A CARTAN-HADAMARD THEOREM FOR MEDIAN METRIC SPACES

BRIAN H. BOWDITCH

ABSTRACT. We show that a complete simply connected path-metric space which is uniformly locally median is a median metric space. We describe some related results.

1. INTRODUCTION

Let (M, σ) be a metric space. Given $a, b, x \in M$, write $a.x.b$ to mean $\sigma(a, b) = \sigma(a, x) + \sigma(b, x)$. Let $I(a, b) = \{x \in M \mid a.x.b\}$ and $\mathrm{Med}(a, b, c) = I(a, b) \cap I(b, c) \cap I(c, a)$ for $a, b, c \in M$.

Definition. We say that (M, σ) is a ***median metric space*** if for all $a, b, c \in M$, $\mathrm{Med}(a, b, c)$ consists of a single element.

In this case, we refer to this element as the ***median*** of a, b, c and denote it by abc. One can show that the metric completion of a median metric space is also a median metric space. Moreover, any connected complete median metric space is geodesic, in particular, a path-metric space. As we note below, median metric spaces arise in many different contexts, and have been much studied.

One can interpret the median property locally as follows:

Definition. We say that a metric space M is ϵ-***locally median*** for $\epsilon > 0$, if $\mathrm{Med}(a, b, c)$ contains precisely one element whenever the diameter of $\{a, b, c\}$ is at most ϵ. We say that M is ***uniformly locally median*** if it is ϵ-locally median for some $\epsilon > 0$.

The main result here is:

Theorem 1.1. *A complete simply connected path-metric space which is uniformly locally is median.*

This will be proven in Section 8. Our argument was inspired by a related result in [ChalCHO] as we mention below.

Conversely, it is not hard to see that any connected median algebra is simply connected. In fact, all the homotopy groups are trivial. An argument shown to be my Elia Fioravanti is given in [Bo4]. Simple connectedness can also be deduced in the complete case from the arguments in Section 3.

Date: First draft: 21st December 2021. Revised: 12th May 2023.

We conjecture that Theorem 1.1 remains true with a weaker hypothesis of "locally median": that is, without requiring a uniform constant $\epsilon > 0$. Some situations in which this is true are discussed in Section 9, and it would seem that any counterexample would need to have some exotic features.

The property of being (uniformly) locally median can be viewed as a kind on "non-positive curvature" condition. In this way, Theorem 1.1 can be thought of as a "Cartan-Hadamard Theorem" for median metric spaces. Related conditions on a metric space are the "local CAT(0)" property, and "local injectivity".

The original Cartan-Hadamard Theorem referred to complete non-positively curved riemannian manifolds (see for example, [BalGS]). This was generalised to (what Gromov has called) locally CAT(0) spaces by work of Aleksandrov and Toponogov (see for example [BrH]). There is also related result for the more general notion of convexity in the sense of Busemann (see [AB]). More recently, a Cartan-Hadamard Theorem for injective metric spaces has been given in [M1].

One obvious example of a median metric space is the real line, $\mathbb{R}$, with the usual metric. The property of being median is closed under l^1 direct products, and so $\mathbb{R}^n$ is also median in the l^1 metric. Infinite dimensional l^1 spaces are also median. Median metric spaces arise in various situations in geometric group theory. For example, CAT(0) cube complexes, with equipped with the l^1 metric on each cube. They arise from spaces with measured walls and have applications to the Haagerup property of groups: see for example, [ChatDH]. The asymptotic cones of various naturally occurring spaces, such as mapping class groups and Teichmüller space, can also be equipped with a median metric. This has various applications to such things as quasi-isometric rigidity of such spaces. A survey of some of these applications is given in [Bo2].

The ternary operation, $[(a, b, c) \mapsto abc] : M^3 \longrightarrow M$, equips M with the structure of a median algebra. (That is to say, it is symmetric and satisfies $aab = a$ and $ab(acd) = ac(abd)$ for all a, b, c, d.) Median algebras have been extensively studied in their own right (see for example, [BanH, R, Bo4].) In this paper, we will only explicitly make use of some basic median algebra identities.

We note that under certain hypotheses (notably connectedness and finite rank) a median metric space has a natural bilipschitz equivalent CAT(0) metric, [Bo1, Z], as well as a bilipschitz equivalent injective metric, [Bo3, M2]. (In the case of $\mathbb{R}^n$, these are respectively the l^2 metric and the l^∞ metric.) From either statement, it follows that such a median metric space is contractible. We suspect this to be true much more generally.

