $x \in T$. By Lemma 3.15, $x \in (\bar{a}, \bar{x}) \subseteq A = [a, m)$. Thus $\bar{a}xm$ so $\langle \bar{a}pm \rangle$ as required.

Thus suppose that $p \notin \bar{T}$. Now $(\bar{a}, p) \subseteq A = (\bar{a}, m)$ so, by Lemma 3.17, $(\bar{a}, m) = (\bar{a}, p) \cup (p, m)$. We want to show that $(\bar{a}, p) \cap (p, m) = \emptyset$. Suppose then that $x \in (\bar{a}, p) \cap (p, m)$. Since $A = (\bar{a}, m)$ is endless, we can find $y \in (\bar{a}, m)$ with $\bar{a}xy$. Now there is some $q \in P$ with $p \prec q$ and with $y \in (\bar{a}, q)$. In summary, we have $\bar{a}xym \wedge \bar{a}xyq \wedge pxm$. Now $pxm \wedge xym$ so $pxy$. Also $xyq$ so $pxq$. Thus $x \in (\bar{a}, p) \cap (p, q)$ contradicting the fact that $\langle \bar{a}pq \rangle$. We conclude that $(\bar{a}, p) \cap (p, m) = \emptyset$. We showed earlier that $m \notin B$, so $m \neq p$. Thus $\langle \bar{a}pm \rangle$ as required.

This finally proves the theorem. ◇

We have already observed that $(P, \langle \rangle)$ contains an isomorphic copy of $T$, namely $\bar{T}$, and can thus be thought of as a kind of completion of $T$. However, as we have already mentioned, the pretree $(P, \langle \rangle)$ is somewhat larger than one might ideally hope for in a completion. There are ways in which one can cut $(P, \langle \rangle)$ down to size if desired. These will be clearer if we give a couple of simple examples.

The case of a finite pretree has already been alluded to indirectly in the proof of Lemma 2.5. Here, completeness is not an issue, but the existence of medians is. Suppose we embed our finite pretree, $T$, in a finite simplicial tree, $\tau$, such that $\tau = \text{hull}(T)$, i.e. every terminal vertex of $\tau$ lies in $T$. We can identify $T$ with $\bar{T}$, and there is a natural bijective correspondence between $P \setminus \bar{T}$ and the set of connected components of $\tau \setminus T$. A component of $\tau \setminus T$ is of one of two types: either it is homeomorphic to a real interval — or it isn't. Now in the latter case (as in the proof of Lemma 2.5), we can assume that it has the form of a wedge of at least three intervals connected at a common endpoint. We can identify each point of $P$ with a point in one of these components. In the former case, we can choose any point, to give a new vertex of degree 2. In the latter case we take the vertex at connecting point of our wedge of arcs. Clearly, vertices of the second type (degree $\geq 3$) are essential if we are to obtain a median pretree. However, those of the first type (degree $= 2$) can be thought of as artifacts of the construction. We would loose nothing by deleting them. In fact, we would achieve the probably desirable result that the completion finite median pretree just gives us back our original pretree.

At the other extreme, we might consider examples of dense linear orders. In this case, the existence of medians is not an issue, whereas completeness is. Consider, for example, the (already complete) closed real interval $[0, 1]$. We can describe $(P, \langle \rangle)$ as $([0, 1] \times \{-, 0, +\}) \setminus \{(0, -), (1, +)\}$ given the lexicographic order (where $- < 0 < +$). Thus, if $x \in T = [0, 1]$, then $\bar{x} = (x, 0)$. The points $(x, -)$ and $(x, +)$ arise respectively as $\text{flow}([0, x), <)$ and $\text{flow}([1, x), >)$. As another example, consider the totally ordered set $T = (0, 1) \cap \mathbf{Q}$, where $(0, 1)$ denotes the open real interval, and $\mathbf{Q}$ the rational numbers. In this case, the completion $(P, \langle \rangle)$ can be identified as a subset of the case described above, namely $([0, 1] \times \{0\}) \cup ((0, 1) \cap \mathbf{Q}) \times \{-, 0, +\}$. In these examples, the points of the form $(x, \pm)$ are superfluous for the purposes of achieving completeness, and could be safely omitted.

Essentially these are the only types of unnecessary points, though the general case is a bit more complicated to describe. It will be clarified by a classification of flows, which we now go on to describe. This discussion will also be relevant to later sections.

We have already seen examples of flows of the type $\text{flow}(a)$ for a point $a \in T$ and $\text{flow}(A)$, where $A$ is an endless directed arc. There is a third type already alluded to in the proof of Theorem 3.19, and our discussion of finite trees, which we now go on to describe.

Suppose $F \subseteq T$ is (for the moment) any subset of $T$. Given $y \in F$, let $N(y) = \{y\} \cup \{x \in F \mid (\exists z \in F)(xyz)\}$.

**Lemma 3.20 :**   *If $x \in N(y) \setminus \{y\}$ and $z \in T \setminus N(y)$, then $xyz$.*

**Proof :**   By hypothesis, there is some $w \in F$ with $xyw$. Since $z \notin N(y)$, we have $\neg zyw \wedge z \neq y$. Thus $xyz$.                                                        $\diamond$

**Lemma 3.21 :**   *$N(y)$ is full.*

Suppose, for contradiction, that $a, b \in N(y)$, $x \in T \setminus N(y)$ and $axb$. If $y \neq a, b$ then by Lemma 3.20, we have $ayx$ and $byx$. But since $axb$, we get the contradiction $yxy$. If $y = b$

so $a \neq y$, we get *ayx* i.e. *abx* which contradicts *axb*.                    $\diamondsuit$

Recall the definition from Section 2:

**Definition :** A subset $F \subseteq T$ is *null* if $(\forall x, y, z \in T)(\neg xyz)$.

Now if $F$ is null then clearly $N(y) \cap F = \{y\}$. In fact, if $x \in N(y)$, then $[x, y] \subseteq N(y)$ and so $[x, y] \cap N(y) = \{y\}$. Also, if $x \neq y$ then $N(x) \cap N(y) = \emptyset$.

**Definition :** A null subset of $F$ is a *maximal null* subset if it is not contained in any strictly larger null subset.

Assuming that $T$ is not a singleton, then a maximal null subset must contain at least two elements (since any 2-element set is null). Note that if $F$ is maximal null, and $x \in T$, then either $(\exists y \in F)(x \in N(y))$ or else, $(\exists y, z \in F)(yxz)$ (otherwise $F \cup \{x\}$ would be a larger null subset).

**Definition :** A *star* is a maximal null subset which is full.

¿From the previous observations, we see that if $F$ is a star, then $T$ can be expressed as a disjoint union: $T = \bigsqcup_{y \in F} N(y)$.

**Lemma 3.22 :** *If $F$ is a star, and $y \in F$, then for $x \in T$, we have $x \in N(y) \Leftrightarrow [x, y] \cap F = \{y\}$.*

**Proof :** We have already shown the implication $(\Rightarrow)$. For $(\Leftarrow)$ suppose $x \notin N(y)$. Then $x \in N(z)$ for some $z \neq y$. Thus $z \in [x, y) \cap F$, so $[x, y] \cap F \neq \{y\}$.                    $\diamondsuit$

We define the map $h = h_F : T \longrightarrow F$ by the condition that $x \in N(h(x))$. In other words, $[x, h(x)] \cap F = \{h(x)\}$.

**Lemma 3.23 :** *If $x, y \in T$, then $[x, y] \subseteq [x, h(x)] \cup [y, h(y)]$. Also, if $h(x) \neq h(y)$, then $[x, y] = [x, h(x)] \cup [y, h(y)]$.*

**Proof :** By Lemma 2.7, we have $[x, y] \subseteq [x, h(x)] \cup [h(x), h(y)] \cup [h(y), y] = [x, h(x)] \cup [y, h(y)]$.

Suppose $h(x) \neq h(y)$. If $z \in [y, h(y)]$, then $z \notin N(x)$, so by Lemma 3.20, we have $h(x) \in [x, z]$. Also by Lemma 3.20, we have $h(x) \in [x, h(y)]$ and $h(y) \in [y, h(x)]$. Putting these facts together, we see that $z \in [x, y]$. Thus $[y, h(y)] \subseteq [x, y]$. Similarly $[x, h(x)] \subseteq [x, y]$.                    $\diamondsuit$

Given a star, $F$, we define a flow, $\mathrm{flow}(F) = \mathrm{flow}_T(F)$ on $T$, which we denote by $xy.F$ for $x, y \in T$ by $xy.F \Leftrightarrow y \in (x, h(x)]$.

**Lemma 3.24 :** *If $F$ is a star, then $\mathrm{flow}(F)$ is a flow on $T$.*

**Proof :**

(F1): If $xy.F \wedge yx.F$, we get $x, y \in [h(x), h(y)] = \{h(x), h(y)\}$, so $x, y \in F$ which is impossible.

(F2): Suppose $xzy$. Then $z \in (x, y) = (x, h(x)] \cup (y, h(y)]$ by Lemma 3.23, so $zy.F \vee zx.F$ as required.

(F3): Suppose $xy.F \wedge z \neq y$. If $h(z) \neq h(x)$, then by Lemma 3.20, we have $xyz$. So suppose $h(z) = h(x)$. Now $y \in (x, h(x)] \subseteq (x, z) \cup [z, h(x)]$, so either $xyz$, or $y \in [z, h(x)] = [z, h(z)]$. But $z \neq y$, so in the latter case, we get $zy.F$. $\diamondsuit$

We shall show (Proposition 3.25) that every flow on a pretree, $T$, is of one of the three types we have already described; namely a flow towards a point of $T$, a flow derived from an endless directed arc, or a flow derived from a star.

In order to describe this classification, we shall refer to the pretree structure on $T \sqcup P$. Recall that every flow on $T$ has the form flow$(p)|T$ for some $p \in P$. In fact it will be enough for us to consider the tree $T \sqcup \{p\}$ (see Lemma 3.2).

Associated to any point $x \in T$ is an arc $[x, p) \subseteq T$. Clearly, any two such arcs must be either cofinal or disjoint. Moreover, if $[x, p) \cap [y, p) = \emptyset$, then $[x, y] = [x, p) \cup [y, p)$. We define a relation $\sim$ on $T$ by $x \sim y \Leftrightarrow [x, p) \cap [y, p) \neq \emptyset$. We see easily that this is an equivalence relation. We write $\mathcal{A} = T/\sim$. Note that if $x \in C \in \mathcal{A}$, then $[x, p) \subseteq C$. We see that $C$ must be full (since if $x, y \in C$ then $[x, y] \subseteq [x, p) \cup [x, y) \subseteq C$). In fact, the same argument shows that the union of any set of equivalence classes is full.

We can split $\mathcal{A}$ as a disjoint union, $\mathcal{A} = \mathcal{E} \sqcup \mathcal{F}$, so that if $x \in \bigcup \mathcal{E}$ then $[x, p)$ is endless, whereas if $x \in \bigcup \mathcal{F}$, then $[x, p)$ has a maximum.

If $x \in \bigcup \mathcal{F}$, we write the maximum of $[x, p)$ as $h(x)$. Thus $[x, p) = [x, h(x)]$. Clearly, if $x, y \in \bigcup \mathcal{F}$, then $x \sim y \Leftrightarrow h(x) = h(y)$. Let $F = \{h(x) \mid x \in \bigcup \mathcal{F}\}$. Clearly, if $x \in F$, then $h(x) = x$. More generally, if $x \in \bigcup \mathcal{F}$, then $[x, h(x)] \cap F = \{h(x)\}$. Note that $F$ is a transversal of $\mathcal{F}$.

Now, if $y, z \in F$, then $[y, z] \subseteq [y, p) \cup [z, p) = \{y\} \cup \{z\}$ and so $(y, z) = \emptyset$. In other words, $F$ is null and full.

Suppose that $\mathcal{E} \neq \emptyset$. Choose $x \in \bigcup \mathcal{E}$, so that $[x, p)$ is endless. Applying lemma 3.11, we see that flow$(p)|T =$ flow$([x, p])|T =$ flow$([x, p] \cap T) =$ flow$([x, p))$. Thus the flow has the form flow$(A)$ where $A$ is an endless directed arc.

Suppose now that $\mathcal{E} = \emptyset$. Let us first consider the case where $|\mathcal{F}| = 1$, so that $F = \{a\}$ for some $a \in T$. Given any $x \in T$, we have $[x, p) = [x, a]$. Thus, if $x, y \in T$, then $xyp \Leftrightarrow y \in (x, p) = (x, a]$, and so flow$(p)|T =$ flow$(a)$.

Finally suppose that $\mathcal{E} = \emptyset$ and $|\mathcal{F}| \geq 2$. We know that $F$ is null and full. Also if $x \in T$, then $[x, p) = [x, h(x)]$ and $[x, h(x)] \cap F = \{h(x)\}$. Note that if $y \in F \setminus \{h(x)\}$ then $h(x) \in [x, p) = [x, y] \cup [y, p) = [x, y]$. If $x \notin F$, then $x \neq h(x)$ and we have $h(x) \in (x, y)$. Thus $F \cup \{x\}$ is not null. This shows that $F$ is, in this case, a maximal null subset. Thus $F$ is a star. Using Lemma 3.22, we see that the definitions of the map $h$ agrees with that given earlier in defining flow$(F)$. Now, given $x, y \in T$, we have $xyp \Leftrightarrow y \in (x, p) = (x, h(x)] \Leftrightarrow xy.F$. Thus, flow$(p)|T =$ flow$(F)$.

We have shown:

**Proposition 3.25 :** *Any flow on a pretree, $T$, has one of the forms* flow$(a)$, flow$(A)$ *or* flow$(F)$, *where $a \in T$, $A$ is an endless directed arc, and $F$ is a star.* $\diamondsuit$

Given $p \in P$, we can associate a pair of cardinals, $E(p) = (|\mathcal{E}|, |\mathcal{F}|)$. Given cardinals $m$ and $n$, write $P(m, n) = \{p \in P \mid E(p) = (m, n)\}$. Thus, $\bar{T} = E(0, 1)$. Those elements of $(P, \langle\rangle)$ which appeared to be superfluous in our earlier examples were either in $P(0, 2)$ or $P(1, 1)$. In fact $(P \setminus (P(0, 2) \cup P(1, 1)), \langle\rangle)$ is a complete median pretree.

We shall return to this point later. First we discuss further the structure of $(P, \langle\rangle)$. We want to consider when two points of $(P, \langle\rangle)$ are $\langle\rangle$-adjacent. This is relevant to the matter raised in the preceding paragraph, and will also be referred to later in the paper.

First, suppose that $F$ is a star, and that $p = $ flow$(F)$. Now, if $x \in F$, we have $(x, p) \neq \emptyset$ (from the construction), and so $\bar{x}$ and $p$ are adjacent. Moreover, if $x, y \in F$ are distinct, then then certainly $(x, p) \cap (y, p) = \emptyset$, and so $\langle \bar{x} p \bar{y} \rangle$.

We now claim that we cannot have two elements of $\bar{T}$ being $\langle\rangle$-adjacent. For suppose $x, y \in T$ and $\bar{x}, \bar{y}$ are $\langle\rangle$-adjacent. Now, $\{x, y\} \subseteq T$ is null and full. Also, it is maximal null, for suppose that $z \in T \setminus \{x, y\}$. Let $m = \mathrm{med}(\bar{x}, \bar{y}, \bar{z})$. If $\{x, y, z\}$ were null, then $m$ would be distinct from $\bar{x}$ and $y$, so $\langle \bar{x} m \bar{y} \rangle$. We conclude that $F = \{x, y\}$ is a star. Let $p = $ flow$(F)$. ¿From the preceding paragraph, we see that $\langle \bar{x} p \bar{y} \rangle$. This proves the claim.

Given $a \in T$, write $L(a) = \{p \in P \mid (a, p) = \emptyset\}$. Thus, $p \in L(a) \Leftrightarrow (\bar{a}, p) = \{a\}$. Note that $L(a) \cap \bar{T} = \emptyset$.

**Lemma 3.26 :** *$L(a)$ is the set of points of $P$ which are $\langle\rangle$-adjacent to $\bar{a}$.*

**Proof :** Suppose $p \in L(a)$, and $\langle \bar{a} r p \rangle$. Now $\{a\} = (\bar{a}, p) = (\bar{a}, r) \cup (r, p)$, so $(r, p) = \{a\}$. In particular, $rap$. Since $\bar{a} \neq r, p$ we deduce $\langle r \bar{a} p \rangle$ contradicting $\langle \bar{a} r p \rangle$.

Conversely, suppose there is some point $x \in (a, p)$. Now, from the earlier discussion, we know that $\bar{x}$ is not $\langle\rangle$-adjacent to $\bar{a}$, so we can assume that $p \neq \bar{x}$. Thus $\langle \bar{a} \bar{x} p \rangle$ and so $p$ is not $\langle\rangle$-adjacent to $\bar{a}$. $\diamondsuit$

**Lemma 3.27 :** *Suppose $p$ and $q$ are distinct points of $L(a)$. Then $\langle p \bar{a} q \rangle$. Moreover, if $r \in P$ with $\langle prq \rangle$, then $r = \bar{a}$.*

**Proof :** We know that $p, q, \bar{a}$ must have a $\langle\rangle$-median. Since $p$ and $q$ are adjacent to $\bar{a}$, the only possibility for this median is $\bar{a}$ itself. Thus $\langle p \bar{a} q \rangle$.

Suppose $\langle prq \rangle$. If $r \neq \bar{a}$ we get the contradiction $\langle p \bar{a} r q \rangle \vee \langle pr \bar{a} q \rangle$. $\diamondsuit$

Note that it follows from Lemma 3.27 that $L(a) \cup \{a\}$ is $\langle\rangle$-full.

**Lemma 3.28 :** *Two points $p, q \in P$ are $\langle\rangle$-adjacent if and only if $(\exists a \in T)(p = \bar{a} \wedge q \in L(a)) \vee (\exists a \in T)(q = \bar{a} \wedge p \in L(a))$.*

**Proof :** The "if" bit has already been done by Lemma 3.26. For the "only if" bit, suppose that $p$ and $q$ are $\langle\rangle$-adjacent. Now $(p, q) \neq \emptyset$, so there is some $a \in (p, q)$. Since $\neg \langle p \bar{a} q \rangle$, without loss of generality, we must have $q = \bar{a}$. Thus, by Lemma 3.27, $p \in L(a)$. $\diamondsuit$

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The picture we have given of $(P, \langle \rangle)$ will be clarified if we give an explicit description of the set $L(a)$ in terms of flows on $T$. Our analysis can be viewed as a generalisation and extension of that concerned with the classification of flows on $T$. (The latter can be viewed as dealing with a terminal vertex, $p$, in the pretree $T \sqcup \{p\}$.) For this reason, we adopt similar notation.

Suppose $a \in T$. There are two natural equivalence relations we can put on $T \setminus \{a\}$. The first, $\sim$, can be defined by $x \sim y \Leftrightarrow [x, a) \cap [y, a) \neq \emptyset$. The second, $\approx$, can be defined by $x \approx y \Leftrightarrow \neg xay$. It is easily verified that these are indeed equivalence relations. Clearly, $x \sim y \Rightarrow x \approx y$. If $x \approx y \wedge x \not\sim y$, then $[x, y] = [x, a) \sqcup (a, y]$. If $x \not\approx y$ then $[x, y] = [x, a) \sqcup \{a\} \sqcup (a, y]$. (The relations $\sim$ and $\approx$ are described in some detail in [AN], where they are referred to respectively as "$R_a$" and "$R_a^*$ relations".)

As before, we write $\mathcal{A} = (T \setminus \{a\})/\sim$, and partition $\mathcal{A}$ as $\mathcal{E} \sqcup \mathcal{F}$ according to whether or not the arcs $[x, a)$ are endless. We can also define a transversal, $F$ to $\mathcal{F}$, and define a map $h : \bigcup \mathcal{F} \longrightarrow F$ such that if $x \in \bigcup \mathcal{F}$, then $[x, a) = [x, h(x)]$ and $[x, h(x)] \cap F = \{h(x)\}$. Also if $x, y \in \bigcup \mathcal{F}$, then $x \sim y \Leftrightarrow h(x) = h(y)$.

Next, we claim that if $x, y \in \bigcup \mathcal{E}$, then $x \approx y \Leftrightarrow \mathrm{flow}([x, a)) = \mathrm{flow}([y, a))$. To see this, suppose first that $x \approx y$. If $x \sim y$, then $[x, a)$ and $[y, a)$ are cofinal so we apply Lemma 3.6. If $x \not\sim y$, then $[x, y] = [x, a) \sqcup (a, y]$. Given any $z, w \in T$, we can find final segments, $A$ and $B$ of $[x, a)$ and $[y, a)$ respectively, such that $z, w \notin A \cup B$. Note that $A \cup B$ is an arc, and so in particular is full. We see that if $a \in A$ and $b \in B$, then $zwa \Leftrightarrow zwb$ (cf. Lemma 4.1). Now, by Lemma 3.7(2), we have $zw.A \Leftrightarrow zwa \Leftrightarrow zwb \Leftrightarrow zw.B$. Thus $\mathrm{flow}([x, a)) = \mathrm{flow}(A) = \mathrm{flow}(B) = \mathrm{flow}([y, b))$ as required. Conversely, suppose $x \not\approx y$. Then $xay$. Thus $xa.[y, a) \wedge \neg xa.[x, a)$, and so $\mathrm{flow}([y, a)) \neq \mathrm{flow}([x, a))$. This proves the claim.

Now let $H$ be a $\approx$-equivalence class of $T \setminus \{a\}$. Let $\mathcal{A}_H = \{N \in \mathcal{E} \mid N \subseteq H\}$. Thus $\mathcal{A}$ gives a partition of $H$ into $\sim$-equivalence classes. Let $\mathcal{E}_H = \mathcal{E} \cap \mathcal{A}_H$ and $\mathcal{F}_H = \mathcal{F} \cap \mathcal{A}_H$. Thus $\mathcal{A}_H = \mathcal{E}_H \sqcup \mathcal{F}_H$.

Let $G_H = (F \cap H) \cup \{a\}$. If $x \in F \cap H$, then $(x, a) = \emptyset$. If $x, y \in F \cap H$, then $x \approx y$ so $\neg xay$, and it follows that $(x, y) = (x, a) \cup (a, y) = \emptyset$. This shows that $G_H$ is null and full.

Suppose $\mathcal{E}_H = \emptyset$. In this case, $G_H$ is maximal null. (Since if $x \in H \setminus F$, then setting $y = h(x) \in F \cap H$, we have $ayx$. If $x \in T \setminus (H \cup \{a\})$ and $y \in F \cap H$, we have $x \not\approx y$ and so $xay$. Either way, $G_H \cup \{x\}$ is not null.) In other words, $G_H$ is a star. We set $p_H = \mathrm{flow}(G_H)$. Note that since $a \in G_H$, we have $(\forall x \in T)(\neg ax.p_H)$.

Suppose that $\mathcal{E}_H \neq \emptyset$. If $x, y \in \bigcup \mathcal{E}_H$, then since $x \approx y$, we have $\mathrm{flow}([x, a)) = \mathrm{flow}([y, a))$. In this case, we write $p_H$ for the flow thus defined. Again, $(\forall x \in T)(\neg xa.p_H)$.

**Lemma 3.29 :** $L(a) = \{p_H \mid H \in (T \setminus \{a\})/\approx\}$.

**Proof :** Suppose $p \notin L(a)$. Then there is some $x \in (a, p)$. Now $axp$ so $ax.p$. Thus we see that $p$ cannot have the form $p_H$ for any $H$.

Conversely, suppose $p \in L(a)$, so $(a, p) = \emptyset$. If $x \in T \setminus \{a\}$, then $[x, p)$ must be equal to either $[x, a]$ or $[x, a)$. Let $H = \{x \in T \setminus \{a\} \mid a \notin [x, p)\}$. If $x, y \in H$, then $\neg xay$ (otherwise $xap \vee yap$), so $x \approx y$. Conversely, if $x \approx y$, then $\neg xay$, so $xap \Leftrightarrow yap$. We see that $H$ is a $\approx$-equivalence class. Note that for each $x \in H$, we have $[x, p) = [x, a)$. We

claim that $p = p_H$.

Suppose that $\mathcal{E}_H \neq \emptyset$. Choose any $x \in \bigcup \mathcal{E}_H$. Now $\mathrm{flow}(p)|T = \mathrm{flow}([x,p])|T = \mathrm{flow}([x,p] \cap T) = \mathrm{flow}([x,p)) = \mathrm{flow}([x,a))$, so $p = p_H$.

Finally, suppose $\mathcal{E}_H = \emptyset$. Thus $G_H = (F \cap H) \cup \{a\}$ is a star. Define $h : T \longrightarrow G_H$ as in the definition of the flow $p_H = \mathrm{flow}(G_H)$. If $x \in H$, then $[x,p) = [x,a) = [x,h(x)]$ by the definition of $h$. If $x \notin H$, then $[x,p) = [x,a)$, by the definition of $H$. Also $[x,a) \cap H = \emptyset$, so $[x,a) \cap G_H = \{a\}$. Thus in this case $a = h(x)$ and again we have $[x,p) = [x,h(x)]$. If follows that $xy.p \Leftrightarrow xyp \Leftrightarrow y \in (x,p) = (x,h(x)] \Leftrightarrow xy.p_H$. Thus $p = p_H$.                    $\diamond$

¿From the above discussion, it's not hard to see that the trimmed down tree $(P \setminus (P(0,2) \cup P(1,1)), \langle\rangle)$ is a complete median pretree as claimed earlier. Moreover, if $T$ is already a complete median pretee, then $P = \bar{T} \cup P(0,2) \cup P(1,1)$, so the above construction doesn't change anything.

Note that if we were only interested in obtaining a median pretree (not necessarily complete), we could also remove all the points of the form $P(2,0)$. In other words, $P' = \bar{T} \cup \bigcup \{P(m,n) \mid m + n \geq 3\}$ is a median pretree. Note that, it $T$ is countable, then so is $P'$. This shows that any countable pretree can be embedded in a countable median pretee (and hence in an **R**-tree — see [Ch1]).

As observed at the beginning of this section, it will be convenient to revert to our original notation for the betweenness relation on the completion. In this case, we shall write $\Phi = \Phi(T)$ instead of $(P, \langle\rangle)$. We shall identify $T$ with $\bar{T}$, so that $T \subseteq \Phi$. In $\Phi$ we drop the angled brackets, $\langle\rangle$, when denoting the betweenness relation. As observed earlier, this is consistent with the betweenness relation already defined on $T$.

We can give a characterisation of $\Phi$ analogous to the pretree $T \sqcup P$ (Proposition 3.4.)

Suppose that $\Theta$ is a pretree, and $T \subseteq \Theta$. We say that this is a *complete embedding* if every flow on $T$ has the form $\mathrm{flow}_\Theta(p)|T$ for a unique element $p \in \Theta$.

Thus $T \subseteq \Phi$ is an example of a complete embedding. Another is $T \subseteq (T \sqcup P) \setminus \bar{T}$.

It is clear from the definition that if $T \subseteq \Theta$ is a complete embedding, then $\Theta$ can be identified set-theoretically with the set of flows on $T$. Thus such embeddings can only differ essentially in the pretree relations on $\Theta$. ¿From the definition, we see that all relations of the form $xyp$ for $x, y \in T$ and $p \in \Theta$ are completely determined, and, in fact, so also are those of the form $pxq$ for $x \in T$ and $p, q \in \Theta$ (Lemma 3.30). The issue then is which relations of the form $prq$ with $r \in \Theta \setminus T$ are satisfied. This is not competely determined. For example, if $\Theta = \Phi$ there will in general be many such relations, but if $\Theta = (T \sqcup P) \setminus \bar{T}$, there are none. However, if we add the additional hypothesis that $\Theta$ be a median pretree (or the weaker hypothesis of being a "saturated pretree" as described below), then this completely identifies $\Theta$ as $\Phi$.

Suppose, for the moment, that $T \subseteq \Theta$ is any complete embedding. the uniqueness hypothesis tells us that for $p, q \in \Theta$ we have that $(\forall a \in T)([a,p] \cap T = [a,q] \cap T) \Rightarrow p = q$. This implies that if $[p,q] = \emptyset$ then $p = q$ (since $[a,p] \subseteq [a,q] \cup [p,q]$ and $[a,q] \subseteq [a,p] \cup [p,q]$). If an embedding satisfies this latter criterion, we refer to it as a *dense* embedding.

**Lemma 3.30 :**   *If $T \subseteq \Theta$ is a dense embedding, and $p, q \in \Theta$ are distinct, then $[p,q] = \bigcap_{x \in T}([x,p] \cup [x,q])$.*

**Proof :** The inclusion ($\subseteq$) follows immediately from the fact that $[p, q] \subseteq [x, p] \cup [x, q]$ for all $x$. For the reverse inclusion, choose any $y \in [p, q] \cap T$. Then $\bigcap_{x \in T}([x, p] \cup [x, q]) \subseteq [y, p] \cup [y, q] = [p, q]$. $\diamond$

In particular, we have $[p, q] \cap T = \bigcap_{x \in T}([x, p] \cup [x, q]) \cap T$. Now we have already observed that relations of the form $xyp$ for $x, y \in T$ and $p \in \Theta$ are determined by the hypotheses of a complete embedding (identifying $\Theta$ with the set of flows). In other words, sets of the form $[x, p] \cap T$ are determined. Lemma 3.30 now shows that intervals of the form $[p, q] \cap T$ are determined.

In order to determine the intervals $[p, q]$ completely, we will need to add another hypothesis.

**Definition :** We shall say that a pretree, $S$, is *saturated* if, given any $x, y, z \in T$, $[x, z] \cap [y, z] = \{z\} \Rightarrow z \in [x, y]$.

(In the terminology of [AN] and [Tr], $S$ is said to have a "true betweenness" relation.)

Clearly, any median pretree is saturated.

**Lemma 3.31 :** *Suppose $T \subseteq \Theta$ is a dense embedding, and that $\Theta$ is a saturated pretree. Then, given $p, q \in \Theta$, we have $[p, q] \setminus T = \{r \in \Theta \setminus T \mid [p, r] \cap [q, r] \cap T = \emptyset\}$.*

**Proof :** The inclusion ($\subseteq$) is clear. For ($\supseteq$), suppose that $r \in \Theta \setminus T$ and $r \notin [p, q]$. Then there is some $s \in [p, r] \cap [q, r)$. Now $[s, r) \subseteq [p, r) \cap [q, r)$. Since $r \notin T$, we have $[s, r) \cap T \neq \emptyset$, and so $[p, r] \cap [q, r] \cap T \neq \emptyset$. $\diamond$

Now if $T \subseteq \Theta$ is a complete embedding, we know that sets of the form $[p, q] \cap T$ are determined. It follows that if $\Theta$ saturated, then $[p, q] = ([p, q] \cap T) \cup ([p, q] \setminus T)$ is determined. In summary, we conclude:

**Proposition 3.32 :** *Suppose $T \subseteq \Theta$ is a complete embedding. Suppose that $\Theta$ is a saturated pretree (for example that it is a median pretree). Then there is a pretree isomorphism from $\Theta$ to $\Phi$ fixing $T$.* $\diamond$

There is another interpretation of the saturation condition on a pretree, which has been described by Truss [Tr]. Given two pretree relations on the same set, $S$, we shall say that one is an *augmentation* of the other if every relation in the latter structure holds in the former. In other words, if we imagine a ternary relation as a subset of $S^3$, then augmentation is just the reverse of set inclusion.

Given $a \in S$, recall the equivalence relations $\sim$ and $\approx$ on $S \setminus \{a\}$ defined earlier in this section. Note that a pretree is saturated if and only if these relations are identical for all $a \in S$.

**Proposition 3.33 :** *A pretree, $S$, is saturated if and only if the pretree relation is maximal among all pretree relations on $S$ (i.e. if $S$ does not admit any non-trivial augmentation.)*

**Proof :** Suppose $S$ is saturated, and $x, y, z \in S$. If there were an augmentation of $S$ including the relation $xyz$, then certains $(x, y) \cap (y, z)$ would be empty (in the augmented, and hence also in the original pretree). But by the saturation hypothesis, the relation $xyz$ must already hold in $S$.

Conversely, suppose $S$ is not saturated. Then there is some $a \in S$ for which the relations $\sim$ and $\approx$ are not the same. Now we add to $S$ all relations of the form $xay$, where $(x \approx y) \wedge (x \not\prec y)$. (In other words, we impose the condition that $[x, a] \cap [y, a] = \emptyset \Rightarrow xay$ in the augmented structure.) One can now verify that $S$ is indeed a pretree in the augmented structure. $\diamond$

Note that by Zorn's Lemma, one can now see easily that any pretree admits an augmentation which is a saturated pretree. Of course, such an augmentation will not in general be unique. (For example, if we continue with the process of adding relations in the manner described above, the result may depend on the order in which the points are chosen.) However, in the case of a complete embedding, $T \subseteq \Theta$, there will be a unique maximal augmentation subject to the constraint that the relations on $T$ do not change and the embedding remains complete. (The construction of the pretree structure on the completion can thus be interpreted as starting with the pretree $\Theta = T \sqcup P \setminus \bar{T}$, and performing the operation described in the proof of Lemma 3.3 simultaneously for all $a \in \Theta \setminus T$.)

In other words, we can describe $\Phi$ as the unique maximal pretree containing $T$ subject to the constraint that $T \subseteq \Phi$ is a complete embedding.

We finish this section with a few remarks concerning "discrete" pretrees and their connection with simplicial trees. This will be used in Section 6, and is also relevant to [Bo2].

**Definition :** A pretree, $T$, is *discrete* if $[x, y]$ is finite for all $x, y \in T$.

A simplicial tree can be defined as a connected graph containing no cycles. The vertex set of a simplicial tree admits a **Z**-metric, where the distance between two points is defined to be the number of edges in the arc connecting them. With this structure, the vertex set is a **Z**-tree, as defined in [Sha]. Moreover every **Z**-tree arises in this way. The notion of a **Z**-tree is, in turn, essentially the same as a discrete median pretree. Given any discrete median pretree, $S$, we can define a metric on $S$ by setting $d(x, y) = |[x, y]| - 1$. It's not hard to see that this satisfies the axioms of a **Z**-tree. (To be precise, it's easy to see that, given $x, y \in S$, a "**Z**-segment" as defined in [Sha] coincides, in this context, with our notion of a closed interval $[x, y]$. The axioms of a **Z**-tree thus demand that, given $x, y, z \in S$, there is some $w \in S$ such that $[x, y] \cap [x, z] = [x, w]$, and that if $[x, y] \cap [x, z] = \{x\}$ then $[x, y] \cup [x, z] = [y, z]$. These are clearly satisfied for a discrete median pretree.) Again, every **Z**-tree arises in this way. We can summarise this by saying that:

**Lemma 3.34 :** *A pretree is a discrete median pretree if and only if it arises as the vertex set of a simplicial tree.* $\diamond$

If a pretree, $T$, is discrete, then so is its completion (as defined by the space of flows). Clearly, this case, there are no flows of type $(m, n)$ for any $m > 0$. Also, all flows of type

$(0, 2)$ are redundant. It's thus natural to form a pretree, only by adding those flows of type $(0, n)$ for $n \geq 3$. This pretree is a discrete median pretree, and is thus the vertex set of a simplicial tree, $\Sigma$. We summarise this by saying:

**Proposition 3.35 :**   *Suppose $T$ is a discrete pretree. Then $T$ naturally embeds in the vertex set, $V(\Sigma)$, of a simplicial tree, $\Sigma$, such that every vertex of $V(\Sigma) \setminus T$ has degree at least 3.* $\diamondsuit$

## 4. Quotients.

In this section, we describe how to construct quotients of pretree, $T$. Our main objective will be to describe a natural quotient, $T/\approx$, which is the largest quotient of $T$ which is dense.

Recall that a subset $Q \subseteq T$ is "full" if $[x, y] \subseteq Q$ for all $x, y \in Q$.

**Lemma 4.1 :**   *Suppose $X, Y \in T$ are full, and $X \cap Y = \emptyset$. Suppose $x, x' \in X$, $y, y' \in Y$ and $z \in T \setminus (X \cup Y)$. Then $xzy \Leftrightarrow x'yz'$.*

**Proof :** We have $z \neq y, y'$ and $\neg yzy'$. Thus $xzy \Leftrightarrow xzy'$. Similarly, $xzy' \Leftrightarrow x'zy'$. $\diamondsuit$

If $X$ and $Y$ are disjoint non-empty full subsets of $T$ and $z \in T$, we write $XzY$ to mean that $z \notin X \cup Y$ and $(\exists x \in X, y \in Y)(xzy)$. By Lemma 4.1, we could have equivalently said $(\forall x \in X, y \in Y)(xzy)$. In the case where $X = \{x\}$, we write $xzY$ for $\{x\}zY$. Note that we can make the usual inferences such as $xzY \wedge xwz \Rightarrow xwY$ etc.

**Definition :** A *full relation* on a pretree is an equivalence relations for which every equivalence class is full.

Suppose $\sim$ is a full equivalence relation on a pretree, $T$. We write $T/\sim$ for the quotient. Given $x \in T$, we write $[x] = [x]_\sim \in T/\sim$ for the equivalence class of $x$. If $X, Y, Z \in T/\sim$, we write $XYZ$ to mean $(\exists y \in Y)(XyZ)$. We can regard this as a ternary betweenness this relation on $T/\sim$.

**Lemma 4.2 :**   *With the betweenness relation thus defined, $T/\sim$ is a pretree. Moreover, if $T$ is a median pretree, then so is $T/\sim$. Also, if $T$ is a complete pretree, then so is $T/\sim$.*

**Proof :** We first verify the axioms of a pretree.

(T1): This is immediate.

(T2): If $XYZ \wedge XZY$, then $(\exists y \in Y, z \in Z)(XyZ \wedge XzY)$. Choose any $x \in X$. By Lemma 4.1, we get the contradiction $xyz \wedge xzy$.

(T3): Suppose $X, Y, Z, W \in T/\sim$, with $XYZ \wedge W \neq Z$. Choose $y \in Y$ so that $XyZ$, and any $x \in X$, $z \in Z$ and $w \in W$. Now apply (T3) for $T$.

This shows that $T/\sim$ is a pretree.

Now suppose that $T$ is a median pretree. Given distinct elements, $X, Y, Z \in T/\sim$, choose any $x \in X$, $y \in Y$, $z \in Z$, let $w = \mathrm{med}(x, y, z)$, and let $W = [w]$. We see readily from the definitions, that $W$ must be a median of $X, Y, Z$. This shows that, in this case, $T/\sim$ is a median pretree.

Now suppose that $T$ is complete pretree (not necessarily a median pretee). Suppose that $(\mathcal{A}, \prec)$ is a directed arc in $T/\sim$. We want to show that $\mathcal{A}$ has a supremum. We can suppose that $\mathcal{A}$ is endless, and has a minimum, say $M \in \mathcal{A}$. Thus if $X, Y \in \mathcal{A}$, then $X \prec Y \Leftrightarrow MXY \vee (X = M \neq Y)$. Choose any $m \in M$.

Suppose that $x \in X \in \mathcal{A}$, and $Y, Z \in \mathcal{A}$ with $X \prec Y$ and $X \prec Z$. Then we have $mxY \Leftrightarrow mxZ$. To see this, we can assume that $Y \prec Z$. Thus $XYZ$, so there is some $y \in Y$ with $XyZ$. Choose any $z \in Z$. Thus $xyz$ and so $mxy \Leftrightarrow mxz$. Thus $mxY \Leftrightarrow mxy \Leftrightarrow mxz \Leftrightarrow mxZ$ as claimed.

Now let

$$A = \{x \in T \mid (\exists X, Y \in \mathcal{A})(x \in X \wedge X \prec Y \wedge mxY)\}.$$

¿From the observation of the preceding paragraph, we see that if $x \in A \cap X$, $X \in \mathcal{A}$, and $Y$ is any point of $\mathcal{A}$ satisfying $X \prec Y$, then $mxY$.

We claim that $A$ is an arc (with minimum $m$).

We first show that it is linear. Note that it is enough to show that if $x, y \in A$, then either $x \in [m, y]$ or $y \in [m, x]$. We can suppose that $x$, $y$ and $m$ are all distinct. Let $X = [x]$ and $Y = [y]$, so $X, Y \in \mathcal{A}$. Choose $Z \in \mathcal{A}$ and with $X \prec Z$ and $Y \prec Z$, and choose some $z \in Z$. Thus $mxz \wedge myz$ so $mxy \vee myx$. This shows that $A$ is linear as claimed.

To show that $A$ is full, it's enough to show that if $x \in T$, $y \in A$ and $mxy$ then $x \in A$. Let $Y = [y]$, and choose any $Z \succ Y$. Then $myZ$ so $mxZ$. Now let $X = [x]$. Either $X = M$, or $MxZ$ so $MXZ$. Thus $X \in \mathcal{A}$ and $X \prec Z$. It follows that $x \in A$.

Since $T$ is complete, there is some $q \in T$ such that $A = [m, q]$. Let $Q = [q] \in T/\sim$. We claim that $\mathcal{A} = [M, Q)$ (i.e. the interval in $T/\sim$).

First we show that $[M, Q) \subseteq \mathcal{A}$. So suppose $MXQ$. Choose $x \in X$ with $MxQ$. Thus $mxq$, so $x \in [m, q) = A$, so by the definition of $A$, we have $X \in \mathcal{A}$ as required.

Finally we show that $\mathcal{A} \subseteq [M, Q)$. Suppose $X \in \mathcal{A} \setminus \{M\}$ and let $x \in X \cap A$. Then $x \in [m, q)$ so $mxq$ so $MXQ$ and $X \in [M, Q)$.

Thus every directed arc in $T/\sim$ has a supremum and so $T/\sim$ is complete. $\qquad \diamondsuit$

If we view a relation on a pretree, $T$, as a subset of $T \times T$, then we see that the set of all relations carries a natural structure as a boolean algebra. We write $\wedge$ and $\vee$ for the binary lattice operations. Thus, if $\sim$ and $\sim'$ are relations on $T$, and $x, y \in T$, then $x(\sim \wedge \sim')y \Leftrightarrow (x \sim y) \wedge (x \sim' y)$ and $x(\sim \vee \sim')y \Leftrightarrow (x \sim y) \vee (x \sim' y)$. More generally, if $\mathcal{S}$ is a nonempty set of relations, then define $\bigwedge \mathcal{S}$ and $\bigvee \mathcal{S}$ by $x(\bigwedge \mathcal{S})y \Leftrightarrow (\forall \sim \in \mathcal{S})(x \sim y)$ and $x(\bigvee \mathcal{S})y \Leftrightarrow (\exists \sim \in \mathcal{S})(x \sim y)$. We write $\leq$ for the partial order on the set of relations: thus $\sim \leq \sim'$ means that $x \sim y \Rightarrow x \sim' y$.

Let $\mathcal{R}$ be the set of all full relations on $T$. Thus $\mathcal{R}$ is closed under the operation $\wedge$. Indeed, if $\mathcal{S} \subseteq \mathcal{R}$ is nonempty, then $\bigwedge \mathcal{S} \in \mathcal{R}$. (This follows from the fact that the intersection of any nonempty set of full subsets of $T$ is full.) Also if $\mathcal{S} \subseteq \mathcal{R}$ is a chain (i.e. linearly ordered by $\leq$) then $\bigvee \mathcal{S} \in \mathcal{R}$.

Note that $\mathcal{R}$ contains a both a minimal and a maximal element, namely the trivial relation, $\sim_0 = \bigwedge \mathcal{R}$ (given by $x \sim_0 y \Leftrightarrow x = y$) and the universal relation $\sim_\infty$ (given by $x \sim_\infty y$ for all $x, y \in T$).

**Definition :** A *codense relation* on $T$ is a full relation, $\sim$, such that $T/\sim$ is dense.

This may be alternatively expressed by saying that if $x, y \in T$ and $x \not\sim y$, then $(\exists z \in T)(xzy \wedge z \not\sim x \wedge z \not\sim y)$.

Let $\mathcal{D} \subseteq \mathcal{R}$ be the set of all codense relations on $T$. Note that if $\mathcal{S} \subseteq \mathcal{D}$, then $\bigwedge \mathcal{S} \in \mathcal{D}$. To see this, write $\cong = \bigwedge \mathcal{S}$, and suppose $x, y \in T$ with $x \not\cong y$. Then $(\exists \sim \in \mathcal{D})(x \not\sim y)$ and so $(\exists z \in T)(xzy \wedge z \not\sim x \wedge z \not\sim y)$. Thus $z \not\cong x \wedge z \not\cong y$. This shows that $\cong \in \mathcal{D}$ as claimed.

Now certainly $\mathcal{D} \neq \emptyset$, since $\sim_\infty \in \mathcal{D}$. Thus we can set $\approx = \bigwedge \mathcal{D}$. We see that $\approx \in \mathcal{D}$. Also if $\sim \in \mathcal{D}$, then $\approx \leq \sim$. Thus $\approx$ is the unique minimal codense relation on $T$.

There are various alternative descriptions of the relation $\approx$. One using a transfinite induction process will be considered shortly. First we give another more explicit description.

By a *rational* (linear) subset of $T$, we mean a linear subset which is order isomorphic to the rational numbers (given either of the two compatible directions on this subset). In other words, it is a countable dense order without endpoints. If $x, y \in T$, we say that $x$ and $y$ are *separated by a rational subset* if $(x, y)$ contains a rational subset.

We write $x \cong y$ if $x$ and $y$ are not separated by a rational subset. Suppose $x \not\cong y$ and let $Q \subseteq (x, y)$ be a rational subset. If $z \in Q$, then $Q \cap (x, z)$ and $Q \cap (y, z)$ are both rational, so $z \not\cong x$ and $z \not\cong y$. Similarly, if $z \neq w \in Q$, the $z \not\cong w$.

We claim that $\cong$ is an equivalence relation. It is clearly reflexive and symmetric. To see that it is transitive, suppose $x \not\cong y$ and that $z \in T \setminus \{x, y\}$. Thus $(x, y) \subseteq (x, z] \cup [z, y)$. Let $Q \subseteq (x, y)$ be a rational subset, and choose any $w \in Q$. Without loss of generality, $w \in (x, z)$ so $(x, w) \subseteq (x, z)$ Thus $(x, w) \cap Q$ is a rational subset of $(x, z)$. We see that $z \not\cong z$. Thus $\cong$ is an equivalence relation as claimed. A similar but simpler argument shows that it is full. Moreover, from the observation of the preceding paragraph, we see that it is codense. Thus $\approx \leq \cong$.

To prove the reverse inequality, suppose that $\sim$ is a codense relation (such as $\approx$). We claim that $\cong \leq \sim$. Suppose that $x, y \in T$ with $x \not\sim y$. We define an order-preserving map $g : \mathbf{Z}[1/2] \cap (0, 1) \longrightarrow (x, y)$ as follows. Choose any $z \in (x, y)$ with $z \not\sim x$ and $z \not\sim y$. Set $h(1/2) = z$. Now find similar points $u \in (x, z)$ and $v \in (z, y)$ and set $h(1/2^2) = u$ and $h(3/2^2) = v$. Now continue inductively in this manner, to obtain a subset of $(x, y)$ order isomorphic to $\mathbf{Z}[1/2] \cap (0, 1)$ and hence to the rationals. We deduce that $\cong \leq \sim$ as claimed.

This shows that $\approx = \cong$. In summary, we have shown:

**Lemma 4.3 :** *Suppose $T$ is a pretree, and that $\approx$ is the minimal codense relation on $T$. If $x, y \in T$, then $x \approx y$ if and only if $x$ and $y$ are not separated by any rational subset.* $\diamondsuit$

Recall that $x, y \in T$ are said to be "adjacent" if $(x, y) = \emptyset$. Let $\simeq$ be the equivalence relation generated by adjacency. In other words, $x \simeq y$ if and only if there is a finite sequence, $x = x_0, x_1, \ldots, x_n = y$, of points of $T$ such that $x_i$ is adjacent to $x_{i-1}$ for each $i \in \{1, 2, \ldots, n\}$. Applying Lemma 2.6 we see that in this case $[x, y]$ is finite. In fact, it we

choose our sequence so that $n$ is minimal, then $[x, y] = \{x_0, x_1, \ldots, x_n\}$. Conversely, it is clear that if $[x, y]$ is finite, then $x \simeq y$. It is easy to see that $\simeq$ is a full relation on $T$. We shall refer to it as the *finite interval relation* on $T$.