We also note Theorem 1.1 has a combinatorial analogue. A connected graph is said to be ***median*** of the vertex set is a median metric space in the combinatorial distance metric. Such graphs have been much studied, and there are many equivalent definitions. A form of the Cartan-Hadamard Theorem in this context is given in [Che]. A different proof in a more general context is given in [ChalCHO].

Our proof broadly follows the strategy of the latter paper, though the details are somewhat different.

The idea behind the proof of Theorem 1.1 is as follows. We first show that it is enough for M to have "approximate medians" for any triple of points: that is a median up to some arbitrarily small additive constant (Proposition 8.13). To construct such such approximate medians, we proceed as follows. Given any $q \in M$, we inductively construct a nested sequence of path-metric spaces, $X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots$, with each inclusion an isometric embedding, and with some basepoint $p \in X_1$. We also construct local isometries $f_n : X_n \longrightarrow M$, with $f_n|X_m = f_m$ whenever $m \leq n$, and with $f_n(p) = q$. Each space X_n has the property that the triple p, x, y has approximate medians for all $x, y \in X_n$. We then set $Y = \bigcup_{n=1}^{\infty} X_n$. The maps f_n combine to give us a map $f : Y \longrightarrow M$. This is a covering map, so if M is simply connected, it is a homeomorphism. Since Y and M are both path-metric spaces (by construction, and by hypothesis respectively) we see that f is an isometry. Since q was arbitrary, it follows that M has approximate medians, as claimed. The fact that M is indeed median calls for another observation, namely Lemma 5.2.

Before starting on this properly, we will describe a procedure for homotoping paths, which will be used in the proof of Theorem 1.1.

2. Some general observations

We first make some simple observations and definitions regarding a general metric space.

Let (M, σ) be a metric space. We will use $N(x, r)$ to denote the open r-neighbourhood of $x \in M$. We use $\operatorname{diam}(A)$ to denote the diameter of a subset $A \subseteq M$.

Given $a, b, c \in M$, write

$$\langle a, b \rangle_c = \frac{1}{2}(\sigma(a, c) + \sigma(b, c) - \sigma(a, b)) \geq 0$$

for the ***Gromov product*** of the pair a, b based at c. Writing $t = \langle b, c \rangle_a$, $u = \langle c, a \rangle_b$ and $v = \langle a, b \rangle_c$, we see that $\sigma(a, b) = t + u$, $\sigma(b, c) = u + v$ and $\sigma(c, a) = v + t$. If $d \in \operatorname{Med}(a, b, c)$, then $\sigma(a, d) = t$. Note also that $a.c.b$ holds if and only if $\langle a, b \rangle_c = 0$.

Lemma 2.1. *Let $a, b, c, d \in M$ with $a.b.c$ and $b.c.d$. Then $\rho(b, c) \leq \rho(a, d)$.*

Proof. We just add the inequalities:

$$\rho(a, b) + \rho(b, c) = \rho(a, c) \leq \rho(a, d) + \rho(c, d)$$
$$\rho(b, c) + \rho(c, d) = \rho(b, d) \leq \rho(a, b) + \rho(a, d)$$

and cancel $\rho(a, b) + \rho(c, d)$. □

If we also have $c.d.a$ and $d.a.b$, then we refer to a, b, c, d as a ***square***. In this case we get, $\sigma(b, c) = \sigma(a, d)$ and $\sigma(c, d) = \sigma(a, b)$.

Given a finite sequence, $\underline{a} = a_0, a_1, \ldots, a_n$, in M, we write $l(\underline{a}) = \sum_{i=1}^{n} \sigma(a_{i-1}, a_i)$. We say that $\underline{a}$ is a ***geodesic sequence***, and write $a_0.a_1.a_2.\cdots.a_n$, if $l(\underline{a}) = \sigma(a_0, a_n)$. Note that this agrees with the notation introduced earlier when $n = 2$. In fact, $\underline{a}$ is geodesic if and only if $a_i.a_j.a_k$ holds whenever $i \leq j \leq k$. Note also that we have a rule of "interpolation". For example, if $a.b.d$ and $b.c.d$ hold, then so does $a.b.c.d$.

By a ***path*** in M we mean a continuous map, $\alpha : I \longrightarrow M$, where $I = [0, T] \subseteq \mathbb{R}$ is a compact real interval. Let $L(\underline{a}) \in [0, \infty]$ be the supremum of $l(\underline{a})$, as $\underline{a}$ varies over all sequences of the form $\alpha(t_0), \alpha(t_1), \ldots, \alpha(t_n)$, where $0 = t_0 \leq t_1 \leq \cdots \leq t_n = T$. This is the ***length*** of α. We say α is ***rectifiable*** if $L(\alpha) < \infty$. Every lipschitz path is rectifiable. Clearly, $L(\alpha) \geq \sigma(\alpha(0), \alpha(T))$. We say that α is ***geodesic*** if $L(\alpha) = \sigma(\alpha(0), \alpha(T))$. One can check that this is equivalent to saying that $\alpha(t).\alpha(u).\alpha(v)$ holds whenever $t \leq u \leq v$. We say that M is a ***geodesic space*** if any two points of M are connected by a geodesic.