Recall that a pretree is "discrete" if every interval is finite. Equivalently, $T$ is discrete if $T/\simeq$ is trivial. For a general pretree, we see that each $\simeq$-equivalence class is a maximal discrete subtree (which gives another means of defining $\approx$). Note that $T$ is dense if and only if $\simeq \, = \, \sim_0$, i.e. each $\simeq$-equivalence class is a singleton.

Suppose now that $\sim$ is any full relation on $T$. Let $\approx$ be the finite interval relation on $T/\sim$. Define a relation, $\sim'$, on $T$ by $x \sim' y \Leftrightarrow [x]_\sim \simeq [y]_\sim$. This is clearly an equivalence relation. In fact, it's easily seen to be a full relation on $T$. ¿From the definition of the pretree structure on $T/\sim$, we see that $x \sim' y$ if and only of the interval $[x, y]$ in $T$ meets only finitely many $\sim$-equivalence classes. ¿From this alternative description, we see easily that if $\sim$ and $\cong$ are two full relations with $\sim \, \leq \, \cong$, then $\sim' \, \leq \, \cong'$. Note also, that $\sim \, = \, \sim'$ if and only if $\sim$ is codense.

We now aim to give an inductive description of the minimal codense relation $\approx$.

To each ordinal number, $\alpha$, we associate a full relation $\sim_\alpha$ by a process of transfinite induction as follows. Set $\sim_0$ to be the trivial relation on $T$ (i.e. equality). If $\alpha = \beta + 1$, set $\sim_\alpha = (\sim_\beta)'$. If $\alpha$ is a limit ordinal, set $\sim_\alpha = \bigvee \{\sim_\beta \mid \beta < \alpha\}$. This gives us a full relation, since, by transfinite induction, all the relations $\sim_\beta$ for $\beta < \alpha$ are full relations. Moreover, if $\gamma \leq \beta < \alpha$, then $\sim_\gamma \, \leq \, \sim_\beta$, so that $\{\sim_\beta \mid \beta < \alpha\}$ is a chain.

We claim that these relations must eventually stabilise, i.e. for some $\alpha$, we have $\sim_{\alpha+1} \, = \, \sim_\alpha$, so that, in fact, $\sim_\beta \, = \, \sim_\alpha$ for all $\beta \geq \alpha$. One elementary way to see this is as follows. Suppose $\sim_\alpha \, \neq \, \sim_{\alpha+1}$. Let $I_\alpha = \{\beta \mid \beta \leq \alpha\}$. Thus $|I_\alpha| = |\alpha| + 1$, where $|.|$ denotes cardinality. We must have $\sim_\beta \, \neq \, \sim_{\beta+1}$ for all $\beta \in I_\alpha$. We define a map $h : T \times T \longrightarrow I_\alpha$ as follows. If $x \not\sim_{\alpha+1} y$, then set $h(x, y) = 0$. If $x \sim_{\alpha+1} y$, then set $h(x, y) = \min\{\beta \mid x \sim_{\beta+1} y\}$. (Thus if $x = y$ then $h(x, y) = 0$.) Now $h$ is surjective, since if $\beta \leq \alpha$ we know that $\sim_\beta \, < \, \sim_{\beta+1}$, so we can find $x, y \in T$ with $x \not\sim_\beta y$ and $x \sim_{\beta+1} y$, and so $h(x, y) = \beta$. It follows that $|\alpha| \leq |T|^2 = |T|$. (In the case where $T$ is finite, we stabilise immediately on $\sim_1 \, = \, \simeq$.) This shows that this process must stabilise as soon as we reach an ordinal of cardinality $|T|$. (Of course one can certainly improve on this statement with a little extra work.)

Suppose now that $\sim_\alpha \, = \, \sim_{\alpha+1}$. We claim that $\sim_\alpha \, = \, \approx$, i.e. the minimal codense relation. Now $\sim_\alpha \, = \, (\sim_\alpha)'$ so it is certainly codense, and so $\approx \, \leq \, \sim_\alpha$. We now show inductively that for all $\beta$, $\sim_\beta \, \leq \, \approx$. Certainly, $\sim_0 \, \leq \, \approx$. Also, if $\sim_\beta \, \leq \, \approx$, then $\sim_{\beta+1} \, = \, (\sim_\beta)' \, \leq \, \approx' \, = \, \approx$. Finally, suppose $\beta$ is a limit ordinal, and that $\sim_\gamma \, \leq \, \approx$ for all $\gamma < \beta$. Then $\sim_\beta \, = \, \bigvee\{\sim_\gamma \mid \gamma < \beta\} \, \leq \, \approx$. We thus conclude that $\sim_\alpha \, = \, \approx$ as claimed.

In summary, we have shown:

**Lemma 4.4 :** *The transfinite inductive process defined above stabilises on the minimal codense relation.* $\diamondsuit$

We finally make some remarks with regards to the completion process described in Section 3. Recall that we can embed any pretree, $T$, in a complete median pretree, $\Phi = \Phi(T)$. This embedding has the property that if $x, y \in \Phi$ are distinct, then $T \cap (x, y) \neq \emptyset$. Let $\approx$ be the minimal codense relation on $\Phi$. Recall that $\approx \, = \, \simeq$ as in Lemma 4.3.

**Lemma 4.5 :**  *If $x, y \in \Phi$, then $x \not\approx y$ if and only if $x$ and $y$ are separated by a rational subset of $T$, i.e. there is a subset $Q \subseteq T \cap (x,y)$ which is order isomorphic to the rationals.*

**Proof :**  We first observe that there is some $z \in T \cap (x,y)$ with $z \not\approx x$ and $z \not\approx y$. To see this choose a rational subset $R \subseteq (x,y)$. Choose any distinct elements $a, b \in R$, and choose any $z \in T \cap (a,b)$. Now $R \cap (x,a) \subseteq R \cap (x,z)$ is rational, so $z \not\approx x$. Similarly, $z \not\approx y$. This gives the required element $z$. Now, as in the proof of Lemma 4.3, we continue the binary subdivision in this manner and construct a subset of $T \cap (x,y)$ order isomorphic to the rationals.  $\diamond$

We shall write $\Psi(T) = \Phi(T)/\approx$. Thus $\Psi(T)$ is complete and dense.

## 5. Continua

In this section we shall be concerned mainly with continua (compact connected hausdorff topological spaces). We shall see (Lemma 5.3) that a continuum carries a natural pretree structure arising from the manner in which it can be separated by the set of cut points. (For this, assumption of compactness is irrelevant.) We shall see that a separable continuum has a natural quotient which is a dendrite. (In general this quotient may be trivial.) Much of the analysis can be done either purely from the topological structure or from the pretree structure. We begin with a purely topological discussion. Some useful references for continua are [HocY], [Ku] and [Na].

Let $M$ be a connected hausdorff topological space.

**Definition :**  A point $a \in M$ is a *cut point* if $M \setminus \{a\}$ is not connected.

Thus, we can write $M \setminus \{a\} = U \sqcup V$, where $U$ and $V$ are nonempty open subsets of $M$. We shall write $UaV$ to represent this situation. Clearly the set $U \cup \{a\}$ is closed in $M$. In fact,

**Lemma 5.1 :**  *If $UaV$, then $U \cup \{a\}$ is connected.*

**Proof :**  Suppose that $U \cup \{a\} = F \sqcup G$, where $F$ and $G$ are closed in $U \cup \{a\}$ and hence in $M$. Without loss of generality, we can assume that $a \in F$ so that $G \subseteq U$. Now $M \setminus G = F \cup V = F \cup (V \cup \{a\})$ and so $M \setminus G$ is also closed. Since $M$ is connected, we deduce that $G = \emptyset$.  $\diamond$

Now suppose that $a$ and $b$ are distinct cut points of $M$, and suppose that $UaU'$ and $VaV'$. Without loss of generality, we have $b \in U'$ and $a \in V'$. Since $U \cup \{a\}$ is connected and $b \notin U \cup \{a\}$, we must have $U \cup \{a\} \subseteq V$ or $U \cup \{a\} \subseteq V'$; (otherwise the sets $(U \cup \{a\}) \cap V$ and $(U \cup \{a\}) \cap V'$ would separate $U \cup \{a\}$). But $a \notin V'$ and so we must have $U \cup \{a\} \subseteq V$. Similarly, $V \cup \{b\} \subseteq U'$. We see that $U \cap V = \emptyset$. Let $W = U' \cap V'$. Now $W \neq \emptyset$ (otherwise $U \cap \{a\}$ and $V \cap \{b\}$ would partition $M$ into two nonempty closed subsets). Thus, $U, V$ and $W$ are nonempty and open in $M$ and $M = U \sqcup \{a\} \sqcup W \sqcup \{b\} \sqcup V$. Note that $U \cup \{a\}$ and $U \cup W \cup \{a, b\}$ are both closed and connected (the latter being

equal to $V' \cup \{b\}$). The same applies to $V \cup \{b\}$ and $V \cup W \cup \{a, b\}$. Also $W \cup \{a, b\}$ is closed, and it's not hard to see that it must be connected. (We omit the argument here, since it will follow from Lemma 5.5.) We shall write $UaWbV$ to represent the situation just described. In summary, we have shown:

**Lemma 5.2 :** *If $a, b \in M$ are distinct cut points, then there are open sets $U$, $V$ and $W$ in $M$ such that $UaWbV$. In fact, if $U'aU''$ and $V'bV''$, then we can choose $U \in \{U', U''\}$ and $V \in \{V', V''\}$.* $\diamond$

At this point, we can define the pretree structure on $M$. Given $x, y, z \in M$, we shall write $xzy$ to mean that there are open sets, $U$ and $V$, of $M$ with $UzV$, $x \in U$ and $y \in V$.

**Lemma 5.3 :** *With the ternary relation thus defined, $M$ is a pretree.*

**Proof :**

(T1): This is immediate.

(T2): Suppose $xab \wedge xba$. There are open sets $U, U', V, V'$ so that $UaU'$ and $VbV'$, with $x \in U \cap V$, $b \in U'$ and $a \in V'$. Applying Lemma 5.3, we have $UaWbV$ , where $W = U' \cap V'$. In particular, $U \cap V = \emptyset$, contradicting $x \in U \cap V$.

(T3): Suppose $xay$ and $z \neq a$. There are open sets $U, V$ with $UaV$, $x \in U$ and $y \in V$. Either $z \in U$ so that $zay$, or else $z \in V$ so that $zax$. $\diamond$

An alternative proof of this fact, using a more general observation about the separating properties of finite subsets of a connected hausdorff space, is given in [Bo5]. It is also described in [W2].

Note that a corollary to Lemma 5.3 is the fact that the set of cut points separating any two given points in $M$ carries a natural linear order. This was mentioned in the introduction. Direct proofs can be found in various places, for example [HocY].

We shall return to the pretree structure later. We first consider further the topology on $M$.

**Definition :** A *branch* of $M$ is a closed subset of $M$ containing at least two elements, whose boundary consists of a single point.

**Lemma 5.4 :** *A closed subset $B \subseteq M$ is branch if and only if it has the form $U \cup \{a\}$, where $a$ is a cut point of $M$ and $U$ is a connected component of $M \setminus \{a\}$. (In this case $\partial B = \{a\}$.)*

**Proof :** Suppose first that $UaV$ and $B = U \cup \{a\}$. Now $V$ cannot be closed (since $M$ is connected), so $a$ cannot lie in the interior of $B$. Thus, $\partial B = B \setminus U = \{a\}$, and so $B$ is a branch.

Conversely, suppose that $B$ is a branch with $\partial B = \{a\}$. Let $U = B \setminus \{a\}$ and $V = M \setminus B$. Since $B$ is closed, $V$ is open. Also $B \setminus \{a\} = B \setminus \partial B$ is open. By hypothesis, $U \neq \emptyset$. Thus $UaV$. $\diamond$

It follows by Lemma 5.1 that any branch is connected. In fact:

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**Lemma 5.5 :**   *Suppose that $K \subseteq M$ is closed and connected and that $B \subseteq M$ is a branch. Then $B \cap K$ is connected.*

**Proof :** Let $\partial B = \{a\}$. Suppose that $B \cap K = F \sqcup G$, where $F$ and $G$ are closed. Without loss of generality, we can suppose that $a \notin F$. Now $K \setminus F = G \cup (K \setminus B) = G \cup (K \setminus (B \setminus \{a\}))$. But $B \setminus \{a\}$ is an open subset of $M$. Thus $K \setminus (B \setminus \{a\})$ is closed, and so $K \setminus F$ is closed. Since $K$ is connected, it follows that either $F = \emptyset$, or $K \setminus F = \emptyset$. In the latter case, we have $K \subseteq F$ so $K = F$ and so $G = \emptyset$.                                                                $\diamond$

(Note this shows that if $UaWbV$, then $W \cup \{a, b\}$ is connected, as claimed earlier.)

The following notion seems to be useful in the present context:

**Definition :**   We say that a subset $F$ of $M$ is *coherent* if it is closed and if $F \cap K$ is connected for every compact connected subset, $K$, of $M$.

Thus, Lemma 5.5 shows that every branch of $M$ is coherent.

Note that if $M$ happens to be compact, then every coherent subset of $M$ must be connected (take $K = M$). However this need not be the case in general.

**Lemma 5.6 :**   *If $F$ and $G$ are coherent subsets of $M$, then $F \cap G$ is coherent.*

**Proof :** If $K \subseteq M$ is compact and connected, then so is $K \cap F$. Since $G$ is coherent, it follows that $K \cap (F \cap G) = (K \cap F) \cap G$ is connected.                    $\diamond$

We say that a set $\mathcal{F}$ of subsets of $M$ is a *chain* if it is totally ordered by inclusion (i.e. if $F, G \in \mathcal{F}$, then $F \subseteq G$ or $G \subseteq F$). Now, in a compact hausdorff space, the intersection of a nonempty chain of closed connected subsets is connected. This is a simple exercise, see for example [HocY]. As a corollary we deduce:

**Lemma 5.7 :**   *If $\mathcal{F}$ is a nonempty chain of coherent subsets of $M$, then $\bigcap \mathcal{F}$ is coherent.*

**Proof :** Suppose $K \subseteq M$ is compact and connected. The sets $K \cap F$, as $F$ varies over $\mathcal{F}$, form a chain of connected subsets of $K$. Thus $K \cap (\bigcap \mathcal{F}) = \bigcap \{K \cap F \mid F \in \mathcal{F}\}$ is connected.                                                                                   $\diamond$

Putting the last two results together, we find that in fact:

**Lemma 5.8 :**   *If $\mathcal{F}$ is a nonempty set of coherent subsets of $M$, then $\bigcap \mathcal{F}$ is coherent.*

**Proof :** Let $\mathcal{F}'$ be the set of all possible intersections of sets in $\mathcal{F}$. Thus $\mathcal{F} \subseteq \mathcal{F}'$ and $\bigcap \mathcal{F} \in \mathcal{F}'$. We partially order $\mathcal{F}'$ by set inclusion. Let $\mathcal{G} \subseteq \mathcal{F}'$ be the set of all coherent sets in $\mathcal{F}'$. By hypothesis, $\mathcal{F} \subseteq \mathcal{G}$. By Lemma 5.7, every chain $\mathcal{H} \subseteq \mathcal{G}$ has a lower bound, namely $\bigcap \mathcal{H}$. Thus, by Zorn's Lemma, $\mathcal{G}$ contains a minimal element, say $G$.

We claim that $G = \bigcap \mathcal{F}$. To see this, suppose that $F \in \mathcal{F} \subseteq \mathcal{G}$. By Lemma 5.6, $F \cap G \in \mathcal{G}$. But $F \cap G \subseteq G$ and $G$ is minimal, so $F \cap G = G$. Thus $G \subseteq F$. It follows that $G \subseteq \bigcap \mathcal{F}$. Thus $\bigcap \mathcal{F} = G \in \mathcal{G}$.                                $\diamond$

Putting together Lemma 5.5 and 5.8, we deduce:

**Lemma 5.9 :** *Any intersection of branches is coherent.* ◇

If $M$ happens to be compact, it follows that an intersection of branches is connected.

We now forget about topology for the moment, and focus on the pretree structure of $M$. Note that some information is lost in doing this — clearly we cannot hope, in general, to recover the topology from the pretree structure alone. For example, points which are not separated by any cut point are indistinguishable in the pretree structure. Also the notion of a "branch" cannot be defined purely in terms of the pretree. However there are various natural pretree constructions which will turn out to be intersections of branches.

Suppose, for the moment, that $M$ is any pretree, and $T \subseteq M$ any subset. Let $\Phi = \Phi(T)$ be the completion of $T$ as defined in Section 3. Thus, we may regard $T$ as a subset of both $M$ and $\Phi$. We define a map $\phi : M \longrightarrow \Phi$ as follows. Given $x \in M$, define $\phi(x) = p$ where $\mathrm{flow}_M(x)|T = \mathrm{flow}_\Phi(p)|T$. This $p$ is uniquely defined (see the discussion of complete embeddings at the end of Section 3). Clearly, $\phi$ restricts to the identity on $T$.

¿From the definition of $\phi$, we have that if $a \in T$ and $x \in M$, then $[a, x] \cap T = [a, \phi(x)] \cap T$. In fact:

**Lemma 5.10 :** *If $x, y \in M$ and $[x, y] \cap T \neq \emptyset$, then $[x, y] \cap T = [\phi(x), \phi(y)] \cap T$.*

**Proof :** Choose any $a \in [x, y] \cap T$. Then $[x, y] \cap T = ([a, x] \cup [a, y]) \cap T = ([a, \phi(x)] \cup [a, \phi(y)]) \cap T = [\phi(x), \phi(y)] \cap T$. ◇

Note, in particular, that for $a \in T$, $\phi(x)a\phi(y) \Rightarrow xay$.

Consider for the moment, a general pretree, $S$. Suppose $a \in S$. Recall from Section 3, that we may define an equivalence relation $\approx\; =\; \approx_S$ on $S \setminus \{a\}$ by $x \approx y \Leftrightarrow \neg xay$. Given $x \in S \setminus \{a\}$, we write $R_S(a, x) = \{a\} \cup [x]_\approx$, where $[x]_\approx$ denotes the $\approx$-equivalence class of $x$. Thus, $R_S(a, x) = \{y \in S \mid \neg xay\}$. Clearly, if $x \approx y$, then $R_S(a, x) = R_S(a, y)$.

Returning to our earlier set-up, suppose that $a \in T$ and $p \in \Phi$. We must have that $R_\Phi(a, p) \cap T \setminus \{a\} \neq \emptyset$. (Otherwise $[p]_\approx \cap T = \emptyset$, so $(\forall b \in T \setminus \{a\})(bap)$. Thus by Lemma 3.1, $\mathrm{flow}_\Phi(p)|T = \mathrm{flow}_\Phi(a)|T$ and so $a = p$, contrary to our assumption.) Now if $b \in R_\Phi(a, p) \cap T \setminus \{a\}$, we have $b \approx_\Phi p$ so $R_\Phi(a, p) = R_\Phi(a, b)$.

Now if $b, b' \in R_\Phi(a, p) \cap T \setminus \{a\}$, then $b \approx_\Phi p \approx_\Phi b'$. Thus $\neg bab'$ and so $b \approx_M b'$. It follows that $R_M(a, b) = R_M(a, b')$. In other words, the set $R_M(a, b)$ is defined independently of the choice of $b \in R_\Phi(a, p) \cap T \setminus \{a\}$. We can thus write it as $R_M(a, p)$. Clearly, $R_\Phi(a, p) \cap T = R_M(a, p) \cap T$.

Now suppose that $x \in R_M(a, p)$, and let $q = \phi(x)$. We claim that $\neg paq$. To see this, choose any $b \in T$ with $p \approx_\Phi b$, so that, by definition, $x \in R_M(a, b)$. Thus, $\neg bax$. By the definition of $f$, we must have $\neg baq$ (since $\mathrm{flow}_\Phi(q)|T = \mathrm{flow}_M(x)|T$). We can suppose that $q \neq a$, so $b \approx_\Phi q$. But $b \approx_\Phi p$ and so $p \approx_\Phi q$. In other words, $\neg paq$ as claimed.

Now given any $p \in \Phi \setminus T$, let $R_M(p) = \bigcap_{a \in T} R_M(a, p)$. Suppose $x \in R_M(p)$, and $q = \phi(x)$. ¿From the previous paragraph, we see that $(\forall a \in T)(\neg paq)$. It follows that $p$ and $q$ must be either equal or adjacent in $\Phi$. (For if $prq$, we must have $r \notin T$. Since we are assuming that $p \notin T$, we would get $(p, r) \neq \emptyset$, and so $(p, q) \cap T \neq \emptyset$.) Now since $p \notin T$, it

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follows by Lemma 3.28 that either $q = p$, or else $q \in T$ and $q$ is adjacent to $p$. In summary, we have shown:

**Lemma 5.11 :** *Suppose $M$ is a pretree, and $T \subseteq M$ is any subset. Let $\phi : M \longrightarrow \Phi$ be defined as above. Suppose $p \in \Phi \setminus T$, and let $R_M(p) \subseteq M$ be defined as above. If $x \in R_M(p)$, then either $\phi(x) = p$, or else $\phi(x) \in T$ and $\phi(x)$ is adjacent to $p$ in $\Phi$.* ◇

The objective, in all of this, is to show that in certain circumstances, the map $\phi$ is "almost surjective". Thus, we would like to have conditions under which the sets $R_M(p)$ are guaranteed to be nonempty. For this we shall need to make further appeal to topology (see Proposition 5.14). We can, however make the following general observation:

**Lemma 5.12 :** *If $T_0 \subseteq T$ is finite, and $p \in \Phi \setminus T$, then $\bigcap_{a \in T_0} R_M(a, p) \neq \emptyset$.*

**Proof :** In fact, we show that $\bigcap_{a \in T_0} R_\Phi(a, p) \cap T_0 \neq \emptyset$. (Recall that $R_\Phi(a, p) \cap T = R_M(a, p) \cap T$.) We are thus reduced to considering the finite pretree $T_0 \sqcup \{p\}$ (which, by Lemma 2.5, can be assumed to be a subset of a finite simplicial tree). Now let $b \in T_0$ be some point adjacent to $p$ in $T_0 \sqcup \{p\}$, i.e. so that $(b, p) \cap T_0 = \emptyset$. If $a \in T_0$, then $\neg bap$, and so $b \in R_\Phi(a, p)$. We see that $b \in \bigcup_{a \in T_0} R_\Phi(a, p) \cap T_0$. ◇

To go any further, we need to reintroduce the topology on $M$. We are assuming that $M$ is a connected hausdorff space. We have defined a pretree structure on $M$ (Lemma 5.3). We shall suppose that $T$ is a subset of the set of cut points of $M$. (Note that this may be defined in terms of the pretree structure: $a \in M$ is a cut point if and only if it is not a terminal point in the pretree.)

**Lemma 5.13 :** *If $a \in T$ and $p \in \Phi$, then $R_M(a, p)$ is an intersection of branches.*

**Proof :** ¿From the definition, we know that $R_M(a, p) = R_M(a, b)$ for some $b \in T$. Let $\mathcal{B}$ be the set of branches $B \subseteq T$ such that $\partial B = \{a\}$ and $b \in B$. We claim that $R_M(a, b) = \bigcap \mathcal{B}$.

Certainly, if $x \in R_M(a, b)$ and $B \in \mathcal{B}$, then we must have $x \in B$ (otherwise, by definition, $xab$). Thus $x \in \bigcap \mathcal{B}$. Conversely, suppose that $x \notin R_M(a, b)$. Then $xab$, so we have open sets $U, V \subseteq M$, with $UaV$, $b \in U$ and $x \in V$. Let $B = U \cap \{a\} \in \mathcal{B}$. Now $x \notin B$, so $x \notin \bigcap \mathcal{B}$. ◇

Thus we see that $R_M(a, p)$ is closed (in fact coherent).
We can now prove:

**Proposition 5.14 :** *Suppose that $M$ is a continuum, and that $T \subseteq M$ is a set of cut points of $M$. Considering $T$ as a pretree, let $\Phi = \Phi(T)$ and let $\phi : M \longrightarrow \Phi$ be the map defined above. If $p \in \Phi$ then either $p \in \phi(M)$, or else $p$ is adjacent (in $\Phi$) to some element of $T$. In all cases, $R_M(p) = \emptyset$.*

**Proof :** Let $\mathcal{R} = \{R_M(a, p) \mid a \in T\}$, so that, by definition, $R_M(p) = \bigcap \mathcal{R}$. By Lemma 5.13, each element of $\mathcal{R}$ is closed. By Lemma 5.12, every finite subset of $\mathcal{R}$ has nonempty intersection. Since we are assuming that $M$ is compact, it follows that $R_M(p) \neq \emptyset$. Choose

some $x \in R_M(p)$. By Lemma 5.11, either $p = \phi(x) \in \phi(M)$, or else $\phi(x) \in T$ and $p$ is adjacent to $\phi(x)$. $\diamond$

As in Section 4, let $\sim_1$ be the (full) equivalence relation on $\Phi$ generated by adjacency. We see immediately that the composition of $\phi$ with the quotient map $\Phi \longrightarrow \Phi/\sim_1$ is surjective.

At the end of Section 4, we defined $\Psi = \Phi/\approx$, where $\approx$ is the minimal codense relation on $\Phi$. Let $\pi : \Phi \longrightarrow \Psi$ be the quotient map, and let $f = \pi \circ \phi : M \longrightarrow \Psi$. We know that $\Psi$ is a quotient of $\Phi/\sim_1$ and so:

**Lemma 5.15 :** *The map $f : M \longrightarrow \Psi$ is surjective.* $\diamond$

We can thus regard $\Psi$ as a topological space, with the quotient topology derived from $M$. Considering it as such, we shall usually denote it by $D_T(M)$. Note that it depends on the set of cut points, $T$, of $M$ which we take. Usually we just take $T$ to be the whole set of cut points. In this way the construction is a natural one, and it is reasonable to abbreviate $D_T(M)$ to $D(M)$ or just $D$. Clearly, $D(M)$ is compact and connected, and we shall see shortly that it must be hausdorff. If it is separable, (for example if $M$ is separable), then it must be a dendrite (Theorem 5.23).

Consider, for the moment, a general pretree, $S$. We say that a full subset $Q \subseteq S$ is *preclosed* if $(\forall x \in S \setminus Q)(\exists y \in S)(xyQ)$. (Recall the notation $xyQ$ from Section 4.)

**Lemma 5.16 :** *Suppose $\sim$ is a full relation on $S$. Let $\pi : S \longrightarrow S/\sim$ be the quotient map. If $Q \subseteq S/\sim$ is preclosed, then $\pi^{-1}(Q) \subseteq S$ is preclosed.*

**Proof :** If $x \in S \setminus \pi^{-1}(Q)$, then $\pi(x) \in S \setminus Q$, so there is some $y \in S$ with $\pi(x)\pi(y)Q$. By the definition of the pretree structure on the quotient, we have $xyQ$. $\diamond$

Suppose now that $S$ is a median pretee. Given distinct points, $x, y \in S$, let $J(x, y) = \{z \in S \mid \mathrm{med}(x, y, z) = y\}$. Thus $J(x, y) = S \setminus H(x, y)$ where $H(x, y)$ was defined in Section 2 (see Theorem 2.14). It is easily seen that $J(x, y)$ is full.

**Lemma 5.17 :** *If $S$ is a dense median pretree, and $x, y \in S$ are distinct, then $J(x, y)$ is preclosed.*

**Proof :** Suppose $z \in S \setminus J(x, y)$. Choose any $w$ in the interval $(y, \mathrm{med}(x, y, z))$. Then we have $zwJ(x, y)$. $\diamond$

Note also that in a dense pretree, all singletons are preclosed.

Now, let's return to our map $\phi : M \longrightarrow \Phi$. Note that if $Q \subseteq \Phi$ is preclosed and $p \in \Phi \setminus Q$, then there is some $a \in T$ such that $paQ$. Recall that if $a, b \in T$ are distinct, then $R_M(a, b)$ is defined as $\{x \in M \mid \neg xab\}$.

**Lemma 5.18 :** *If $Q \subseteq \Phi$ is preclosed, then $\phi^{-1}(Q) = \bigcap\{R_M(a, b) \mid a, b \in T, abQ\}$.*

**Proof :** Suppose $x \in \phi^{-1}(Q)$ and $a, b \in T$ with $abQ$. Now $\phi(x) \in Q$, so $ab\phi(x)$. Thus $abx$, so $\neg bax$ and so $x \in R_M(a, b)$.

Conversely, suppose $x \notin \phi^{-1}(Q)$. We can find $a \in T$ with $\phi(x)aQ$, and then $b \in T$ with $abQ$. Thus $\phi(x)ab$, so $xab$, and so $x \notin R_M(a, b)$. ◇

Now, applying Lemma 5.13 and Lemma 5.9, we deduce:

**Lemma 5.19 :** *If $Q \subseteq \Phi$ is preclosed, then $\phi^{-1}(Q) \subseteq M$ is coherent (hence closed and connected).* ◇

We now return to our map $f : M \longrightarrow D(M)$. Recall that $f$ is defined as a composition of $\phi : M \longrightarrow \Phi$ and $\pi : \Phi \longrightarrow \Psi \equiv D(M)$.

**Lemma 5.20 :** *If $x, y \in D$ are distinct, then $J(x, y)$ is closed in $D$.*

**Proof :** By Lemma 5.17, $J = J(x, y)$ is preclosed. Thus, by Lemma 5.16, $\pi^{-1}(J) \subseteq \Phi$ is also preclosed. Thus, by Lemma 5.19, $f^{-1}(J) = \phi^{-1}\pi^{-1}(J)$ is coherent, and hence closed. Thus $J(x, y)$ is closed. ◇

Note that sets of the form $H(x, y) = D \setminus J(x, y)$ are open in $D$. It follows easily that $D$ is hausdorff, hence a continuum.

Also, since singletons in $D$ are closed, essentially the same argument shows that:

**Lemma 5.21 :** *If $x \in D$, then $f^{-1}(x)$ is coherent (hence closed and connected).* ◇

Now, it's not hard to show that if we map a compact space (such as $M$) into a normal space (such as $D$) in such a way that the preimage of any point is connected, then, in fact the preimage of any closed connected set is connected.

We summarise what we have shown so far:

**Proposition 5.22 :** *Suppose that $M$ is a continuum, and let $D$ be the quotient space as defined above. Then $D$ is a continuum. Moreover, the preimage of any subcontinuum of $D$ by the quotient map is a subcontinuum of $M$.* ◇

Note that, as remarked in the introduction, there is a more direct way to define $D$ as the quotient $M/\sim$, where $x \not\sim y$ if and only if there is a set of cut points of $M$, separating $x$ and $y$, which is order-isomorphic to the rational numbers in the natural linear order. The fact that this is equivalent to our earlier definition is an easy consequence of Lemma 4.2.

Finally, if $M$ is separable, then so is $D$. Thus, applying Theorem 2.14, we deduce:

**Theorem 5.23 :** *If $M$ is a separable continuum, then $D$ is a dendrite.* ◇

As remarked in the introduction, if $M$ is a separable continuum with the property that any pair of distinct points are separated by a third point of $M$, then it is immediate that the above equivalence relation is just equality on $M$. It follows that $M$ itself is a

dendrite, as defined in Section 2, and so, for example, can be realised as a compact **R**-tree. We see that this weak definition is equivalent to the apparently much stronger one.

As we have observed elsewhere, this construction can be expressed more succinctly by defining the equivalence relation directly in terms of the (linear) order type of the set of points which separate two given points of $M$. (See the discussion in the introduction.) The fact that this is equivalent to the above construction follows using Lemma 4.3. Also, with this approach, we see easily that the quotient is a dendrite by the apparently weaker definition.

We have chosen this more convoluted path to Theorem 5.23 for a number of reasons. It gives us more information, and shows directly that the quotient is a dendrite by the apparently stronger definition. Also, this approach will allow us to show that, in certain circumstances, the result is non-trivial — as in Theorem 6.1.

## 6. Groups.

In this section, we consider convergence group actions on continua. The notion of a convergence group was defined by Gehring and Martin [GeM1]. Accounts in more general contexts can be found in [Tu1] and [Bo5].

Our main result of this section will be:

**Theorem 6.1 :**  *Suppose that $\Gamma$ is a one-ended finitely generated group which admits a minimal convergence action on a continuum, $M$. If $M$ has a cut point which is not a parabolic fixed point, then the quotient $D(M)$ is non-trivial (i.e. not a point).*

Here, a "parabolic fixed point" should be interpreted as one whose stabiliser is infinite and contains no loxodromic elements. Such a subgroup either contains a parabolic element or is an infinite torsion group. In either case, it has no other fixed points in $M$. (For most applications we will assume that $\Gamma$ has no infinite torsion subgroup, so one need not worry to much about this point.)

Recall that the quotient, $D(M)$, was defined with reference to a set, $T \subseteq M$, of cut points of $M$. For terminological convenience, we have usually taken this to be the set of all cut points, though, in fact, any $\Gamma$-invariant set of cut points will do. Clearly, if the quotient with respect to one $\Gamma$-invariant of cut points is non-trivial, then the quotient with respect to any larger set will be non-trivial. With this in mind, we see that to prove the result, we can assume that no point of $T$ is a parabolic fixed point. (Take $T$ to be the $\Gamma$-orbit of such a point.)

Note that with the hypotheses of the theorem, $M$ will necessarily be separable, and so the quotient $D(M)$ will be a dendrite (Theorem 5.23). Since the construction is natural, it will be equivariant. Moreover, the induced action on $D$ will also be a convergence action.

One of the main applications of this theorem is to the case of a one-ended word-hyperbolic group $\Gamma$ where $M = \partial\Gamma$. Thus, if $\partial\Gamma$ contains a cut point we see that it must have an equivariant quotient which is a dendrite. The result also applies to "uniform convergence actions" on continua as we describe later.

We should elaborate a bit on some of the terms used in the statement of Theorem 6.1. We begin by giving a definition of a convergence action.

Let $M$ be (for the moment) any compact hausdorff topological space. The *space of distinct (ordered) triples* in $M$ is the cartesian product $M \times M \times M$ minus the large diagonal. This is locally compact and hausdorff.

**Definition :** We say that an action of a group, $\Gamma$, by homeomorphism on $M$ is a *convergence action* (or that $\Gamma$ is a *convergence group*) if the induced action on the space of distinct triples of $M$ is properly discontinuous.

(This is equivalent to what is termed a "discrete convergence action" in the original paper [GeM1]. The term "discrete" has frequently been omitted in the subsequent literature.)

Note that, if $M$ is metrisable, this definition can be rephrased in terms of sequences. It is equivalent to asserting the following. Suppose that $(\gamma_i)_i$ is a sequence of distinct elements of $\Gamma$, that $x, y, z, x', y', z' \in M$, and that $(x_i)_i$, $(y_i)_i$ and $(z_i)_i$ are sequences in $M$ with $x_i \to x$, $y_i \to y$, $z_i \to z$, $\gamma_i x_i \to x'$, $\gamma_i y_i \to y'$ and $\gamma_i z_i \to z'$. Then, either at least two of $x, y, z$ must be equal, or at least two of $x', y', z'$ must be equal. (If $M$ is not metrisable, then we simply replace the word "sequence" by "net".)

In fact, the definition of a convergence action is usually given in the following equivalent form. We shall again phrase it in terms of sequences. In doing so, we are tacitly assuming that $M$ is metrisable. All the arguments we shall give based on this definition can be readily translated to the general case by rephrasing everything in terms of nets, as is described in [Bo5]. However, since the arguments are exactly the same, and since all the spaces we are principally interested in are metrisable anyway, it does not seem worthwhile introducing this additional terminology into the proceedings here.

Suppose then again that $\Gamma$ acts by homeomorphism on a compact hausdorff (metrisable) space $M$. The action of $\Gamma$ is a convergence action if and only if for every sequence, $(\gamma_n)_n$, of distinct elements of $\Gamma$, there is a subsequence, $(\gamma_i)_i$, and points $\lambda$ and $\mu$ such that if $K$ is any closed subset of $M \setminus \{\lambda\}$, and $U \subseteq M$ is any open set containing $\mu$, then $\gamma_i(K) \subseteq U$ for all but finitely many $i$. This is often expressed by saying that maps $\gamma_i|(M \setminus \{\lambda\})$ converge locally uniformly to (the constant map which sends all of $U \setminus \{\lambda\}$ to) $\mu$. Note that $\lambda$ may or may not be equal to $\mu$.

Note that, in the above definition, if we set $K' = M \setminus U$, then $K'$ is a closed subset of $M \setminus \{\mu\}$, and the hypothesis can be rephrased as saying that $K' \cap \gamma_i K = \emptyset$ for all but finitely many $i$. This is a more symmetrical formulation — we see that the convergence hypothesis is satisfied for the sequence $(\gamma_i^{-1})_i$ on swapping $\lambda$ and $\mu$.

The equivalence of the above definitions is shown in [GeM2] in the case where $M$ is a topological sphere. This argument seems to work unaltered for metrisable Peano continua. A proof in the general case is given in [Bo5]. Here we shall work mostly with the second definition.

Typical examples of convergence groups are kleinian groups (i.e. groups acting properly discontinuously and isometrically on hyperbolic $n$-space). Such groups act as convergence groups on the ideal sphere, and were the principle motivating examples of [GeM1]. Many of the dynamical properties of kleinian groups can be interpreted in this broader

context.

Other examples are (word) hyperbolic groups acting on their boundaries. The fact that such groups are convergence groups, by the convergence subsequence definition, is proven directly in [F] or [Tu1]. In fact, in this case, one can say more — namely that the action on the boundary is a "uniform" convergence group. This was observed by Gromov. For a proof see for example [Bo5]. A convergence group acting on a compactum, $M$, is said to be *uniform* if the induced action on the space of distinct triples of $M$ is cocompact in addition to being properly discontinuous. It turns out that this is equivalent to demanding that every point of $M$ is a conical limit point — see [Tu2] or [Bo7]. In fact, hyperbolic groups are the only examples of uniform convergence groups [Bo7]. It thus gives a means of characterising hyperbolic groups dynamically. We note that a uniform convergence group has no parabolic elements and no infinite torsion subgroups. For dynamical proofs of these assertions, see [Tu2]. Geometric arguments, in the context of hyperbolic groups, can be found, for example, in [GhH].

Other examples of convergence groups are relatively hyperbolic groups acting on their boundaries. Relatively hyperbolic groups were introduced by Gromov [Gr] and generalise the notion of geometrically finite kleinian groups. For some discussion in relation to convergence groups, see [Bo8].

Theorem 6.1 is applicable to hyperbolic groups, as well as to relatively hyperbolic groups with mild restrictions on the class of groups that can arise as maximal parabolic groups. In the latter case, one has to take care in handling maximal parabolic groups that are two-ended. This is discussed in [Bo6]

We now return to the general set-up.

We suppose that $\Gamma$ acts a convergence group on the compact hausdorff space $M$. The most basic result is the classification of elements of $\Gamma$. Given $\gamma \in \Gamma$, write fix$(\gamma)$ for the set of fixed points of $\gamma$ in $M$.

**Definition :** An element $\gamma \in \Gamma$ is *elliptic* if it has finite order. It is *parabolic* if it has infinite order and fix$(\gamma)$ consists of a single point. It is *loxodromic* if it has infinite order and fix$(\gamma)$ consists of a pair of points.

The following result can be found in [GeM1] or [Tu1]:

**Lemma 6.2 :** *Suppose that $\Gamma$ acts as a convergence group on a compact hausdorff space, $M$, with at least three points. Then every element of $\Gamma$ is elliptic, parabolic or loxodromic.*
$$\diamondsuit$$

It's not hard to see from the following discussion that any iterate of a parabolic will be parabolic, and that any iterate of a loxodromic will be loxodromic. Note also that if $\gamma \in \Gamma$ is loxodromic, and $F \subseteq M$ is a closed invariant set, then either $F \subseteq$ fix$(\gamma)$ or fix$(\gamma) \subseteq F$.

Suppose that $\gamma \in \Gamma$ is loxodromic. We know that there are points, $\lambda, \mu \in M$, such that some subsequence of $(\gamma^n)_{n \in \mathbf{N}}$, restricted to $M \setminus \{\lambda\}$ converges locally uniformly to $\mu$. Now, it's easy to see that $\lambda$ and $\mu$ must be precisely the fixed points of $\gamma$. Let $U$ be any open set with $\mu \in U$ and $\lambda \notin \bar{U}$. It follows that there is some $m \in \mathbf{N}$ with $\gamma^m(U) \subseteq U$. Since $\mu$ is fixed by $\gamma$, the intersection $W = \bigcap_{i=0}^{m-1} \gamma^i(U)$ is non-empty. Moreover, $\gamma(W) \subseteq W$. Now,

if $K \subseteq M \setminus \{\mu\}$ is closed, and $V \ni \mu$ is open, there exist $p, q \in \mathbf{N}$ such that $\gamma^p(K) \subseteq W$ and $\gamma^q(W) \subseteq U$. Thus $\gamma^n(K) \subseteq V$ for all $i \geq p + q$. In words, the forward iterates of $\gamma$ restricted to $M \setminus \{\lambda\}$ converge locally uniformly to $\mu$, without the necessity of passing to a subsequence. Since the situation is symmetric with respect to simultaneously replacing $\gamma$ by $\gamma^{-1}$, subsets of $M$ with their complements, and swapping $\lambda$ and $\mu$, we see that the backward iterates of $\gamma$ restricted to $M \setminus \{\mu\}$ converge locally uniformly to $\lambda$. We refer to $\mu$ as the *attracting fixed point* of $\gamma$, and $\lambda$ as the *repelling fixed point*.

Note that the cyclic group $\langle \gamma \rangle$ acts properly discontinuously on $M \setminus \{\lambda, \mu\}$. In fact, it acts cocompactly on this set. To see this, choose disjoint closed neighbourhoods, $F$ and $F'$, of $\mu$ and $\lambda$ respectively. Let $U = M \setminus F$ and $V = M \setminus F'$, so that $U \cup V = M$. Now there is some $m > 0$ such that $U \cup \gamma^m(V) = M$. Let $K = \bar{U} \cap \gamma^m(\bar{V})$. Now it's not hard to see that the images of $K$ under $\langle \gamma^m \rangle$, and hence under $\langle \gamma \rangle$, must cover $M \setminus \{\lambda, \mu\}$.

A corollary of the above observation is that if $\gamma$ is loxodromic with $\mathrm{fix}(\gamma) = \{\lambda, \mu\}$, then $\langle \gamma \rangle$ has finite index in the group $G = \{g \in \Gamma \mid \{\lambda, \mu\} \subseteq \mathrm{fix}(g)\}$. To see this, fix some closed set $K$, whose $\langle \gamma \rangle$-orbit covers $M \setminus \{\lambda, \mu\}$, and choose any $x \in M \setminus \{\lambda, \mu\}$. Now given any $g \in G$, there is some $n(g) \in \mathbf{Z}$ such that $\gamma^{n(g)} g(x) \in K$. Since $\Gamma$ acts properly discontinuously on the space of distinct triples, we see that $\{\gamma^{n(g)} g \mid \gamma \in G\}$ is finite. Thus $[G, \langle \gamma \rangle] < \infty$ as claimed.

We also have

**Lemma 6.3 :** *If $\gamma \in \Gamma$ is loxodromic, then any element $\Gamma$ which fixed one of the fixed points of $\gamma$ must fix both of the fixed points of $\gamma$.*

**Proof :** Let $\mathrm{fix}(\gamma) = \{x, y\}$. We claim that if $h \in \Gamma$ and $x \in \mathrm{fix}(h)$, then $y \in \mathrm{fix}(h)$.

Without loss of generality, we can assume that $y$ is the attracting fixed point of $\gamma$. Suppose that $h^{-1}(y) = z \neq y$. Now choose any compact set $K \subseteq M \setminus \{x, y\}$, whose images under $\gamma$ cover $M \setminus \{x, y\}$, and choose any $w \in M \setminus \{x, y\}$.

Now given $n \in \mathbf{N}$, there is some $m(n) \in \mathbf{Z}$ such that $\gamma^{m(n)}(h\gamma^{-n}(w)) \in K$. Now $\gamma^{m(n)} h\gamma^{-n}(\gamma^n z) = \gamma^{m(n)} h(z) = \gamma^{m(n)}(y) = y$, and $\gamma^{m(n)} h\gamma^{-n}(x) = x$. Thus, the images of the triples $(x, w, \gamma^n(z))$ under $\gamma^{m(n)} h\gamma^{-n}$ remain in a compact region of the space of distinct triples. Also $\gamma^n(z) \to y$ as $n \to \infty$. Thus, since $\Gamma$ acts properly discontinuously on the space of distinct triples, we must have that the set $\{\gamma^{m(n)} h\gamma^{-n} \mid n \in \mathbf{N}\}$ is finite. We can thus find some $p > 0$, and $q \in \mathbf{Z}$ such that $\gamma^q h = h\gamma^p$. But now, $h(\gamma^p z) = \gamma^q(hz) = \gamma^q(y) = y = h(z)$, and so $\gamma^p z = z$. But $\gamma$ has no fixed points outside $\{x, y\}$, so we get a contradiction. $\diamond$

Putting this result together with the preceding remarks, we see that if $x \in M$ is the fixed point of a loxodromic, then its stabiliser is virtually cyclic. In other words:

**Proposition 6.4 :** *The stabiliser of any point of $M$, is either virtually cyclic, or consists entirely of elliptics and parabolics.* $\diamond$

We say that a subgroup of $\Gamma$ is *loxodromic* if it is virtually cyclic and contains a loxodromic. We say that a subgroup is *parabolic* if it has a unique fixed point in $M$.

In [Tu1] it is shown that any subgroup of $\Gamma$ is loxodromic, parabolic, or contains a free group of rank two. In particular, we see that the only possibility for an infinite torsion

subgroup is a parabolic group. ¿From Proposition 6.4, we see that a parabolic group is either an infinite torsion group or contains a parabolic element.

One can go on from here to describe a partition if $M$ in to a limit set and discontinuity domain, as for kleinian groups (see [GeM1]). However, we shall only be interested in "minimal" actions, where the discontinuity domain is empty. More precisely:

**Definition :** An action of $\Gamma$ on $M$ is *minimal* if $M$ has no proper nonempty closed $\Gamma$-invariant subset.

In particular, the orbit of any point under $\Gamma$ is dense (and so, if $\Gamma$ is countable, then $M$ is separable).

Note that with the definitions we have given, any group action on a singleton or a pair will qualify as a convergence action. (In the latter case it may be possible to have an infinite order element which swaps the two points, so our classification fails in this case.) Since we are only really interested in continua, a two-point space will never arise. However it is convenient to allow a singleton, since it saves us from having to make special qualifications about trivial quotients.

We make one more general observation about convergence actions:

**Lemma 6.5 :** *Suppose that $\Gamma$ acts as a convergence group on a compact hausdorff space, $M$. Suppose that $\sim$ is an equivalence relation on $M$ which gives a $\Gamma$-invariant upper-semicontinuous decomposition of $M$ (so that the quotient $M/\sim$ is hausdorff). Then, the induced action of $\Gamma$ on $M/\sim$ is also a convergence action.*

**Proof :** Let $D = M/\sim$, and let $f : M \longrightarrow D$ be the quotient map. Suppose that $(\gamma_n)_n$ is a sequence of distinct elements of $\Gamma$. Let $(\gamma_i)_i$ be a subsequence, and let $\lambda, \mu \in M$ be such that $\gamma_i|M\setminus\{\lambda\}$ converges locally uniformly to $\mu$. Suppose $K \subseteq D\setminus\{f(\lambda)\}$ and $K' \subseteq D\setminus\{f(\mu)\}$ be closed subsets of $D$. Thus, $f^{-1}(K) \not\ni \lambda$ and $f^{-1}(K') \not\ni \mu$ are closed subsets of $M$. Thus, for all sufficiently large $i$, we have $f^{-1}(K' \cap \gamma_i K) = f^{-1}(K) \cap \gamma_i f^{-1}(K') = \emptyset$. Thus, $K' \cap \gamma_i K = \emptyset$. This gives the convergence condition for $D$. $\diamondsuit$

(An alternative proof of this is given in [Bo5].) Note that an element which is parabolic for the action on $M$ will also be parabolic for the action on $N$; whereas an element which is loxodromic for the action on $M$ may be either parabolic or loxodromic for the action on $N$. In the last case, the preimage of a loxodromic fixed point in $N$ will be single loxodromic fixed point in $M$.