Given $a, b \in M$, we say that $c \in M$ is a ***midpoint*** of a, b if $\sigma(a, c) = \sigma(b, c) = \sigma(a, b)/2$. Clearly in any geodesic space, midpoints always exist. Conversely, the following is well known:

Lemma 2.2. *If M is complete, and any two points have a midpoint, then M is geodesic.*

This is easily seen by taking iterated midpoints to so as construct an isometric embedding of an interval in the diadic rationals (up to rescaling), and then taking the closure of the image.

(We remark that Menger's Theorem tells us that, in fact, it is enough to assume that M is complete, and that $I(a, b) \neq \{a, b\}$ for all $a, b \in M$. This is a more subtle fact, and we will not need it here.)

Recall that a ***net*** in M consists of directed set, $(D, \leq)$, together with a map, $\omega : D \longrightarrow M$. We say that ω ***converges*** to $a \in M$ if for all $\epsilon > 0$, there is some $x \in D$ such that $\sigma(\omega(y), a) \leq \epsilon$ for all $y \geq x$. We write $\omega \to a$. A net, $\omega : D \longrightarrow \mathbb{R}$ is ***non-increasing*** if $\omega(y) \leq \omega(x)$ whenever $y \geq x$. Note that any non-increasing net $\omega : D \longrightarrow [0, \infty)$ converges.

We will need the following observation.

Lemma 2.3. *Suppose that D is a directed set, and that $\delta : D \longrightarrow [0, \infty)$ is a non-increasing net with $\delta \to 0$. Suppose that M is a complete metric space, and that $\omega : D \longrightarrow M$ is a net with the property that if $x, y \in D$, with $x \leq y$, then $\sigma(\omega(x), \omega(y)) \leq \delta(y) - \delta(x)$. Then ω converges to some $a \in M$. Moreover, $\sigma(\omega(x), a) \leq \delta(x)$ for all $x \in D$.*

Proof. We can certainly find some increasing sequence, $(x_n)_{n\in\mathbb{N}}$, in D, with $\delta(x_n) \to 0$. If $m \geq n$, then $\sigma(\omega(x_n), \omega(x_m)) \leq \delta(x_n) - \delta(x_m) \leq \delta(x_n)$. In particular, $(\omega(x_n))_{n\in\mathbb{N}}$ is Cauchy, and so converges on some $a \in M$. Letting $m \to \infty$

above, we get $\sigma(\omega(x_n), a) \leq \delta(x_n)$ for all n. Now suppose $\epsilon > 0$. Choose n so that $\delta(x_n) \leq \epsilon/2$. If $y \geq x_n$, then $\sigma(\omega(y), \omega(x_n)) \leq \delta(x_n) - \delta(y) \leq \epsilon/2$. Thus $\sigma(\omega(y), a) \leq \sigma(\omega(y), \omega(x_n)) + \sigma(\omega(x_n), a) \leq 2(\epsilon/2) = \epsilon$. This shows that $\omega \to a$.

Finally, if $x \in D$, then for any $y \geq x$, we have $\sigma(\omega(y), \omega(x)) \leq \delta(x) - \delta(y) \leq \delta(x)$. Since $\omega(y)$ converges on a, we get $\sigma(a, \omega(x)) \leq \delta(x)$ as required. □

Recall that $\mathrm{Med}(a, b, c) = I(a, b) \cap I(b, c) \cap I(c, a)$.

Definition. We say that M is ***modular*** if $\mathrm{Med}(a, b, c) \neq \varnothing$ for all $a, b, c \in M$.