Returning to Theorem 6.1, we should explain the term "one-ended". To any finitely generated group, $\Gamma$, one can associate a space of ends, namely the space of ends of the corresponding Cayley graph. It's not hard to see that this is well defined, independently of the choice of finite generating set. In fact, there are just four possibilities for this space, namely:

(1) $\Gamma$ is finite and has no ends.

(2) $\Gamma$ has one end (in some sense the "generic" situation).

(3) $\Gamma$ has two ends and is virtually infinite cyclic.

(4) The space of ends ends is a Cantor set, and Γ splits non-trivially as an amalgamated free product or an HNN-extension over a finite subgroup.

(The fact that these are the only possibilities was shown by Hopf [Hop]. The fact that in Case (4) Γ splits over a finite subgroup is due to Stallings [St].)

In particular, we note that if Γ is one-ended, then it does not split over a finite subgroup.

We shall want to relate this to group actions on trees.

By a "graph of groups", we shall mean an ordered triple, $(V, E, G)$, where $(V, E)$ is a finite or countable connected graph, and $G$ associates to each vertex, $v \in V$, a "vertex group" $G(v)$, and to each edge $e \in E$, an "edge group" $G(e)$. Moreover, if $v$ is incident on the edge $e$, then we are given a particular monomorphism from $G(e)$ into $G(v)$. To such a graph of groups, we may associate the fundamental group, $\pi_1(V, E, G)$.

Suppose that $e \in E$. Now, if $e$ lies in a cycle in $(V, E)$, then $\pi_1(V, E, G)$ can be written as an HNN-extension over $G(e)$. If $e$ does not lie in a cycle, then removing $e$ splits the graph into two connected subgraphs, $(V_1, E_1)$ and $(V_2, E_2)$. Now we can restrict $G$ to these subgraphs to give us two graphs of groups, $(V_1, E_1, G_1)$ and $(V_2, E_2, G_2)$. In this case, $\pi_1(V, E, G)$ splits as an amalgamated free product over $G(e)$, namely

$$\pi_1(V, E, G) \cong \pi_1(V_1, E_1, G_1) *_{G(e)} \pi_1(V_2, E_2, G_2).$$

As described at the end of Section 3, a "simplicial tree" is a connected graph with no cycles. We recall (Lemma 3.34) that (the vertex set of) a simplicial tree is essentially the same structure as a discrete median pretree.

Suppose $(S, I)$ is a simplicial tree, with vertex set $S$, and edge set $I$. Suppose Γ acts without edge inversions on $S$. Given $p \in S$ or $f \in I$, write $\Gamma(p)$ and $\Gamma(f)$ for the vertex and edge stabilisers respectively. Thus, if $p, q \in S$ are the endpoints of the edge $f$, then $\Gamma(f) = \Gamma(p) \cap \Gamma(q)$. Let $V = S/\Gamma$ and $E = I/\Gamma$, so that $(V, E)$ is the quotient graph. There is a natural graph of groups, $(V, E, G)$, such that if $v \in V$ and $e \in E$, then $G(v)$ and $G(e)$ are naturally isomorphic to the stabilisers, $\Gamma(\tilde{v})$ and $\Gamma(\tilde{e})$, where $\tilde{v}$ and $\tilde{e}$ are lifts of $v$ and $e$ to $(S, I)$. Moreover, we have $\Gamma \cong \pi_1(V, E, G)$.

**Lemma 6.6 :** *Suppose that Γ acts without edge inversions on a simplicial tree such that all the edge stabilisers are finite. If Γ is a one-ended finitely generated group, then Γ fixes some vertex of the tree.*

**Proof :** Let $(V, E, G)$ be the quotient graph of groups. Thus, $G(e)$ is finite for all $e \in E$. Since Γ does not split as an HNN-extension over any $G(e)$, we see that $(V, E)$ cannot contain any cycles, and so is itself a tree.

Now suppose that $e \in E$. Let $(V_1, E_1, G_1)$ and $(V_2, E_2, G_2)$ be as above. Now the splitting over $G(e)$ must be trivial, and so, without loss of generality, we have $\pi_1(V_1, E_1, G_1) \subseteq G(e)$. We put an arrow on the edge $e$ pointing from $(V_1, E_1)$ to $(V_2, E_2)$.

Now, it's easy to see that at each vertex, $v \in V$, there is at most one incident edge pointing away from $v$. If all incident edges point towards $v$, we refer to $v$ as a "sink". Now, either there is a (unique) sink, or else there is an infinite ray of edges all of which

are directed towards infinity. In the former case, we see that $G(v)$ must support the whole of $\pi_1(V, E, G)$, and so the stabiliser of a lift (in fact *the* lift) of $v$ to $S$ will be the whole group $\Gamma$. In the latter case, we could write $\Gamma$ as an ascending chain of finite subgroups. But since $\Gamma$ is finitely generated, we would deduce that $\Gamma$ itself were finite. $\diamondsuit$

**Lemma 6.7 :** *Suppose that $\Gamma$ acts on a bipartite tree, $S = S_0 \sqcup S_1$, preserving each of $S_0$ and $S_1$. Suppose that whenever two distinct edges $e$ and $e'$ meet at a vertex in $S_0$, then either $\Gamma(e)$ or $\Gamma(e')$ is finite. Suppose that $\Gamma$ is finitely generated and one-ended. Then $\Gamma$ fixes some vertex of $S$.*

**Proof :** We modify the proof of the last lemma. Let $(V, E, G)$ be the quotient graph of groups. Thus $V = V_0 \sqcup V_1$ where $V_0 = S_0/\Gamma$ and $V_1 = S_1/\Gamma$. Let $E_\infty$ be the set of edges $e \in E$ for which $\Gamma(e)$ is infinite.

Now any cycle in $(V, E)$ would have to contain a vertex in $V_0$ and hence an edge $e \in E \setminus E_\infty$. Thus $\Gamma$ would split as an HNN-extension over $G(e)$. We conclude that $(V, E)$ is a tree.

Now, we can put arrows on the edges of $E \setminus E_\infty$, as in the proof of Lemma 6.6. In this case, the same argument leads us to a connected subgraph of groups $(V', E', G')$, with $E' \subseteq E_\infty$, and with $\pi_1(V, E, G)$ supported on $\pi_1(V', E', G')$. Now, if $V$ consists of a single vertex, we are done. The other possibility is that it consists of a single vertex, $v \in V_1$, together with a set of adjacent vertices in $V_0$.

Suppose $w \in V' \cap V_0$, and let $e \in E'$ be edge joining $w$ to $v$. Now, we must have $G(w) \subseteq G(e)$, otherwise, there would be at least two edges of the simplicial tree $S$, both incident on the same lift of $w$ and which both project to $e$. But since the stabilisers of these edges are infinite, we contradict the hypotheses of the lemma. It follows that we have $\Gamma(w) \subset G(v)$ for all $w \in V' \cap V_0$. Thus $\pi_1(G, E, V)$ is supported on $G(v)$, and so $v$ is fixed by $\Gamma$. $\diamondsuit$

We now set about the proof of Theorem 6.1. Let $M$ be a continuum. Let $T \subseteq M$ be any set of cut points. We begin with a refinement of Lemma 5.11, which gives a complete description of the sets $R_M(p)$ for $p \in \Phi \setminus T$. Write $\Lambda(p)$ for the set of elements of $\Phi$ which are adjacent to $p$. Thus $\Lambda(p) \subseteq T$. Note that, as observed in Section 3, the set $\{p\} \cup \Lambda(p)$ is full. Also, if $a, b \in \Lambda(p)$ are distinct, then $apb$.

**Lemma 6.8 :** $R_M(p) = \phi^{-1}(p) \cup \Lambda(p)$.

**Proof :** Suppose $x \in R_M(p) \setminus \phi^{-1}(p)$. Let $a = f(x)$. By Lemma 5.11, we have $a \in \Lambda(p)$. Now $x \in R_M(a, p) = R_M(a, b)$ for some $b \in T$ with $\neg pab$. Thus $\neg xab$. Since $a = \phi(x)$, we have $\text{flow}(a)|T = \text{flow}(x)|T$. Since $\neg bax$, we must have $x = a$, and so $x \in \Lambda(p)$. This shows that $R_M(p) \subseteq \phi^{-1}(p) \cup \Lambda(p)$.

Conversely, suppose $p \in \Phi \setminus T$. If $p \in \phi^{-1}(p)$, then $\text{flow}(x)|T = \text{flow}(p)|T$. Since $x \notin T$, we have $bax \Leftrightarrow bap$ for all $b, a \in T$, and so $x \in R_M(p)$.

Finally, suppose $b \in \Lambda(p)$. Given $a \in T$, we have $\neg bap$ and so, by definition, $R_M(a, p) = R_M(a, b)$. In particular, $b \in R_M(a, b)$. Thus $b \in R_M(p)$. $\diamondsuit$

Note that we always have that $R_M(p) \neq \emptyset$ (Proposition 5.14).

If $a \in T$, we shall define $R_M(a) = \{a\}$. We have thus associated to each point $p \in \Phi$ a non-empty closed (connected) subset of $M$. Any two such sets are either disjoint, or intersect in a single point of $T$.

Note that $R_M(p)$ might a singleton in essentially three cases, namely, if $p \in T$, if $p \in \Phi \setminus T$ and $p$ is not adjacent to any element of $\Phi$, or if $p \in \Phi \setminus T$, and $p$ is adjacent to a single element of $\Phi$.

Let $\Phi'$ be the set of those elements of $\Phi \setminus T$ which are adjacent of precisely one element of $T$. (These are all of type $P(1,1)$ as described in the classification of flows in Section 3.) The elements of $\Phi'$ play no useful role in the argument, so it will be convenient to forget about them as far as possible. We set $\Phi_0 = \Phi \setminus \Phi'$. Note that $\Phi/\simeq$ can be identified with $\Phi_0/\simeq$, where $\simeq$ is the finite interval relation as defined in Section 4.

Suppose now that $\Gamma$ acts as a convergence group on $M$. We assume that $T \subseteq M$ is a $\Gamma$-invariant set of cut points. The map $\phi : M \longrightarrow \Phi$ is natural and thus equivariant. We get an induced action of $\Gamma$ on $\Phi$. These actions must agree on $T$, so there is no ambiguity in writing $\gamma(p)$ for the image of $p \in \Phi$ under $\gamma$.

Given an element $\gamma \in \Gamma$, we write $F(\gamma)$ for the set of fixed points of $\gamma$ in $\Phi$. Note that every point of $F(\gamma) \setminus \Phi_0$ is adjacent to a (unique) point of $F(\gamma) \cap T$.

**Lemma 6.9 :** *Suppose that $\gamma$ is loxodromic, then either*

*(1) $F(\gamma) \cap \Phi_0$ consists of a single point, two adjacent points, or a single point of $\Phi_0 \setminus T$ together with two adjacent points of $T$, and each element of $F(\gamma) \setminus \Phi_0$ is adjacent to some element of $F(\gamma) \cap T$.*

*(2) $F(\gamma)$ consists of two points of $p, q \in \Phi_0 \setminus T$, both of which are terminal in $\Phi$, and such that the interval $[p, q]$ is infinite.*

**Proof :** First, we observe that $F(\gamma) \cap \Phi_0 \neq \emptyset$. To see this, note that if $\mu \in \text{fix}(\gamma)$, then $f(\mu) \in \Phi$ is fixed by $\gamma$. If it should happen that $f(\mu) \notin \Phi_0$, then the unique adjacent point of $T$ will also be fixed by $T$. In any case, we get at least one fixed point in $\Phi_0$.

We next claim that it is not possible for $F(\gamma) \cap \Phi_0$ to contain an element of $T$ together with two adjacent elements. For suppose, to the contrary, that $p, q \in \Phi_0 \setminus T$, and $a \in \Lambda(p) \cap \Lambda(q) \subseteq T$, and that $a, p, q \in F(\gamma)$. Now the sets $R_M(p)$ and $R_M(q)$ are closed and $\gamma$-invariant, and $R_M(p) \cap R_M(q) = \{a\}$. But since $p, q \in \Phi_0 \setminus T$, we have that these sets must each contain at least two elements. Thus, $\text{fix}(\gamma) \subseteq R_M(p) \cap R_M(q)$. Thus $\text{fix}(\gamma) = \{a\}$ contradicting the supposition that $\gamma$ is loxodromic.

Note also that $F(\gamma) \cap T$ can contain at most two points.

Suppose, for the moment, that for all $p, q \in F(\gamma) \cap \Phi_0$, we have that $[p, q]$ is finite. In this case $[p, q]$ must be fixed pointwise by $\gamma$, Lemma 3.28 tells us that it must consists of an alternating sequence of elements of $T$ and $\Phi \setminus T$. Note also that $[p, q] \subseteq \Phi_0$. Thus, $[p, q] \subseteq F(\gamma)$. We see that $|[p, q] \setminus T| \leq 1$ and that $|[p, q] \cap T| \leq 2$. Thus, $|[p, q]| \leq 3$, and if $|[p, q]| = 3$ then $p, q \in T$. Now, since this applies to any interval in $F(\gamma) \cap \Phi_0$, we see that if $F(\gamma) \cap \Phi_0$ contains more than two points, it must consist of a point $p \in \Phi_0 \setminus T$, together with a subset of the set $\Lambda(p)$ of adjacent points. Since $\Lambda(p) \subseteq T$, this subset has at most two elements.

We can thus assume that $F(\gamma) \cap \Phi_0$ contains two elements $p$ and $q$ for which $[p, q]$ is infinite. Now $R_M(p)$ and $R_M(q)$ are disjoint closed $\gamma$-invariant subsets of $M$, and so must each consist of a single point, i.e. $R_M(p) = \{p\}$ and $R_M(q) = \{q\}$.

Now, suppose that $r \in F(\gamma) \cap \Phi_0 \setminus \{p, q\}$. Without loss of generality, $[p, r]$ is infinite, so the same argument shows that $R_M(r) = \{r\}$. Since $\gamma$ has only two fixed points in $M$, we get a contradiction. We conclude that $F(\gamma) \cap \Phi_0 = \{p, q\}$.

We next show that $p, q \in \Phi_0 \setminus T$ and that $p$ and $q$ are terminal in $\Phi$.

Now, $(p, q)$ and hence $(p, q) \cap T$ are infinite. Define $N(p) = \bigcap \{R_M(a, b) \mid a, b \in T, qabp\}$. Note that if $abp$, then $\neg bap$, so $R_M(a, b) = R_M(a, p)$. We conclude that $R_M(p) \subseteq N(p)$. In particular, $N(p) \neq \emptyset$. Also $N(p)$ is closed and $\gamma$-invariant. We similarly define $N(q)$. Note that $N(p) \cap N(q) = \emptyset$. (Choose $a, b, c, d \in T$ with $pabcdq$. Then $N(p) \subseteq R_M(b, a) \cap R_M(c, d) = \emptyset$.) It follows that $N(p)$ and $N(q)$ are both singletons.

Now suppose, for contradiction, that $p \in T$. Then $N(p) = \{p\}$. Choose any $c \in (p, q)$. Since $p$ is a cut point of $M$, there is some $x \in M$ with $xpc$. It follows easily that $apx$ for all $a \in (p, q) \cap T$. Now suppose $qabp$. We must have $abx$ so $\neg bax$ and so $x \in R_M(a, b)$. Thus $x \in N(p) = \{p\}$ and so $x = p$. This contradiction shows that $p \notin T$. Similarly $q \notin T$.

A similar argument shows that $p$ and $q$ are terminal in $\Phi$. For suppose, to the contrary, that $p$ is not terminal. There is some $c \in \Phi$ with $cpq$. We can assume that $c \in T$. Now, if $a, b \in T$ with $qabp$, we must have $qabpc$. In particular $abc$, so $\neg bac$, and so $c \in R_M(a, b)$. We see that $c \in N(p) = \{p\}$. We get the contradiction that $c = p$.

Finally, suppose that $F(\gamma)$ contains two elements $r, s$ with $[r, s]$ infinite. If either of these does not lie in $\Phi_0$, it must be adjacent to an element of $F(\gamma) \cap T$. But this would contradict what we have already shown. We conclude that in this case we have $F(\gamma) = \{r, s\} \subseteq \Phi_0$. $\diamondsuit$

**Lemma 6.10 :** *Suppose $\gamma$ is a parabolic whose fixed point, $a \in M$, does not lie in $T$. Then, $F(\gamma)$ consists of a single point of $\Phi_0 \setminus T$.*

**Proof :** We know that $F(\gamma) \cap T = \emptyset$. It follows immediately that $F(\gamma) \subseteq \Phi_0$. As in the proof of Lemma 6.9, we see that $F(\gamma) \neq \emptyset$. Also, if $p, q \in F(\gamma)$ are distinct, then $[p, q]$ is infinite (otherwise it would contain a point of $F(\gamma) \cap T$). Now, $R_M(p)$ and $R_M(q)$ are disjoint closed $\gamma$-invariant sets, contradicting the fact that $\gamma$ is parabolic. This shows that $F(\gamma)$ contains just one point. $\diamondsuit$

(We remark that if $\gamma$ is parabolic with fixed point $a \in T$, the $F(\gamma)$ consists of the point $a$ together with a set of adjacent points of $\Phi$. We shall not need this fact here.)

The proof of Theorem 6.1 will proceed by transfinite induction. Recall that $D(M) \equiv \Psi = \Phi/\approx$, where $\approx$ is the minimal codense relation on $\Phi$, as defined in Section 4. Also, $\approx = \sim_\alpha$ for some ordinal $\alpha$. The idea will be to show inductively that no element of $\Phi/\sim_\beta$ is fixed by the whole of $\Gamma$. In particular, it follows that $\Phi/\sim_\alpha$ is non-trivial.

The hypothesis that $M$ has a cut point is used to get the induction started. (Recall that $\sim_0$ is just equality on $\Phi$.) We assume that $\Gamma$ acts minimally on $M$.

**Lemma 6.11 :** *No element of $\Phi$ is fixed by $\Gamma$.*

**Proof :** First, we claim that if $p \in \Phi$, then $R_M(p) \neq M$. Suppose, to the contrary, that $R_M(p) = M$. Clearly $p \notin T$. Let $a$ be any point of $T$. Now, $a$ is a cut point, so we can find $x, y \in M$ with $xay$. Now, $x, y \in R_M(p) \subseteq R_M(a, p)$. By definition, $R_M(a, p) = R_M(a, b)$ for some $b \in T \setminus \{a\}$. We have $bax \vee bay$. But since $x, y \in R_M(a, b)$, we have $\neg bax \wedge \neg bay$. This proves the claim.

But now, if $p \in \Phi$ were fixed by $\Gamma$, $R_M(p)$ would be a closed nonempty $\Gamma$-invariant subset. By the minimality hypotheses, we would deduce that $R_M(p) = M$. $\diamondsuit$

Now, let $\simeq \; = \; \sim_1$ be the finite interval relation on $\Phi$ (Section 4). Recall that a $\simeq$-equivalence class, $X$, is a (maximal) discrete full subset. Since $X$ is full, it must itself be a median pretree, and so by Lemma 3.34, it can be thought of as a simplicial tree.

At this point we need to assume that $\Gamma$ is one-ended. ¿From the observation following the statement of Theorem 6.1, we can assume that the stabiliser of any point of $T$ is either finite or loxodromic (see Proposition 6.4).

**Lemma 6.12 :** *No element of $\Phi/\simeq$ is fixed by $\Gamma$.*

Suppose, for contradiction, that $\Gamma$ preserves setwise the $\simeq$-equivalence class, $X \subseteq \Phi$. Now $S = X \cap \Phi_0$ is full and nonempty (since any element of $X$, which is not terminal in $X$ must be adjacent to at least two elements of $X$). Thus, $S$ is a discrete median pretree, and so by Lemma 3.34, it is a simplicial tree. Let $S_0 = S \cap T$ and $S_1 = S \setminus T$. Thus $S = S_0 \sqcup S_1$, and $\Gamma$ preserves $S_0$ and $S_1$. By Lemma 3.28, this a bipartite partition, (i.e. each edge of $S$ has one endpoint in each of $S_0$ and $S_1$). We aim to verify the hypotheses of Lemma 6.6.

Suppose that $a \in S_0$ is adjacent to distinct points $p, q \in S_1$, and that each of the edge stabilisers, $\Gamma(a) \cap \Gamma(p)$ and $\Gamma(a) \cap \Gamma(q)$ is infinite. Now, $\Gamma(a)$ is infinite, and hence must be loxodromic. (See the remark before the statement of the lemma.) Thus, $\Gamma(a) \cap \Gamma(p)$ and $\Gamma(a) \cap \Gamma(q)$ each have finite index in $\Gamma(a)$. It follows that there must be some infinite order element $\gamma \in \Gamma(p) \cap \Gamma(q)$. Now $\gamma$ is loxodromic and fixes $p$, $a$ and $q$. But $a \in T$ and $p, q \in \Phi_0$, contradicting Lemma 6.9. This shows that at least one of $\Gamma(a) \cap \Gamma(p)$ or $\Gamma(a) \cap \Gamma(q)$ must be finite.

Now, the hypotheses of Lemma 6.6 are satisfied, so we see that $\Gamma$ must fix some element of $S \subseteq \Phi$, which contradicts Lemma 6.11. $\diamondsuit$

Before we finish the proof of Theorem 6.1, we need a lemma about complete median pretrees. Suppose that $\Theta$ is a complete median pretree, and that $X$ and $Y$ are disjoint nonempty full subsets, and that $X \cup Y$ is also full.

Suppose that there is some $p \in Y$ with the property that for some $x \in X$, $[x, p] \cap Y = \{p\}$, and so $[x, p] \cap X = [x, p)$. Now, if $z \in X$, we have $[x, z] \subseteq X$. Thus $\mathrm{med}(x, z, p) \in [x, p) \cap [z, p)$ and so $[x, p)$ and $[z, p)$ are cofinal. In particular, we see that, in fact, $[x, p] \cap Y = \{p\}$ for all $x \in X$. Suppose that there is some $q \in Y$ such that $[x, q] \cap Y = \{q\}$. Then since $\mathrm{med}(x, p, y) \in [p, q] \subseteq Y$, we see that $p = q$. In other words, the point $p \in Y$, if it exists, is uniquely determined. We write it as $p = p(Y, X)$.

Suppose that $x \in X$ and $y \in Y$. Let $q = \mathrm{med}(x, y, p)$. Then we see easily that $[x, q] \cap Y = \{q\}$ and so, in fact, $q = p$. This shows that $[x, y] \cap Y = [p, y]$.

**Lemma 6.13 :**   *Suppose that $\Theta$ is a complete median pretree, and that $X, Y \in \Theta$ are disjoint, full and nonempty subsets with $X \cup Y$ full. Then either $p(X, Y)$ or $p(Y, X)$ exists.*

**Proof :** Choose any $x \in X$ and $y \in Y$. Now $[x, y] \cap X$ and $[x, y] \cap Y$ gives a partition of $[x, y]$ into two non-empty subarcs. By completeness, $[x, y] \cap X$ has the form $[x, p]$ or $[x, p)$ for some $p \in \Theta$. In the former case, we must have $p \in X$. Since $[y, p] \subseteq [x, y]$, we must clearly have $[y, p] \cap X = \{p\}$ and so $p = p(X, Y)$. In the latter case, we can assume that $p \in [x, y]$. (For if not, let $q = \text{med}(x, y, p) \in [x, y]$. Then $[x, p) = [x, q]$ and we are reduced to the first case.) Thus, $p \in Y$, and so $[x, p] \cap Y = \{p\}$. Thus $p = p(Y, X)$.                    $\diamond$

**Proof of Theorem 6.1 :**   We prove, by transfinite induction on the ordinal $\alpha$, that $\Gamma$ cannot fix any element of $\Phi/\sim_\alpha$. Lemma 6.12 has already done this for $\simeq = \sim_1$, so we can assume that $\alpha > 1$.

Suppose first that $\alpha$ is a limit ordinal, and suppose that the $\sim_\alpha$-equivalence class $\Xi \subseteq \Phi$ is preserved setwise by $\Gamma$. Let $\{\gamma_1, \ldots, \gamma_n\}$ be a finite generating set for $\Gamma$. Choose any $x \in \Xi$. Now for each $i \in \{1, \ldots, n\}$, we have $x \sim_\alpha \gamma_i(x)$, and so, by the definition of $\sim_\alpha$ as $\bigvee\{\sim_\beta \mid \beta < \alpha\}$, we must have $x \sim_{\beta_i} \gamma_i(x)$ for some $\beta_i < \alpha$. Let $\beta = \max\{\beta_1, \ldots, \beta_n\} < \alpha$. Now, $x \sim_\beta \gamma(x)$ for each $\gamma \in \{\gamma_1, \ldots, \gamma_n\}$ and hence for all $\gamma \in \Gamma$. Thus, the $\sim_\beta$-equivalence class containing $x$ is preserved by $\Gamma$. In other words $\Phi/\sim_\beta$ contains a point fixed by $\Gamma$, contrary to the inductive hypothesis.

Thus we can assume that $\alpha = \beta + 1$ is a successor ordinal. Now $\Phi/\sim_\alpha = (\Phi/\sim_\beta)/\simeq$, where $\simeq$ denotes the finite-interval relation in $\Phi/\sim_\beta$. Suppose, for contradiction, that the $\sim_\alpha$-equivalence class $\Xi \subseteq \Phi$ is preserved by $\Gamma$. Now, $\Xi/\sim_\beta \subseteq \Phi/\sim_\beta$ is a finite-interval-equivalence class, and hence a simplicial tree. Write $\Sigma = \Xi/\sim_\beta$. Thus $\Sigma$ admits a $\Gamma$-action, which by the inductive hypotheses has no $\Gamma$-invariant vertex. We first show that there are no edge inversions on $\Sigma$.

Suppose that $X, Y \in \Sigma$ are adjacent vertices. Thus $X \cup Y$ is a full subset of $\Phi$. We claim that the points $p(X, Y)$ and $p(Y, X)$, as described by Lemma 6.13, cannot both exist. For if they did, they would clearly have to be adjacent in $\Phi$, and thus be identified by the finite-interval relation $\sim_1$. Since we are assuming that $\alpha > 1$ and so $\beta \geq 1$, they would also be identified by $\sim_\beta$, and so could not lie in distinct $\sim_\beta$-equivalence classes. Thus, applying Lemma 6.13, we see that precisely one of the points $p(X, Y)$ or $p(Y, X)$ exists. It follows that no element of $\Gamma$ can swap $X$ and $Y$. We have thus shown that there are no edge inversions on $\Sigma$.

The next objective will be to show that if the stabiliser of an edge of $\Sigma$ is infinite, then one of the incident vertices will be terminal in $\Sigma$.

Suppose then that $X, Y \in \Sigma$ are adjacent. We can suppose that $p = p(Y, X) \in Y$ exists. Now if the edge stabiliser, $\Gamma(X) \cap \Gamma(Y)$ were infinite, it would have to contain an infinite order element, $\gamma$. Moreover, $\gamma$ must fix $p$.

Let $W$ be the set of point $x \in \Phi$ such that $\neg x p X$. Thus $X \subseteq W$, and $Y \cap W = \emptyset$. Moreover if $x, y \in W$, then $[x, p)$ and $[y, p)$ are cofinal (since $\text{med}(x, y, p) \neq p$). Since $X$ and $p$ are $\gamma$-invariant, so is $W$. We claim that $W$ contains a fixed point of $\gamma$.

Suppose, to the contrary, that $W$ contains no fixed point of $\gamma$. We construct a $\gamma$-invariant arc in $W$ as follows. Choose any $y \in X$, and choose $x \in [y, p) \cap [\gamma(y), p) \subseteq X$.

Now, $\gamma(x) \in \gamma([y, p)) = [\gamma(y), p)$. Since $x$ and $\gamma(x)$ are, by assumption, distinct, and both lie in $[\gamma(y), p)$, we must have either $x \in (p, \gamma(x))$ or $\gamma(x) \in (p, x)$. Without loss of generality (replacing $\gamma$ by $\gamma^{-1}$ and $x$ by $\gamma(x)$ if necessary), we can assume that $x \in (p, \gamma(x))$. Now, $\gamma^n(x) \in \gamma^n(p, \gamma(x)) = (p, \gamma^{n+1}(x))$, and so $(p, \gamma^n(x)) \subseteq (p, \gamma^{n+1}(x))$. It follows that $A = \bigcup_{n=0}^{\infty}(p, \gamma^n(x))$ is an arc. Clearly, $\gamma(A) = A$. Also, since the arcs $(p, \gamma^n(x))$ are strictly increasing, it's easy to see that $A$ is endless. Since $\Phi$ is complete, there is some $z \in \Phi$ such that $A = (p, z)$. Moreover, since $\Phi$ is a median pretree, this $z$ is determined uniquely, and so $\gamma(z) = z$. Now, since $y \in A = (p, z)$, we have $pyz$ and so $\neg zpy$. Since $y \in X$ we have $\neg zpX$, and so $z \in W$. Thus $W$ must have contained a fixed point of $\gamma$ after all.

Let $q \in W$ be a fixed point of $\gamma$. Since $q \notin Y$, we have $p \not\sim_\beta q$ and so $p \not\sim_1 q$. In other words, $[p, q]$ is infinite.

Now Lemmas 6.9 and 6.10, we see that we see that $\gamma$ must be loxodromic. In fact, we are in case (2) of Lemma 6.9, and so, in particular, $p$ is terminal in $\Phi$.

Now it follows that, in fact, $Y = \{p\}$. For suppose $y \in Y$. Choose any $x \in X$. Then $[x, y] \cap Y = [y, p]$. In particular, $p \in [x, y]$, so since $p$ is terminal, and $p \neq x$, we must have $p = y$. It now follows immediately that $Y$ must be a terminal point of $\Phi/\sim_\beta$, and so in particular of $\Sigma = \Xi/\sim_\beta$.

In summary, we have shown that if an edge of $\Sigma$ is stabilised by an infinite group, then one its enpoints must be terminal. Now if we delete from $\Sigma$ each such edge together with its terminal endpoint, we obtain a simplicial tree $S \subseteq \Sigma$. all of whose edge stabilisers are finite. By Lemma 6.5, we see that $\Gamma$ must fix some vertex of $S$, i.e. some element of $\Phi/\sim_\beta$, contrary to the inductive hypothesis.

In summary, we conclude that for each ordinal $\alpha$, no vertex of $\Phi/\sim_\alpha$ is fixed by $\Gamma$. In particular, $\Phi/\sim_\alpha$ is non-trivial. Now, by Lemma 4.4, the minimal codense relation on $\Phi$ has the form $\sim_\alpha$ for some ordinal $\alpha$. We deduce that the quotient by the minimal codense relation is non-trivial. $\diamondsuit$

As explained in the introduction, one of the main applications we have in mind is to the boundaries of hyperbolic groups.

Suppose that $\Gamma$ is a (word) hyperbolic group in the sense of Gromov [Gr] (see also [GhH]). The boundary, $\partial\Gamma$, of $\Gamma$ is a compact metrisable topological space. Moreover, $\Gamma$ acts on $\partial\Gamma$ as a convergence group (without parabolics) in the sense of [GeM1] (see [F,Tu1,Bo5]). Also the orbit of every point under $\Gamma$ is dense, and so the action is minimal. If $\Gamma$ is one-ended, then $\partial\Gamma$ is a continuum. Moreover, a hyperbolic group cannot contain an infinite torsion subgroup. Thus, if $\partial\Gamma$ has a (global) cut point, we see that Theorem 6.1 gives an equivariant quotient which is a non-trivial dendrite. This proves Theorem 0.1 described in the introduction.

As mentioned in the introduction, we can use this construction together with that of [L] or [Bo3] to obtain a splitting of $\Gamma$ over a two-ended subgroup. (For this we need to assume that $\Gamma$ if finitely presented and has no infinite torsion subgroup.) With certain additional hypotheses, using an idea of Swarup [Swa] one can show that every global cut point must be a parabolic fixed point see [Bo6]. This gives the result [Swa] that a one-ended hyperbolic group has no global cut point. In fact, the argument can be applied to the case

of relatively hyperbolic groups (in particular, geometrically finite kleinian groups) to show that if the boundary (or limit set) is connected, then every global cut point is a parabolic fixed point. For this one need to place certain mild restrictions on the class of groups that can occur as maximal parabolic groups. (It is sufficient to assume that they are one or two-ended, finitely presented, and not infinite torsion groups. Probably only the last of these assumptions is really important.) These restrictions are redundant for geometrically finite kleinian groups (or indeed geometrically finite groups acting on hadamard manifolds of pinched negative curvature). One can go on to show that such boundaries are locally connected. The details are set out in [BoS,Bo9,Bo10]. In the case of kleinian groups, this has for some time been an open problem in dimension greater than 3.

Recently, Swenson [Swe] suggested an alternative route through some of the dendrite constructions described above. Thus, starting with a convergence action on a continuum, one uses the pretree structure to construct a "non-nesting" action on a real tree, and then applies the result of [L]. Although the construction is less natural than the quotient dendrite, it avoids the necessity of demonstrating non-triviality directly, so it is may prove to be both simpler and more powerful. Finding an optimal route through these various construction so as to give the strongest possible result remains an ongoing project.

It is natural to wonder what hypotheses on a minimal convergence action on a continuum are necessary in order to force local connectness. Indeed, I know of no counterexample for finitely generated groups, though it would indeed be remarkable if this were sufficient. In this connection, it is worth mentioning the classical conjecture that the limit set of a finitely generated 3-dimensional kleinian group is locally connected if it is connected. (There are certainly counterexamples if one drops the assumption of finite generation.) Some progress has been made of this question by Cannon and Thurston, Minsky and others, though the techniques employed are very different to those discussed here (see, for example, [Min]). I don't know of any work on the problem in higher dimensions. One might wonder if a dynamical approach might bring any new insights, though it is unclear where to begin, or how one might bring the finite generation hypothesis into play.

## 7. Dendrons.

We shall use the term "dendron" to mean a compact real tree, where a "real tree" as defined in Section 1 can be described as a uniquely arc-connected locally arc-connected topological space. (Thus a "dendrite" is just a separable dendron. For most of this section, the separability assumption will not be needed.)

We shall be principally interested here in isometric group actions on $\mathbf{R}$-trees. Such an action naturally gives rise to an action on a dendron. We shall see that the latter action is a convergence action if and only if the action on the $\mathbf{R}$-tree satisfies a certain "edge-discreteness" condition (Proposition 7.2).

For the purpose of analysing boundaries of hyperbolic groups with cut points, the main interest is in going in the opposite direction. Thus to a convergence action on a dendron (or dendrite), one would aim to associate an isometric action on an $\mathbf{R}$-tree. We discuss this further later in this section.

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Before formulating the theorems, we recall some results about $\mathbf{R}$-trees, and give some examples of the phenomena we are interested in.

There are many equivalent ways to define an $\mathbf{R}$-tree. A common way is to regard it as a special case of an $\Lambda$-tree where $\Lambda$ is an ordered abelian group. They were introduced in this form in [MorS]. (See also [Sha1,Sha2,Mor,Pa2] for more discussion.) We shall see some other ways of defining $\mathbf{R}$-trees later. The term "edge" is sometimes used to mean a non-trivial closed interval, particularly with reference to "edge-stabilisers". Thus, a common hypothesis is to place restrictions on the possible groups that can arise as edge-stabilisers.

Suppose $\Gamma$ acts isometrically on an $\mathbf{R}$-tree. Suppose $\Gamma$ is finitely generated and acts freely. A theorem of Rips tells us that $\Gamma$ is a free product of surface groups and free abelian groups (see, for example [GaLP]). An extension of this classification theorem has been given by Bestvina and Feighn [BeF2]. In particular they show that if $\Gamma$ is finitely presented and acts stably with cyclic edge stabilisers, then $\Gamma$ splits over a small subgroup. Here "stable" means that given a nested decreasing sequence of edges, the corresponding edge stabilisers are eventually constant. A "small" group is one which does not contain a free group on two generators.

Central to this section will be notion of the "compactification" of an $\mathbf{R}$-tree. An elegant idea for describing such a compactification can be found in a paper of Ward [W1], and a more intuitive construction was described by Pearson [Pe]. Of course, these papers were written before the notion of an $\mathbf{R}$-tree was formulated, and apply to certain kinds of real trees described purely as topological spaces. (Note that the metric structure is not directly relevant here.) A general method explicitly for $\mathbf{R}$-trees was described in [MayNO]. Note that the term "compactification" should be interpreted broadly, in that the subspace topology on the $\mathbf{R}$-tree will in general be coarser than the original metric topology. It seems that the notion of a pretree gives a particularly natural context in which to describe these ideas, and we give an overview of these constructions in terms of pretrees later.

First, it will be helpful to illustrate these these notions with reference to a well-known example — that of a surface group acting on an $\mathbf{R}$-tree which is the leaf-space of a measured lamination (or of a measured foliation). In fact, a theorem of Skora [Sk] tells us that all minimal free surface group actions arise in this way (see also [O]). For a discussion of laminations on surfaces, see for example [CaEG].

Suppose, then, that $\Sigma$ is a closed compact surface of genus at least 2. Let $\Gamma = \pi_1(\Sigma)$. We choose some hyperbolic structure on $\Sigma$. Suppose we have a measured lamination on $\Sigma$ with support $L$, such that each connected component of $\Sigma \setminus L$ is a topological disc. We lift the lamination to the hyperbolic plane $\mathbf{H}^2$. Let $T$ be the set of strata of the lifted lamination, where a stratum is either a non-boundary leaf, or the closure of a complementary region. (The latter type is a finite-sided polygon.) Thus, $T$ gives a $\Gamma$-invariant partition of $\mathbf{H}^2$.

We put a metric, $d$, on $T$ by defining the distance between two strata to be the transverse measure across the set of geodesics which separate them. It turns out that $(T, d)$ is an $\mathbf{R}$-tree. The induced action of $\Gamma$ on $T$ is free.

Now, given a stratum $x \in T$, let $f(x)$ be its closure in the disc $\mathbf{H}^2 \cup \partial \mathbf{H}^2$. Let $D$ be the set of such closures, together with the set of points of $\partial \mathbf{H}^2$ which do not lie in any such closure. Thus $D$ gives an upper semicontinuous decomposition of $\mathbf{H}^2 \cup \partial \mathbf{H}^2$. We give

$D$ the quotient topology. Thus, $D$ is a continuum. In fact, $D$ is a dendrite. Note that $f : T \longrightarrow D$ is an injective map, and it's not hard to see that it's continuous. Moreover, $f(T)$ is dense in $D$, and $D \setminus f(T)$ consists entirely of terminal points. We can thus view $D$ as a kind of compactification of $T$, although the subspace topology on $T$ is coarser than the original metric topology.

Note that $D$ can also be viewed as a quotient of $\partial \mathbf{H}^2$, since the closure of every stratum meets $\partial \mathbf{H}^2$ (in a finite set). Now $\Gamma$ is word-hyperbolic, and we can identify $\partial \Gamma$ with $\partial \mathbf{H}^2$. We thus have an example of an equivariant quotient of the boundary of a hyperbolic group which is a non-trivial dendrite. Moreover, $\Gamma$ acts on $D$ as a convergence group. Of course, this is quite different from the sort of picture we would expect from our cut-point construction, though it should guard against attempts to find some immediate contradiction.

The dendrites arising from this construction have been well-known for some time; they arise as the limit sets of Bers boundary groups. In fact, it's known from the work of Thurston, Bonahon and Minsky among others, that if a surface group acts properly discontinuously on $\mathbf{H}^3$ in such a way that the discontinuity domain is connected, and the quotient manifold, $\mathbf{H}^3$, has a lower bound on injectivity radius, then the limit set is a dendrite of the type described, and that the natural actions of $\Gamma$ are conjugate. One would conjecture that the bound on injectivity radius should be unnecessary. One would also suspect that all dendrites of this type arise as limit sets of such groups.

We shall want to explore more generally the relationship between actions on $\mathbf{R}$-trees and actions on dendrons, as illustrated by this example. The topological relationship can be described by the following compactification theorem (cf. [MayNO]).

**Proposition 7.1 :** *Given an $\mathbf{R}$-tree, $T$, there is a dendron $D$ and a continuous injective map $f : T \hookrightarrow D$ such that $f(T)$ is dense in $D$. Moreover, if $D'$ is another dendron, and $f' : T \hookrightarrow D'$ is another such map, then there is a unique homeomorphism $g : D \longrightarrow D'$ such that $f' = g \circ f$.*

Note that only the topology of $T$ is relevant here. In fact we could replace the term "$\mathbf{R}$-tree" with "real tree" provided we assume that $T$ contains no embedded long line. We shall write $D(T)$ for the dendron thus defined. It's not hard to see (Lemma 7.6) that $D(T) \setminus T$ consists entirely of terminal points of $D(T)$. Note that if $T$ is separable, then so is $D(T)$, and so in this case $D(T)$ is a dendrite.

¿From the naturality of the construction, we see that any homeomorphism of an $\mathbf{R}$-tree, $T$, extends to a homeomorphism of $D(T)$. (Since we are dealing with two different topologies, the word "extends" should be interpreted on the level of sets.) In particular, any isometric action of a group $\Gamma$ on $T$ gives rise to an action on $D(T)$ by homeomorphism. We aim to describe when the latter action is a convergence action.

To do this, let $E(T) = T \times T \setminus \{(x, x) \mid x \in T\}$ be the set of edges. Thus, formally, an edge is an ordered pair of distinct points $(x, y)$, though we shall usually imagine it as the closed interval $[x, y]$. We give $E(T)$ the product topology. We define a notion of "edge-discreteness" for a group acting isometrically on $T$. There are several equivalent ways to do this as we describe later. For the moment, we say that $\Gamma$ is "edge-discrete"

if all the edge stabilisers are finite, and no $\Gamma$-orbit of an edge accumulates in $E(T)$. We shall see (Proposition 7.20) that if edge-discreteness fails, then either some edge stabiliser is infinite, or there is some closed subset of $T$ isometric to the real line, whose setwise stabiliser contains a dense set of translations. Relating this to the compactification, we shall show:

**Proposition 7.2 :** *Suppose the group $\Gamma$ acts isometrically on an $\mathbf{R}$-tree $T$. Then $\Gamma$ is edge-discrete if and only if the induced action on $D(T)$ is a convergence action.*

The "if" bit of the above result is fairly elementary (in view of the fact that edge-discreteness is defined in terms of the metric topology, which is finer than the compactified topology). The "only if" bit is somewhat more involved.

A natural question to ask is: what happens if we drop the assumption that $\Gamma$ acts isometrically? In other words, given a convergence action on a dendron, $D$, can we re-construct an isometric action on an $\mathbf{R}$-tree, $T$, for which $D = D(T)$. We can begin by removing all the terminal points from $D$ and looking for an equivariant path-metric on the complement, $T$. (Note that we cannot expect such a metric to be continuous.)

We shall see (Proposition 7.26) that it is possible to find an equivariant metric on $T$, such that $T$ is topologically a real tree. A result of Mayer and Oversteegen [MayO] now shows that $T$ can indeed be given the structure of an $\mathbf{R}$-tree. Unfortunately, the latter construction is not canonical, in the sense that it's not clear that it can be made equivariant.

In summary we can ask:

**Question :** Suppose a group $\Gamma$ acts as a convergence group on a dendron $D$. Is this action induced by an isometric action on an $\mathbf{R}$-tree $T$, for which $D = D(T)$?

(One might try restricting this question in various ways, for example by supposing that $\Gamma$ is finitely presented and acts minimally.)

Later in this section, we make a start on this question (Proposition 7.26). In [Bo3], we use this result to construct an action of $\Gamma$ on an $\mathbf{R}$-tree in the case where $\Gamma$ is finitely presented, although it is not necessarily of the type asked for in the question. However, it is sufficient to show that $\Gamma$ splits over a finite or two-ended subgroup. (At least provided we add the hypothesis that $\Gamma$ does not contain an infinite ascending chain of finite subgroups.) Putting this together with Theorem 6.1, we obtain the result (Corollary 0.2) that the boundary of a strongly rigid one-ended hyperbolic group has no global cut point, and hence is locally connected. (Recall that, in this context, a " strongly rigid" group is one which does not split over a two-ended subgroup.)

A situation in some sense at the opposite extreme to the rigid case was considered by Martinez [Mar]. In that paper, it was shown that if $\Gamma$ is a one-ended hyperbolic group which is an amalgamated free product of two free or surface groups over an infinite cyclic subgroup, then $\partial\Gamma$ is locally connected. Using a combination of these ideas, it was shown in [Bo4] that any strongly accessible one-ended hyperbolic group has locally connected boundary [Bo4]. Swarup [Swa] showed how, in fact, these arguments could be carried over

to the general case, using a relative version the splitting result of [L] or [Bo3], together with (standard) accessibility over two-ended subgroups of [BeF1]. These ideas are carried further in [Bo6], with applications to relatively hyperbolic groups, as discussed in Section 6. It remains an interesting question as to whether every hyperbolic, or indeed every finitely presented group must be strongly accessible (over finite and two-ended subgroups).

We now get on with a more detailed account of the compactification process for **R**-trees.

There are many equivalent ways of defining an **R**-tree, other than as a special case of $\Lambda$-tree. Perhaps the simplest definition is to say that an **R**-tree is a path-metric space which contains no subset homeomorphic to a circle. Alternatively, it is a path-metric space (or in fact any connected metric space) which is 0-hyperbolic, i.e. $(\forall x, y, z, w \in T)(d(x,y) + d(z,w) \leq \max\{d(x,z) + d(y,w), d(x,w) + d(y,z)\})$. (In other words, the largest two of the three quantities $d(x,y) + d(z,w)$, $d(x,z) + d(y,w)$ and $d(x,w) + d(y,z)$ are always equal.) In either of these other definitions we could replace "path-metric space" by "length space", where the latter means that every pair of points can be joined by a geodesic. In fact, since there is no embedded circle, we see that every pair of points are connected by a unique arc, which must therefore be geodesic. Moreover, such a space is locally connected, and hence a real tree, in the sense of Section 1 (a uniquely arc-connected, locally arc-connected topological space). In other words, an **R**-tree is a real tree with a particular path-metric. The equivalence of the above definitions is shown in [MorS] or [Bo1]. One can also show that, in fact, any connected 0-hyperbolic metric space is an **R**-tree.

¿From the point of view of compactification, we are not really interested in the metric structure, so for the most part, we will talk about real trees. Note that not every real tree can be given the structure of an **R**-tree; for example, the long line is a real tree. However, **R**-trees can be characterised topologically as metrisable real trees [MayO].

A method for compactifying a real tree can be found in a paper of Ward [W1]. Since the idea is easy to describe, we shall begin with a sketch of this process. Suppose that $T$ is a real tree. Let $E(T)$ be the set of edges, as defined above. Let $X(T) = \prod_{(x,y) \in E(T)} [x,y]$ be the cartesian product, with the product topology. Thus $X(T)$ is compact by Tychonoff's theorem. We can think of an element of $\rho \in X(T)$ as a map $\rho : E(T) \longrightarrow T$ such that $\rho(x,y) \in [x,y]$ for all $x, y$. We define a map $\pi : T \longrightarrow X$ by setting $\pi(a)(x,y) = \text{med}(x,y,a)$. Now it's not hard to see that $\pi$ is a continuous injection. We can thus compactify $T$ by taking the closure, $D$, of $\pi(T)$ in $X(T)$, and identifying $T$ with $\pi(T)$. Of course, the subspace topology may be coarser than the original.

In general, $D$ need not be a real tree. For example, if $T$ were the long line, $D$ is obtained by adding two endpoints. These endpoints are not connected by a real interval, so $D$ is not arc connected in the topological sense. However, this is essentially the only thing that can go wrong.