In this case, we refer to any element of $\mathrm{Med}(a, b, c)$ as a ***median*** of a, b, c. We note:

Lemma 2.4. *If M is a connected modular metric space, then any two points have a midpoint.*

Proof. Let $a, b \in M$. Define a map, $f : M \longrightarrow [0, \sigma(a, b)]$ by $f(x) = \langle b, x \rangle_a$. Then f is continuous, and $f(a) = 0$, $f(b) = \sigma(a, b)$. Therefore, there is some $x \in M$ with $f(x) = \sigma(a, b)/2$. Let $c \in \mathrm{Med}(a, b, x)$. Then c is a midpoint of a, b. □

Together with Lemma 2.2, this immediately gives:

Proposition 2.5. *A complete connected modular metric space is geodesic.*

Suppose now that M is a median metric space. We have noted that the ternary operation, $[(a, b, c) \mapsto abc] : M^3 \longrightarrow M$ gives M the structure of a median algebra. This means that it is symmetric in a, b, c and that $aab = a$ and $(abc)de = (ade)(bde)c$ for all $a, b, c, d, e \in M$. (This is equivalent to the definition given in the introduction.) Using Lemma 2.1 one can show that $\sigma(abc, abd) \leq \sigma(c, d)$ for all $a, b, c, d \in M$. (In other words, the median operation is 1-lipschitz with respect to the induced l^1 metric on M^3.) We also have $I(a, b) = \{abx \mid x \in M\} = \{x \in M \mid abx = x\}$. This is the ***median interval*** from a to b.

We will make use of a number of basic properties of median algebras. In particular, we have the following.

Fix a basepoint, $p \in M$. Write $a \wedge b = pab$. Then $(a \wedge b) \wedge c = a \wedge (b \wedge c)$. We write $a \leq b$ to mean that $a \wedge b = a$. Then $\leq$ is a partial order on M, with minimum p. Note that if $a \leq b$ and $a \leq c$ hold, then $a \leq b \wedge c$. We also note that if $a.c.b$ holds, then $a \wedge b \leq c$. (Since writing $m = a \wedge b$, we have $m \wedge c = pmc = pm(abc) = (pma)(pmb)c = mmc = m$.) Note that Lemma 2.1 implies that if $a, b, c, d \in M$ with $b, d \leq a$ and $c = b \wedge d$, then $\sigma(b, c) \leq \sigma(d, a)$ and $\sigma(c, d) \leq \sigma(a, b)$.

(For elabolaration on the above facts, see [BanH, R, Bo4].)

Lemma 2.6. *If $a, b, c \in M$, then $\sigma(a, a \wedge b) \leq \sigma(a, a \wedge c) + \sigma(c, b \wedge c)$.*

Proof. $\sigma(a, pab) \leq \sigma(a, pa(abc)) + \sigma(pa(abc), pab) = \sigma(a(pab)a, a(pab)(pac)) + \sigma(a(pab)c, a(pab)(pbc)) \leq \sigma(a, pac) + \sigma(c, pbc)$. □

Given a nonempty finite subset, $A = \{a_0, a_1, \dots a_n\} \subseteq M$, write

$$\bigwedge A = a_0 \wedge a_1 \wedge \cdots \wedge a_n.$$

Suppose $c \in M$ with $a.c.b$ for some $a, b \in A$. Then $\bigwedge A \leq c$. (Since, by the earlier observation, we have $(\bigwedge A) \wedge c = (\bigwedge A) \wedge a \wedge b \wedge c = (\bigwedge A) \wedge a \wedge b = \bigwedge A$.) In other words, $\bigwedge(A \cup \{c\}) = \bigwedge A$. More generally, we note:

Lemma 2.7. *Suppose $A \subseteq M$ is finite and non-empty, $a, b \in A$ and $c \in M$. Then $\sigma(\bigwedge A, \bigwedge(A \cup \{c\}) \leq \langle a, b\rangle_c$.*

Proof. Let $m = abc$. By the above observation, we have $\bigwedge A = (\bigwedge A) \wedge m$. Thus $\sigma(\bigwedge A, (\bigwedge A) \wedge c) \leq \sigma(m, c) = \langle a, b\rangle_c$. $\square$

For future reference, we make the following definition:

Definition. A subset, $\Omega \subseteq M$ is ***starlike*** about p if $a \wedge b \in \Omega$ for all $a, b \in \Omega$.

Note that this is equivalent to saying that if $a, b \in M$ with $a \leq b$ and $b \in \Omega$, then $a \in \Omega$.

We note that the relation $\leq$ can also be defined in an arbitrary metric space with basepoint p, by writing $a \leq b$ to mean $p.a.b$. Again, this is easily seen to be a partial order on M, and it agrees with the above notation if M is median.

We finally note that the above arguments can also be applied to an ϵ-locally median metric space, provided that our constructions never take us outside a set of diameter at most ϵ. This will easily be seen to be the case in our applications. For example, if we adjoin all the medians to a set, then its diameter increases by at most a factor of 2. The verification of any given median identity only involves iterating the median operation some finite number of times. For example, the derivation of the identity $(abc)de = (ade)(bde)c$ is somewhat complicated, but can be carried out in some set of diameter at most some fixed multiple of that of the original set $\{a, b, c, d, e\}$. (It would be an exercise to figure out what this number is.) Therefore, after shrinking ϵ by some fixed universal multiple, there would be no loss in assuming that this identity holds whenever $\operatorname{diam}\{a, b, c, d, e\} \leq \epsilon$.