Note that a real tree has a natural pretree structure. In this context, an "arc" is defined to be a full linearly ordered subset (Section 2). This leads to the definition:

**Definition :** A real tree is *short* if every arc is homeomorphic to a connected subset of

**R**.

Note that every **R**-tree is short. In fact:

**Lemma 7.3 :**  *Every arc in an **R**-tree is isometric to a connected subset of **R**.*

**Proof :** Suppose that $T$ is an **R**-tree, and $A \subseteq T$ is an arc (in the pretree sense). Choose any $a \in A$, and a direction $<$ on $A$. Define a map $f : A \longrightarrow \mathbf{R}$ by $f(x) = d(a, x)$ of $x \geq a$ and $f(x) = -d(a, x)$ if $x \leq a$. It's not hard to verify that $f$ is an isometry onto its range, and that the range is connected. $\diamond$

It turns out that Ward's compactification of a short real tree is always a real tree, and hence a dendron. This is a bit tricky to see directly. (In the original paper, Ward puts additional hypotheses on the tree which ensures that the topology on $T$ itself does not change.) We shall thus use an approach to compactification which is more explicit, but which will take longer to describe.

As already mentioned, the subspace topology of the compactified topology on a real tree is in general coarser that the original topology, and admits a simple description in terms of the pretree structure. This topology is described in several places in the literature, usually in terms of pseudotrees. A very general account of topologies on pseudotrees can be found in [Ni], which includes many other references to this subject. However, pretrees seem a more natural context in which to phrase these ideas, at least in relation to real trees. Although many of the constructions can be dealt with more generally, for the sake of simplicity, we shall restrict attention here to "real pretrees".

**Definition :** A pretree, $T$, is a *real pretree* if given any pair of distinct points $x, y \in T$, the interval $[x, y]$ is order isomorphic to a real closed interval. A real pretree is *short*, if every arc is separable (and hence order isomorphic to a full subset of **R**).

**Lemma 7.4 :**  *A real pretree is median.*

**Proof :**  By essentially the same argument as Lemma 2.11 (which showed that every complete dense pretree is median). $\diamond$

Obviously, every (short) real tree, $T$, is a (short) real pretree. Note that if $x, y \in T$, then from the definition of the pretree structure, the pretree interval $[x, y]$ is the same as the real arc $[x, y]$ joining $x$ to $y$, so there is no clash of notation.

**Lemma 7.5 :**  *Suppose that $S$ and $T$ are real trees and $f : S \longrightarrow T$ is a continuous bijection. Then $f$ is a pretree isomorphism.*

**Proof :** Suppose that $x, y \in S$ and $x \neq y$. Then $[x, y]$ is homeomorphic to a real closed interval with endpoints $x$ and $y$. Thus $f([x, y])$ is also homeomorphic to a real closed interval, with endpoints $f(x)$ and $f(y)$. Thus $f([x, y]) = [f(x), f(y)]$. This shows that $f$ is a pretree isomorphism. $\diamond$

Suppose that $T$ is a real tree and $S \subseteq T$. Then $S$ is full if and only if it is connected, and hence subtree. In such a case, $S$ is topologically dense in $T$ if and only if it is (weakly or strongly) dense in the pretree sense. We note:

**Lemma 7.6 :** *Suppose $T$ is a real tree, and $S \subseteq T$ is a dense subtree. Then every point of $T \setminus S$ is terminal.* $\diamond$

We can now get on with describing the compactification. Suppose that $T$ is any real tree (or for the moment, any dense median pretree). Given $x, y \in T$, $x \neq y$, we define, as in Section 5, $J(x, y) = \{z \in T \mid \mathrm{med}(x, y, z) = y\}$.

**Definition :** If $T$ is a real pretree, the *order topology* on $T$ is defined by taking as subbase for the closed sets the collection $\{J(x, y) \mid (x, y) \in E(T)\}$.

There are various other ways of describing this topology. A particularly useful one is in terms of preclosed subsets. Recall from Section 5, that a full subset $Q \subseteq T$ is "preclosed" if $(\forall x \in T \setminus Q)(\exists y \in T)(xyQ)$. We shall take the term "preclosed" to imply full. (Thus, in the case of a real tree, the preclosed subsets are precisely the topologically closed subtrees.)

Now, any set of the form $J(x, y)$ is preclosed. Conversely, it's easily seen that any preclosed subset is an intersection of sets of the form $J(x, y)$, and so we could alternatively define the order topology by taking the collection of all preclosed subsets as subbase for the closed sets.

We note:

**Lemma 7.7 :** *Any non-empty intersection of preclosed subsets is preclosed.*

**Proof :** Let $\mathcal{F}$ be a collection of preclosed sets, and let $G = \bigcap \mathcal{F}$. Clearly $G$ is full. Suppose $x \notin G$. There is some $F \in \mathcal{F}$ such that $x \notin G$. Thus $(\exists y \in T)(xyF)$. Since $G \subseteq F$ we have $xyG$. $\diamond$

We can thus define the *preclosure*, $\bar{Q}$, of a full set $Q \subseteq T$ to be the intersection of all preclosed sets containing $Q$. It's not hard to see that $\bar{Q} = \{x \in T \mid \neg(\exists y \in T)(xyQ)\}$.

Suppose that $Q \subseteq S \subseteq T$ with $Q$ and $S$ full in $T$, and $Q$ preclosed in $S$. Then $Q = \bar{Q} \cap S$. Conversely, if $Q, S \subseteq T$ are full, and $Q$ is preclosed in $T$, then $Q \cap S$ is full and preclosed in $S$. This shows:

**Lemma 7.8 :** *If $S$ is a full subset of the real pretree $T$, then the order topology on $S$ is the same as the subspace topology of the order topology on $T$.* $\diamond$

Now note that the order topology on a linear pretree is the same as the usual topology for a totally ordered set. Putting this observation together with Lemma 7.8, we get:

**Lemma 7.9 :** *If $T$ is real pretree with the order topology, and $x, y \in T$ are distinct, then $[x, y]$ is homeomorphic to a real closed interval.* $\diamond$

In fact:

**Proposition 7.10 :** *Suppose $T$ is a real pretree in the order topology. Then $T$ is a real tree. Moreover, the induced pretee structure agrees with the original.*

**Proof :** For the moment, we use $[x, y]$ to denote the pretree interval.

We first note that the order topology is hausdorff. To see this, given distinct points $x, y \in T$, choose any $z \in (x, y)$. We then have $x \in T \setminus J(x, z)$ and $y \in T \setminus J(y, z)$.

Now, Lemma 7.9 tells us that $T$ is (topologically) arc connected. We show that it is locally arc connected.

Suppose $x \in T$ and $V \subseteq T$ is a neighbourhood of $x$. We can assume that $V$ has the form $T \setminus (F_1 \cup \cdots \cup F_n)$, where $F_1, \ldots, F_n \subseteq T$ are full and preclosed. For each $i$, let $G_i = \{z \in T \mid [x, z] \cap F_i \neq \emptyset\}$. Now $G_i$ is also full and preclosed. Let $U = T \setminus (G_1 \cup \cdots \cup G_n)$. Thus $U \subseteq V$ is a neighbourhood of $x$. If $y \in U$, then $[x, y] \subseteq U \subseteq V$.

We have thus verified the hypotheses of Lemma 2.19, and so it follows that $T$ is a real tree. $\diamond$

If $T$ is a real tree, then the order topology on $T$ will in general be coarser than the original topology. However, we have:

**Lemma 7.11 :** *If $T$ is a dendron, then the topology agrees with order topology.*

**Proof :** The identity map from the original topology to the order topology is compact to hausdorff, and thus a homeomorphism. $\diamond$

Note that by Lemma 7.8, we could weaken the hypothesis of Lemma 7.11 to say that $T$ is a subtree of a dendron with the subspace topology.

In view of Proposition 7.10, we should think of a real tree as more general notion than a real pretree (rather than the other way around, as one might at first imagine).

We can observe:

**Lemma 7.12 :** *A dendron is complete as a pretree.*

**Proof :** There are various ways to see this. For example, the preclosure of an arc is again an arc. Since it is closed in the dendron it must be compact, and is thus easily seen to be an interval. $\diamond$

Note that this shows that a dendron is necessarily a short real tree. We shall omit the details, since we are primarily interested in the converse to Lemma 7.12; namely:

**Proposition 7.13 :** *A complete real pretree is compact in the order topology.*

**Proof :** By the Alexander Subbase Theorem [Ke], to show that a space is compact, it's enough to show that if a given intersection of closed subbase elements is empty, then some finite subset of these subbase elements has empty intersection.

Suppose then that $T$ is complete, and that $\mathcal{F}$ is a collection of preclosed subsets such that the intersection of every finite subset of $\mathcal{F}$ is non-empty. We can assume that $\mathcal{F}$

is closed under finite intersection (by replacing $\mathcal{F}$ by the set of all finite intersections of elements of $\mathcal{F}$ if necessary, and noting that an intersection of preclosed sets is preclosed).

Suppose, for contradiction, that $\bigcap \mathcal{F} = \emptyset$. Choose any $a \in T$, and let

$$A = \{a\} \cup \{x \in T \mid (\exists F \in \mathcal{F})(axF)\}.$$

We claim that $A$ is an arc.

First, we note that $A$ is linear. To see this, suppose $x, y \in A \setminus \{a\}$ are distinct. There exist $F, G \in \mathcal{F}$ with $axF$ and $ayG$. Choose any $z \in F \cap G \in \mathcal{F}$. We have $axz \wedge ayz$ and so $axy \vee ayx$. It follows that $A$ is linear. Now it's easy to see that $A$ is full, since if $x \in A$, then clearly $[a, x] \subseteq A$.

Now, by completeness, there is some $b \in T$ such that either $A = [a, b]$ or $A = [a, b)$. Since $\bigcap \mathcal{F} = \emptyset$, there is some $F \in \mathcal{F}$ with $b \notin F$. Since $F$ is preclosed, $(\exists z \in T)(xzF)$. Now if $abz$, we would have $azF$, and so $z \in A$. But $A \subseteq [a, b]$, so get the contradiction that $azb$.

It follows that $\neg abz$. Since $T$ is a dense median pretree, we can find $x, y \in T$ with $axyb \wedge zxyb$. Now since $y \in A$, there is some $G \in \mathcal{F}$ with $ayG$. We thus have $xyG \wedge yxF$. Choosing any $w \in F \cap G$, we get the contradiction $xyw \wedge yxw$. $\diamondsuit$

We remark that in any pretree, a finite intersection full subsets is non-empty if and only if any pair of these subsets has non-empty intersection. This follows by a simple induction argument (cf. Helly's theorem for convex subsets of euclidean space). We thus see that for a complete real pretree with the order topology, any set of subbase elements which meet pairwise has non-empty intersection — a property known as "supercompactness". In other words, we see that all dendrons are supercompact (see [Ni]).

In summary, we have reduced the problem of compactifying a real tree to one of completing a real pretree. The following construction, essentially the same as that described in [Pe], gives a fairly explicit description of this completion.

**Definition :** A *ray* in a real pretree is a directed arc with no supremum.

Given a real pretree, $T$, let $\mathcal{A}$ be the set of rays in $T$. The relation of cofinality, which we denote here by $\sim$, is an equivalence relation on $\mathcal{A}$. We may identify the quotient $Q = \mathcal{A}/\sim$ as a subset of the set, $P$, of flows on $T$. We can thus regard $T \sqcup Q$ as a subpretree of the pretree $T \sqcup P$ defined in Section 3. We write $D(T) = T \sqcup Q$.

It's easy to give an explicit description of the pretree relation on $D(T)$. First note that every point of $Q = D(T) \setminus T$ is terminal. If $x, y \in T$ and $p \in Q$, then we have $xyp$ if and only if there is some representative $A \in \mathcal{A}$ of $p$ such that $xyA$. Finally, if $x \in T$ and $p, q \in Q$, then we have $pxq$ if and only if there are representatives, $A, B \in \mathcal{A}$ of $p$ and $q$ respectively, such that $AxB$.

**Lemma 7.14 :** $D(T)$ *is complete.*

**Proof :** Suppose that $A \subseteq D$ is a directed arc, with minimum $a$ and with no supremum in $D$. We can suppose that $a \in T$. Now if $p \in A \cap Q$, then since $p$ is terminal, it would have to be a supremum of $A$. We can thus suppose that $A \subseteq T$, and so $A \in \mathcal{A}$. Let $p \in Q$ be the corresponding point. ¿From the definition of the pretree structure, it's easily verified that $A = [a, p)$ and so $p$ a supremum. $\diamond$

**Lemma 7.15 :**  *If $T$ is a short real pretree, then $D(T)$ is a real pretree.*

**Proof :** If $x, y \in D(T)$ are distinct, then $(x, y)$ is an arc in $T$, and thus order-isomorphic to $\mathbf{R}$. It thus follows that $[x, y]$ is order isomorphic to a closed real interval. $\diamond$

In summary, we have shown:

**Proposition 7.16 :**  *If $T$ is a short real pretree, then $D(T)$ is a dendron in the order topology. Moreover, $T$ is dense in $D(T)$.*

**Proof :** By Lemmas 7.14 and 7.15, and Propositions 7.10 and 7.13. $\diamond$

In order to prove uniqueness part of the compactness theorem as we originally stated it (Proposition 7.1), we need one more observation:

**Lemma 7.17 :**  *Suppose that $D$ is a complete real tree, and that $T \subseteq D'$ is a dense subtree. Then we can extend the inclusion of $T$ into $D'$ to a pretree isomorphism of $D(T)$ onto $D'$.*

**Proof :** By Lemma 7.6, we know that every point of $D' \setminus T$ is terminal. It's now easy to see that the points of $D' \setminus T$ are in natural bijective correspondence with cofinality classes of rays in $T$. Moreover the pretree structure on $D'$ must be what one would expect. $\diamond$

**Proof of Proposition 7.1 :**  Suppose $T$ is an $\mathbf{R}$-tree. By Lemma 7.3, $T$ is a short real tree. By Proposition 7.16, $T$ embeds continuously in a dendron $D(T)$.

Now suppose that $D'$ is another dendron, and that $f' : T \hookrightarrow D'$ is a continuous injective map, with $f'(T)$ dense in $D'$. By Lemma 7.12, $D'$ is complete. By Lemma 7.5, $f'$ is a pretree isomorphism onto $f'(T)$. By Lemma 7.17, $f'$ extends to a pretree isomorphism of $D(T)$ onto $D'$. By Lemma 7.11, this extension is a homeomorphism.

The uniqueness of the extension is trivial. $\diamond$

We can now relate all this back to first construction we described, based on the paper of Ward [W1]. Thus, given a real tree, $T$, we have a continuous injection, $\pi : T \longrightarrow X(T)$, of $T$ into the infinite product $X(T)$. By unravelling the definitions, it's not hard to see that a subbase for the closed sets in the induced topology on $T$ is given by $\{J(x, y) \mid (x, y) \in E(T)\}$. In other words, the induced topology is precisely the order topology on $T$.

Note that if $T$ happens to be complete, then the image $\pi(T)$ will be closed in $X(T)$. With some work, one can verify this directly (giving another proof of Proposition 7.13), though in retrospect, we know already, by Proposition 7.13, that $\pi(T)$ is compact.

Now, given any short real tree, $T$, we can extend the map $\pi : T \longrightarrow X(T)$ to a continuous map $D(T) \longrightarrow X(T)$. To do this, first embed $D(T)$ in $X(D(T))$ and then project back into $X(T)$ by forgetting those intervals which have some endpoint in $D(T) \setminus T$. It's easily seen that this extension remains injective. Since $D(T)$ is compact, it is a homeomorphism onto its range. Since $T$ is dense in $D(T)$, we see that it can be described as the closure of $\pi(T)$ in $X(T)$.

We have thus shown that both compactifications give the same result. In particular, the result of the Ward compactification is indeed a dendron.

Before we leave the subject of compactification, we note that in the special case of a separable $\mathbf{R}$-tree, there is a simple intuitive way of describing this process which we briefly outline. We again reduce the question of compactification to one of "completion", this time interpreted in the metric space sense.

We begin by observing:

**Lemma 7.18 :** *The metric completion of an $\mathbf{R}$-tree is an $\mathbf{R}$-tree.*

**Proof :** This follows easily from the description of an $\mathbf{R}$-tree as a 0-hyperbolic path-metric space. (Note that the completion of a path-metric is always a path-metric.) $\diamondsuit$

Given an $\mathbf{R}$-tree, $(T, d)$, let $\mu$ be the 1-dimensional Hausdorff measure on $T$. Thus, if $x, y \in T$, then $d(x, y) = \mu([x, y])$.

Now, if $(T, d)$ has finite length, i.e. $\mu(T) < \infty$, then it's easily seen that $T$ is precompact (totally bounded), and so its metric completion will be compact.

Now, given any $\mathbf{R}$-tree $(T, d)$, the idea is to find another metric $d'$ on $T$, with $d' \le d$ and such that $(T, d')$ is an $\mathbf{R}$-tree of finite length. It will be easiest to deal with the case where $(T, d)$ has no terminal points.

To do this, let $(a_i)_{i \in \mathbf{N}}$ be a sequence of points of $T$, whose image in $(T, d)$ is dense. For each $n \in \mathbf{N}$, let $T_n = \bigcup_{i=0}^{n}[a_0, a_i]$. Thus, $(T_n)_{n \in \mathbf{N}}$ is an increasing sequence of finite trees which exhaust $T$. Passing to a subsequence, we may as well assume that $A_n = T_n \setminus T_{n-1}$ is non-empty and thus a half-open interval for all $n \ge 1$. Thus $T = \bigsqcup_{i=0}^{n} A_i$, where $A_0 = \{a_0\}$. By subdividing these intervals if necessary, we can also suppose that $\mu(A_i) \le 1$ for all $i$. We now define a new metric $d'$ by rescaling the metric $d$ on $A_i$ by a factor of $2^{-i}$. In other words, if $x, y \in T$, define $d'(x, y) = \sum_{i=1}^{\infty} 2^{-i}\mu(A_i \cap [x, y])$. Note that this is in fact a finite sum, since we have $x, y \in T_n$ for some $n$. Now $d'$ is a path metric. Moreover, we see that for each $n$, $(T_n, d')$ is homeomorphic to $(T_n, d)$ from which it follows easily that $(T, d')$ contains no embedded circle. Thus $(T, d')$ is an $\mathbf{R}$-tree. Moreover it has length at most 2, and so its completion is compact, and hence a dendrite.

The case where $(T, d)$ does have terminal points is easily dealt with, by first removing them, performing the above construction, and putting them back in at the end. We omit the details here.

Before leaving the general subject of topologies on pretrees, there is one further remark we could make. Note that, given a real pretree $T$, the order topology can be thought of as the unique coarsest topology on $T$, with respect to which $T$ is a real tree (since preclosed sets are necessarily closed in any real tree). There is also a unique finest topology in this sense. It can be defined by declaring a subset of $T$ to be open if its intersection with any

closed pretree interval is open in the order topology on that interval. One can easily verify that this is indeed a topology, and that $T$ is a real tree with this topology. However, we shall have no use for this topology here.

We now want to move on to consider group actions on **R**-trees. We begin by recalling the classification of isometries. Suppose $(T, d)$ is an **R**-tree.

**Definition :** An isometry $\gamma$ of $T$ is *loxodromic* if it has no fixed point in $T$.

**Lemma 7.19 :** *Suppose that $\gamma$ is loxodromic. Then there is a $\gamma$-invariant subset $l \subseteq T$ which is isometric to the real line.*

**Proof :** Choose any $x \in T$. If $d(x, \gamma x) \geq d(x, \gamma x)$, then the midpoint of the segment $[x, \gamma x]$ would be fixed by $\gamma$. Thus, we must have $d(x, \gamma x) < d(x, \gamma^2 x)$. Let $y = \mathrm{med}(x, \gamma x, \gamma^2 x)$. Now it's not hard to see that $\gamma y \in (y, \gamma^2 y)$, and so the set $l = \bigcup_{n \in \mathbf{Z}}[y, \gamma y]$ is isometric to the real line. $\diamond$

It's easy to see that the set $l$ is unique with this property. We denote it by $l(\gamma)$, and refer to it as the *axis* of $\gamma$. It is a closed subset of $T$. In fact, it is the unique minimal nonempty $\gamma$-invariant subtree. It is translated some positive distance by $\gamma$ which we denote by $m(\gamma)$. Thus $m(\gamma) = \min\{d(x, \gamma x) \mid x \in T\}$. Note that any iterate of a loxodromic is loxodromic with the same axis. In fact, $m(\gamma^n) = nm(\gamma)$.

Note, that if $x \in T$, and $d(x, \gamma^2 x) = 2d(x, \gamma x)$, then $\gamma x \in [x, \gamma^2 x]$, and so, if $d(x, \gamma x) > 0$, $\gamma$ is loxodromic, with $x \in l(\gamma)$. Moreover, $m(\gamma) = d(x, \gamma x)$.

If $\gamma$ is not parabolic, then its fixed point set is a non-empty subtree. In this case we set $m(\gamma) = 0$. If $\gamma$ has infinite order and has a unique fixed point, we refer to it as a *parabolic*.

We are primarily interested in group actions which have finite edge-stabilisers. In this case every group element will be loxodromic, parabolic or of finite order. We shall refer to an element of finite order as *elliptic*. Note that, in this case, every iterate of a parabolic is parabolic.

Recall that the set of edges, $E(T)$, of $T$ is defined as $E(T) = \{(x, y) \in T^2 \mid x \neq y\}$. (We usually imagine such an edge as a closed interval, $[x, y]$.) We define a metric, $\delta$, on $E(T)$ by $\delta((x, y), (x', y')) = \max(d(x, x'), d(y, y'))$. Thus an isometry, $\gamma$, of $T$ also acts an isometry of $(E(T), \delta)$. A useful property to note is that if $e = (x, y) \in E(T)$ and $a \in [x, y]$, then $d(a, \gamma a) \leq \delta(e, \gamma e)$. This follows by the convexity of the distance function $d$.

Suppose $\Gamma$ acts isometrically on $T$.

**Definition :** We say that $\gamma$ is *edge-discrete* if there does not exist an edge $e \in E(T)$ together with a sequence $(\gamma_i)_{i \in \mathbf{N}}$ of distinct elements of $\Gamma$ such that $(\gamma_i(e))_{i \in \mathbf{N}}$ is Cauchy with respect to the metric $\delta$.

This is one of several equivalent definitions we could have chosen. Given Proposition 7.20, there are a number of apparently stronger alternatives. For example, we could say

that $\Gamma$ is edge-discrete if it has finite edge stabilisers and no $\Gamma$-orbit in $E(T)$ accumulates in $E(T)$. This is the definition we mentioned earlier.

Now suppose $l \subseteq T$ is a subset isometric to the real line. Such a set is necessarily closed, and we shall refer to it simply as a *line*. Suppose $\Gamma$ acts by isometry on $T$. We write $G \leq \Gamma$ for the subgroup which preserves $l$ setwise, and respects its orientation. We write $H \triangleleft G$ for the pointwise stabiliser of $l$, and set $B(l) = G/H$. Thus $B(l)$ acts faithfully by translation on $l$, and so can be regarded as a subgroup of the additive group $\mathbf{R}$. This subgroup is well defined once we have chosen an orientation on $l$. Note that if $\Gamma$ has finite edge stabilisers, then certainly $H$ is finite, so $G$ is a finite extension of an abelian group.

**Proposition 7.20 :**  *Suppose $\gamma$ acts on $T$ with finite edge stabilisers, then either $\Gamma$ is edge-discrete, or there is a line $l \subseteq T$ such that $B(l)$ is a dense subgroup of the additive reals.*

Before we begin proving this, we make a few general observations.

Fix an edge $e = (x, y) \in E(T)$, and write $\partial e = \{x, y\}$. Given an action of $\Gamma$ on $T$, and some $r > 0$, write $G(e, r) = \{\gamma \in \Gamma \mid \delta(e, \gamma e) \leq r\}$. Suppose $d(x, y) \geq 2r$. Given $\gamma \in G(e, r)$, define $\theta(\gamma) = \frac{1}{2}(d(x, \gamma y) - d(y, \gamma x))$.

A more intuitive way of describing $\theta(\gamma)$ is as follows. Choose any $a \in [x, y]$ with $d(a, \partial e) \geq r$. Since $d(x, \gamma x) \leq d(x, a)$, we see that $a \notin [x, \gamma x]$. Similarly, $a \notin d(y, \gamma y)$. Thus $a \in [\gamma x, \gamma y]$. Similarly, $\gamma a \in [x, y]$. Thus $a, \gamma a \in [x, y] \cap [\gamma x, \gamma y]$. Note also that $d(a, \gamma a) \leq \delta(e, \gamma e)$.

Suppose $a \in [x, \gamma a]$. Then $d(a, \gamma a) = d(a, x) - d(a, \gamma x)$. Now $\gamma a \in [y, a]$, and so we also have $-d(a, \gamma a) = d(a, y) - d(a, \gamma y)$. Thus $2d(a, \gamma a) = d(a, x) + d(a, \gamma y) - d(a, y) - d(a, \gamma x) = d(x, \gamma y) - d(y, \gamma x)$. Thus $\theta(\gamma) = d(a, \gamma a)$. Similarly, if $\gamma a \in [x, a]$ we find that $\theta(\gamma) = -d(a, \gamma a)$.

Note that if $\gamma \in G(e, r)$, then $\gamma^{-1} \in G(e, r)$, and $\theta(\gamma^{-1}) = -\theta(\gamma)$. Also $|\theta(\gamma)| \leq \delta(e, \gamma e)$.

Now suppose that $\beta, \gamma \in G(e, r/2)$, so that $\gamma\beta \in G(e, r)$. As before, choose any $a \in [x, y]$ with $d(a, \partial e) \geq r$. We have $d(a, \beta a) \leq r/2$ and so $d(\beta a, \partial e) \geq r/2$. Thus, $a, \beta a, \gamma\beta a \in [x, y] \cap \beta[x, y] \cap \gamma\beta[x, y]$. If $\theta(\beta) \geq 0$ and $\theta(\gamma) \geq 0$, then $\theta(\beta) + \theta(\gamma) = d(a, \beta a) + d(\beta a, \gamma\beta a) = d(a, \gamma\beta a) = \theta(\gamma\beta)$. By similar arguments, we see that the same identity holds whatever the signs of $\theta(\beta)$ and $\theta(\gamma)$. Thus, $\theta$ is a kind of "local homomorphism" to $\mathbf{R}$. Note also that $|\theta(\beta) - \theta(\gamma)| = |\theta(\gamma^{-1}\beta)| \leq \delta(e, \gamma^{-1}\beta e) = \delta(\beta e, \gamma e)$.

Now if $\gamma \in G(e, r/2)$, then $\theta(\gamma^2) = 2\theta(\gamma)$. Thus, if we choose $a$ as above, then $d(a, \gamma^2 a) = 2d(a, \gamma a)$ and so $m(\gamma) = d(a, \gamma a)$. We conclude that $m(\gamma) = |\theta(\gamma)|$.

Finally note that if $\delta(e, \gamma e) < r$ and $\theta(\gamma) = 0$, then $\gamma$ stabilises an edge: choose any distinct $a, b \in (x, y)$ with $d(a, \partial e) \geq r$ and $d(b, \partial e) \geq r$, then $a$ and $b$ are fixed by $\gamma$. In fact we notice that all such $\gamma$ can be assumed to stabilise the same edge.

**Proof of Proposition 7.20 :**  Suppose that $\Gamma$ has finite edge-stabilisers but is not edge-discrete. There is an edge $e = (x, y)$ and a sequence $(\gamma_i)_{i \in \mathbf{N}}$ of distinct elements of $\Gamma$ such that $(\gamma_i e)_{i \in \mathbf{N}}$ is Cauchy in $(E(T), \delta)$.

Choose any $r < d(x, y)/2$. We can suppose that $\gamma_i \in G(e, r/8)$ for all $i \in \mathbf{N}$ (by

replacing $e$ by $\gamma_k e$ and each $\gamma_i$ by $\gamma_{i+k}\gamma_k^{-1}$ for large enough $k$ if necessary).

Let $a, b \in [x, y]$ with $d(x, a) = d(y, b) = r$. Let $K \leq \Gamma$ be the stabiliser of the edge $(a, b)$. By hypothesis, $K$ is finite.

Let $\theta : G(e, r) \longrightarrow \mathbf{R}$ be the map defined above. Note that if $\gamma \in G(e, r)$ and $\theta(\gamma) = 0$, then $\gamma$ fixes $a$ and $b$, so $\gamma \in K$. Now, given any $\beta, \gamma \in G(e, r/4)$, the commutator $[\beta, \gamma] = \beta\gamma\beta^{-1}\gamma^{-1}$ lies in $G(e, r)$. Moreover $\theta([\beta, \gamma]) = \theta(\beta) + \theta(\gamma) - \theta(\beta) - \theta(\gamma) = 0$, and so $[\beta, \gamma] \in K$.

Now, given $i, j \in \mathbf{N}$, we have $|\theta(\gamma_i) - \theta(\gamma_j)| \leq \delta(\gamma_i e, \gamma_j e)$ and so the sequence $(\theta(\gamma_i))_{i \in \mathbf{N}}$ is Cauchy. We can suppose that the numbers $\theta(\gamma_i)$ are all distinct and non-zero (since $K$ is finite).

Suppose $\beta \in G(e, r/4)$ and $k \in \mathbf{N}$. Consider the commutators $[\beta, \gamma_i] \in K$ for $i > k$. There exist $j > i > k$ with $[\beta, \gamma_i] = [\beta, \gamma_j]$. We see that $\beta$ commutes with $\gamma_j^{-1}\gamma_i$. Note that $\gamma_j^{-1}\gamma_i \in G(e, r/4)$.

Using this idea, we may inductively define a sequence $\beta_n \in G(e, r/4)$ as follows. Set $\beta = \gamma_0$. Given $\beta_n = \gamma_{j(n)}^{-1}\gamma_{i(n)}$ with $j(n) > i(n)$, set $\beta_{n+1} = \gamma_{j(n+1)}^{-1}\gamma_{i(n+1)}$ where $j(n+1) > i(n+1) > j(n)$ and $\beta_{n+1}$ commutes with $\beta_n$. Note that $\theta(\beta_n) = \theta(\gamma_{i(n)}) - \theta(\gamma_{j(n)})$ tends to zero, but is always non-zero. We see that each $\beta_n$ is loxodromic, and that $m(\beta_n) = |\theta(\beta_n)|$.

Now since $\beta_{n+1}$ commutes with $\beta_n$, we see that $\beta_{n+1}$ must fix setwise the axis $l(\beta_n)$. Thus $l(\beta_{n+1}) = l(\beta_n)$. Thus, by induction, $l(\beta_n) = l$ is constant for all $n$. Since it is translated arbitrarily small distances, we see that the group $B(l)$ must be a dense subgroup of $\mathbf{R}$. $\diamondsuit$

We now want to relate this to convergence groups. Proposition 7.1 gives us a continuous injective map $f : T \hookrightarrow D$ of $T$ into a dendron $D$. We identify $T$ with $f(T)$, so $T \subseteq D$ is a dense subtree. We can extend an action of a group $\Gamma$ on $T$ by isometry to an action of $\Gamma$ on $D$ by homeomorphism.

Proposition 7.2 asserts that the action on $D$ is a discrete convergence action if and and only if the action on $T$ is edge discrete. Now one direction is more or less immediate, given Proposition 7.20. We thus concentrate on showing that the extension of an edge discrete action is a convergence action. (Note that given this, it's easy to see that the terms "loxodromic" and "parabolic", as defined in the two contexts, agree.)

We shall do this by showing the action on the space of distinct triples is properly discontinuous. As in Section 6, we shall phrase our argument in terms of sequences. Thus, technically, this only deals with the case where the dendron, $D$, is metrisable. This is the principal case of interest to us. (Note, for example, that if $T$ is separable then $D$ is separable, and hence metrisable.) One can obtain a proof in the general case simply by rephrasing everything in terms of nets (cf. [Bo5]).

Our argument is complicated slightly by having to deal with two different topologies. If $(x_i)_i$ is a sequence in $D$ and $x \in D$, we shall write $x_i \to x$ to mean that $x_i$ converges to $x$ in the compact topology. Note that if all these points happen to lie in $T$, then the assertion that $d(x, x_i) \to 0$ is, in general, stronger (since the metric topology on $T$ is finer). However, if all these points are constrained to lie in some interval of $T$, then these statements are equivalent (since both topologies agree on any interval).

Suppose, then, that $\Gamma$ acts isometrically on $T$, giving an induced action by homeomorphism on $D$. Suppose that the action on $T$ is edge-discrete. We want to show that

the action on distinct triples in $D$ is properly discontinuous. We begin with the following observation:

**Lemma 7.21 :** *Suppose $(\gamma_i)_i$ is a sequence of distinct elements of $\Gamma$. Suppose $x, x' \in D$, $z \in T \setminus \{x\}$ and $z' \in T \setminus \{x'\}$. Suppose that $(x_i)_i$ is a sequence in $D$ and $(z_i)_i$ is a sequence in $T$. Then, it is impossible to have simultaneously that $x_i \to x$, $\gamma_i x_i \to x'$, $d(z, z_i) \to 0$ and $d(z', \gamma_i z_i) \to 0$.*

**Proof :** Suppose that these sequences do indeed converge in the manner described. Choose some $\epsilon > 0$, and let $w$ and $w'$ be the points in the intervals $(x, z)$ and $(x', z')$, respectively, with $d(w, z) = d(w', z') = \epsilon$. Such points certainly exist, provided we choose $\epsilon$ small enough.

Now for all sufficiently large $i$, we have $w \notin [x, x_i] \cup [z, z_i]$ and $w' \notin [x', \gamma_i x_i] \cup [z', \gamma_i z_i]$. It follows that $w \in [x_i, z_i]$ and $w' \in [\gamma_i x_i, \gamma_i z_i]$. Now $|d(\gamma_i w, \gamma_i z_i) - \epsilon| = |d(w, z_i) - \epsilon| \leq d(z, z_i)$ and $|d(w', \gamma_i z_i) - \epsilon| \leq d(z', \gamma_i z_i)$. Since $w', \gamma_i w \in [\gamma_i x_i, \gamma_i z_i]$, we see that $d(w', \gamma_i w) = |d(w', \gamma_i z_i) - d(\gamma_i w, \gamma_i z_i)| \leq d(z', \gamma_i z_i) + d(z, z_i) \to 0$. We also have $d(z', \gamma_i z) \leq d(z', \gamma_i z_i) + d(\gamma_i z, \gamma_i z_i) = d(z', \gamma_i z_i) + d(z, z_i) \to 0$. It follows that the sequence of edges, $(\gamma_i(z, w))_i$, is Cauchy, contradicting the edge-discreteness of $\Gamma$.     $\diamond$

**Proof of Proposition 7.2 :** We show that the action on distinct triples of $D$ is properly discontinuous. Suppose, for contradiction, that we can find sequences, $(x_i)_i$, $(y_i)_i$ and $(z_i)_i$ in $D$ with $x_i \to x$, $y_i \to y$, $z_i \to z$, $\gamma_i x_i \to x'$, $\gamma_i y_i \to y'$ and $\gamma_i z_i \to z'$, where $x, y, z \in D$ are all distinct, and $x', y', z' \in D$ are all distinct.

Let $m = \mathrm{med}(x, y, z)$, $m_i = \mathrm{med}(x_i, y_i, z_i)$ and $m' = \mathrm{med}(x', y', z')$. By continuity of the median (Lemma 1.3), we have that $m_i \to m$ and $\gamma_i m_i \to m'$. Now, without loss of generality, we have $m \neq x$, $m \neq y$ and $m' \neq x'$. Thus, replacing $z$ by $m$, $z_i$ by $m_i$, and either $z'$ or $y'$ by $m'$, we can assume that $z \in (x, y)$, $z_i \in (x_i, y_i)$ and either $z' \in (x', y')$ or $y' \in (x', z')$.

Suppose, first, that $z' \in (x', y')$. For all sufficiently large $i$, we have that $z_i \notin [x, x_i] \cup [y, y_i]$ and $\gamma_i z_i \notin [x', \gamma_i x_i] \cup [y', \gamma_i y_i]$. Now $z_i \in (x_i, y_i)$ and so $z_i \in (x, y)$. Also $\gamma_i z_i \in (\gamma_i x_i, \gamma_i y_i)$ and so $\gamma_i z_i \in (x', y')$. Now, the metric topology on $(x, y)$ agrees with the subspace topology from $D$. Since $z, z_i \in (x, y)$ and $z_i \to z$, we see that $d(z, z_i) \to 0$. Similarly, since $z', \gamma_i z_i \in (x', y')$ and $\gamma_i z_i \to z'$ we have $d(z', \gamma_i z_i) \to 0$. Since $x_i \to x$ and $\gamma_i x_i \to x'$, we get a contradiction to Lemma 7.21.

Finally, suppose that $y' \in (x', z')$. For all sufficiently large $i$, we have that the intervals $[x', \gamma_i x_i]$, $[y', \gamma_i y_i]$ and $[z', \gamma_i z_i]$ are mutually disjoint. Since $y' \in (x', z')$, this tells us, in particular, that $\gamma_i z_i \notin [\gamma_i x_i, \gamma_i y_i]$. This contradicts the fact that $z_i \in (x_i, y_i)$.     $\diamond$

We now return to the question formulated earlier of constructing isometric actions on trees from convergence actions on dendrons.

We shall want to restrict attention to metrisable dendrons. We saw in Section 1 that every dendrite (i.e. separable dendron) is metrisable (Lemma 1.7), though this need not be the case in general. (Consider for example the real pretree formed by taking uncountably many copies of a real closed interval, and connecting them all together at one endpoint.

This pretree is complete, and hence compact in the order topology. However the connecting point has no countable base.)

We shall want to consider different kinds of metrics on a real pretree, $T$. In general, we use the term "metric" in the elementary sense (as a map $d : T^2 \longrightarrow \mathbf{R}$). Two metrics are *equivalent* if they induce the same topology. We are only really interested in metrics for which $(T, d)$ is topologically a real tree, inducing the original pretree structure. We shall examine hypotheses under which this is the case.

**Definition :** A metric, $d$, on a real pretree, $T$, is *monotone* if, given $x, y, z \in T$ with $z \in [x, y]$, we have $d(x, z) \leq d(x, y)$.

Note that this is equivalent to saying that the metric ball $N(x, r) = \{y \in T \mid d(x, y) \leq r\}$ is full for all $x \in T$ and $r \geq 0$.

**Lemma 7.22 :** *Suppose that $T$ is a real pretree, and that $d$ is a monotone metric on $T$ such that for all $x, y \in T$, the interval $([x, y], d)$ is compact. Then $(T, d)$ is a real tree, inducing the the original pretree structure on $T$.*

**Proof :** First, we claim that $([x, y], d)$ is homeomorphic to a closed real interval. To see this, suppose that $a, b \in [x, y]$ and $z \in (a, b)$. Choose $0 < r < \min\{d(z, a), d(z, b)\}$. Thus, $a, b \notin N(z, r)$. Since $N(z, r)$ is full, we have $N(z, r) \cap [x, y] \subseteq (a, b)$. Thus, $(a, b)$ is open in $([x, y], d)$. Similarly, intervals of the form $[x, a)$ and $[y, a)$ for $a \in [x, y]$ are open in $([x, y], d)$. Since these form a base for the order topology on $[x, y]$, we see that the metric topology is finer that the order topology. Since the metric topology is, by hypothesis, compact, and the order topology is hausdorff, we see that they must agree. Since $T$ is a real pretree, $[x, y]$ is homeomorphic to a closed real interval in the order topology. This proves the claim.

Now, since the metric balls $N(x, r)$ are full, we see that the local connectedness hypothesis of Lemma 2.19 is satisfied. We conclude that $(T, d)$ is a real tree. $\diamondsuit$

(In our principal application of this lemma (Proposition 7.26), we shall see directly that pretree intervals in the metric topology are homeomorphic to real closed interval, so we could bypass much of the above argument.)

**Definition :** A metric, $d$, on a real pretree, $T$, is *convex* if, given $x, y, z \in T$ with $z \in [x, y]$, we have $d(x, y) = d(x, z) + d(z, x)$.

Now, clearly a convex metric, $d$, is monotone, and so by Lemma 7.22, $(T, d)$ is a real tree. But now it is clear that $(T, d)$ satisfies the axioms of an $\mathbf{R}$-tree. This gives us yet another characterisation of an $\mathbf{R}$-tree, namely as real pretree with a convex metric. Note that, as we have already observed, any path metric on a real tree is necessarily a convex metric. Also, from [MayO], we know that any metrisable real tree admits an equivalent path metric. Unfortunately (for us) their construction is not a canonical one.

Suppose, now, that $D$ is a metrisable dendron which admits a convergence action by some group $\Gamma$. Choose (for the moment) any metric $d$ on $D$. Let $R$ be the set of terminal

points of $D$, and let $T = D \setminus R$. Thus, $(T, d)$ is a real tree, which is a bounded metric space. Moreover, $\Gamma$ acts by homeomorphism on $T$.

**Lemma 7.23 :** *Given any $x, y \in T$ and $r > 0$, the set $\{\gamma \in \Gamma \mid d(\gamma x, \gamma y) \geq r\}$ is finite.*

**Proof :** Suppose not. Let $(\gamma_i)_{i \in \mathbf{N}}$ be a sequence of distinct elements of $\Gamma$ with $d(\gamma_i x, \gamma_i y) > r$. Passing to a subsequence, we can find $\lambda, \mu \in D$ so that the maps $\gamma_i | (D \setminus \{\lambda\})$ converge locally uniformly to the point $\mu$. Let $U$ be a connected neighbourhood of $\mu$ with diameter less than $r$. Now, since $x, y \notin R$, we can find $x', y' \in D \setminus \{\lambda\}$, with $[x, y] \subseteq [x', y']$. Now, for all sufficiently large $i$, we have $\gamma_i x', \gamma_i y' \in U$. Since $U$ is convex, it follows that $\gamma_i x, \gamma_i y \in U$, and so $d(\gamma_i x, \gamma_i y) \leq r$, contradicting the hypothesis on $(\gamma_i)$. $\diamond$

Thus, given $x, y \in T$, we can define $d'(x, y) = \max\{d(\gamma x, \gamma y) \mid \gamma \in \Gamma\}$. Clearly, $d'$ is $\Gamma$-invariant, and $d \leq d'$.

**Lemma 7.24 :** *$d'$ is a metric.*

**Proof :** We only really need to verify the triangle inequality. Suppose $x, y, z \in T$. then there is some $\gamma \in \Gamma$ such that $d'(x, y) = d(\gamma x, \gamma y)$. Thus $d(x, y) = d(\gamma x, \gamma y) \leq d(\gamma x, \gamma z) + d(\gamma z, \gamma y) \leq d'(\gamma x, \gamma z) + d'(\gamma z, \gamma y) = d'(x, z) + d'(z, y)$. $\diamond$

**Lemma 7.25 :** *If $x, y \in T$, then $([x, y], d')$ is homeomorphic to a closed real interval (where $[x, y]$ denotes an interval in the original tree $(T, d)$).*

**Proof :** In other words, we want to show that the metrics $d$ and $d'$ are equivalent on $[x, y]$. Since $d \leq d'$, it's enough to show that, given any $\epsilon > 0$, there is some $\eta > 0$ such that if $a, b \in [x, y]$ with $d(a, b) \leq \eta$ then $d'(a, b) \leq \epsilon$.

Suppose, for contradiction, that this fails. There are sequences, $(a_i)_{i \in \mathbf{N}}$ and $(b_i)_{i \in \mathbf{N}}$ of points of $[x, y]$ with $d(a_i, b_i) \to 0$ and $d'(a_i, b_i) \geq \epsilon$. Thus, there are elements $\gamma_i \in \Gamma$ with $d(\gamma_i a_i, \gamma_i b_i) \geq \epsilon$.

Now, if $\{\gamma_i \mid i \in \mathbf{N}\}$ were finite, we could assume that $\gamma_i = \gamma$ is constant for all $i$. But $\gamma$ maps $[x, y]$ $d$-homeomorphically onto $\gamma([x, y])$, and so $\gamma | [x, y]$ is uniformly $d$-continuous. We get the contradiction that $d(\gamma_i a_i, \gamma_i b_i) \to 0$.

Thus, passing to a subsequence, we can assume that the elements $\gamma_i$ are all distinct. Thus, again passing to a subsequence, we can fine $\lambda, \mu \in D$ such that the maps $\gamma_i | (D \setminus \{\lambda\})$ converge uniformly to $\mu$. Now, since $x, y \notin R$, we can find $x', y' \in D \setminus \{\lambda\}$, with $[x, y] \subseteq [x', y']$. We choose some connected neighbourhood $U$ of $\mu$ of diameter less that $\epsilon$. Now for all sufficiently large $i$, we have $\gamma_i x', \gamma_i y' \in U$, and so $\gamma_i a_i, \gamma_i b_i \in U$. We thus get the contradiction that $d(\gamma_i a_i, \gamma_i b_i) < \epsilon$. $\diamond$

So far, this works for any metric on $D$. To go any further, we shall want to assume that $d$ is monotone. This is possible, since by the theorem of Bing/Moise referred to in Section 1, [Bi1,Bi2,Moi], any metrisable Peano continuum admits a path metric. In particular, $D$ admits a convex metric, which is certainly monotone. (This also follows from the result of [MayO].)

Now, the property of monotonicity is clearly preserved in passing from the metric $d$ to $d'$. By Lemmas 7.22 and 7.25, it thus follows that, in this case, $(T, d')$ is topologically a real tree.

Now, since $d \leq d'$, the inclusion map $f : (T, d') \hookrightarrow (D, d)$ is a continuous injection, and $f(T)$ is dense in $D$. Thus, $D$ is the unique compactification of the real tree $(T, d')$ as described by Proposition 7.1.

Finally recall that a metrisable real tree has the topological structure of an **R**-tree [MayO].

In summary, we have shown:

**Proposition 7.26 :** *Suppose that $D$ is a metrisable dendron, and that $\Gamma$ acts homeomorphically as a convergence group on $D$. Then, there is an **R**-tree $T$, with an action of $\Gamma$ by homeomorphism, and a $\Gamma$-equivariant continuous injection $f : T \hookrightarrow D$, with $f(T)$ dense in $D$. Moreover, $T$ admits a $\Gamma$-invariant monotone metric which is equivalent to the **R**-tree metric.* $\diamond$

In other words, $T$ admits one metric which is convex, and another (equivalent) metric which is monotone and $\Gamma$-invariant. Thus we arrive at the natural question, can the convex metric itself be made $\Gamma$-invariant? In [Bo3], we succeed in constructing a convex pseudometric, sufficient for our purposes with regards to hyperbolic groups.

In the above construction, we made a choice of monotone metric, $d$, on the dendron $D$. It turns out that the induced topology on $(T, d')$ is independent of this choice. This follows from the following observation. We start with a dendron $D$, admitting a convergence action of a group $\Gamma$, and let $T$ be the subpretree consisting of the set of all nonterminal points.

**Lemma 7.27 :** *Suppose that $d$ is any monotone metric on $D$ (inducing the original topology), and that $d_0$ is any $\Gamma$-invariant monotone metric on $T$. Then, the identity map from $(T, d_0)$ to $(T, d')$ is continuous.*

**Proof :** Suppose not. Then we can find sequences of points $x_i, y_i \in T$, with $d_0(x_i, y_i) \to 0$ and $d'(x_i, y_i)$ bounded below. Now (after applying appropriate elements of $\Gamma$), we can suppose that $d(x_i, y_i) = d'(x_i, y_i)$. In particular $d(x_i, y_i)$ is bounded below, whereas $d_0(x_i, y_i)$ tends to 0. Now, on passing to a subsequence, we can suppose that $x_i \to x \in D$ and $y_i \to y \in D$ (in the topology on $D$). Since $d(x_i, y_i)$ is bounded below, $x \neq y$. Now choose any distinct points $a, b \in (x, y)$. Thus, for all sufficiently large $i$, we have $a, b \in (x_i, y_i)$. But now, by monotonicity, we get that $d_0(x_i, y_i) \geq d_0(a, b) > 0$, contradicting the fact that this sequence tends to 0. $\diamond$

**Corollary 7.28 :** *Suppose $D$, $\Gamma$ and $T$ are as above. Suppose that $d_1$ and $d_2$ are monotone metrics on $D$ (inducing the original topology). Then, the metrics $d'_1$ and $d'_2$ on $T$ are equivalent.*

**Proof :** Apply Lemma 7.27 with $d = d_1$ and $d_0 = d'_2$, and with $d = d_2$ and $d_0 = d'_1$. $\diamond$

This shows that the topology we obtain on $T$ in this way is a canonical one. It can be

described as the coarsest topology on $T$ which can be induced by a $\Gamma$-invariant monotone metric.

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