We finally note:

Lemma 2.8. *If M is a ϵ-locally median path-metric space, and $a, b \in M$ with $\sigma(a, b) < \epsilon/2$, then a, b are connected by a (unique) geodesic in M.*

The proof follows that of Lemma 2.4.

3. Retracting paths

In this section, we equip $[0, \infty)^2 \subseteq \mathbb{R}^2$ with the l^1 metric. We will use the notation $\underline{x} = (x_1, x_2)$ for $\underline{x} \in [0, \infty)^2$. Taking $\underline{0}$ as our basepoint, we have $\underline{x} \wedge \underline{y} = (\min(x_1, y_1), \min(x_2, y_2))$, and $\underline{x} \leq \underline{y}$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$.

Let (M, σ) be a median metric space with basepoint $p \in M$. Let $I = [0, T] \subseteq \mathbb{R}$, and let $\alpha : I \longrightarrow M$ be a 1-lipschitz path. We write $a = \alpha(0)$ and $b = \alpha(T)$. Thus $\sigma(a, b) \leq T$.

The main aim of this section is to construct a canonical homotopy of α to a point. More precisely, we will construct a compact subset $\Omega \subseteq [0, \infty)^2$, starlike about $\underline{0}$, and 1-lipschitz maps, $\beta : I \longrightarrow \Omega$ and $\phi : \Omega \longrightarrow M$, with $\phi \circ \beta = \alpha$. The the topological boundary of $\beta(I)$ is $[0, \infty)^2$ is $\beta(I)$: this means that $\Omega = \{\underline{w} \in [0, \infty)^2 \mid (\exists t \in I)(\underline{w} \leq \beta(t))\}$. If $\underline{x}, \underline{y} \in \Omega$, then $\phi(\underline{x} \wedge \underline{y}) \leq \phi(\underline{x}) \wedge \phi(\underline{y})$. Moreover, if there is some $\underline{z} \in \Omega$ with $\underline{x}, \underline{y} \leq \underline{z}$, then $\phi(\underline{x} \wedge \underline{y}) = \phi(\underline{x}) \wedge \phi(\underline{y})$. Note that (except in very special "degenerate" cases) Ω is a topological disc.

Let $\mathcal{P} = \mathcal{P}(I)$ be the set of all finite subsets of I which contain $\{0, T\}$. We view $\mathcal{P}$ as a directed set under inclusion. Given $P \in \mathcal{P}$, we will generally write $P = \{t_0, t_1, \ldots, t_n\}$ where $0 = t_0 < t_1 < \cdots < t_n = T$. We write $a_i = \alpha(t_i)$, and let $\underline{a}$ be the sequence $a_0, a_1, \ldots, a_n$ in M. We set $L(P) = l(\underline{a})$ as defined in Section 2. Thus $L : \mathcal{P} \longrightarrow [0, T]$ is a non-decreasing net in $[0, T]$. It converges to $L_0 := L(\alpha) \leq T$, that is, the rectifiable length of α.

Given $i \in \{1, \ldots, n\}$, let $l_{1,i} = \sigma(a_i, a_i \wedge a_{i-1})$ and $l_{2,i} = \sigma(a_{i-1}, a_i \wedge a_{i-1})$. Thus, $\sigma(a_{i-1}, a_i) = l_{1,i} + l_{2,i}$. Write $L_j(P) = \sum_{i=1}^{n} l_{j,i}$. Thus $L(P) = L_1(P) + L_2(P)$.

Lemma 3.1. *If $P, Q \in \mathcal{P}$ with $P \subseteq Q$, then $L_j(P) \leq L_j(Q)$.*

Proof. Adding one point at a time, we can suppose that $Q = P \cup \{u\}$ for some $u \in I$. Now $u \in (t_{i-1}, t_i)$ for some i. Let $c = \alpha(u)$. Then $L(P \cup \{u\}) = l(\underline{b})$, where $\underline{b}$ is the sequence $a_0, a_1, \ldots, a_{i-1}, c, a_i, \ldots, a_n$. By Lemma 2.6, we have $\sigma(a_i, a_{i-1} \wedge a_i) \leq \sigma(a_i, a_i \wedge c) + \sigma(c, c \wedge a_{i-1})$. All the other terms defining $L_1(P \cup \{u\})$ remain unchanged, and so $L_1(P) \leq L_1(P \cup \{u\})$. Similarly, $L_2(P) \leq L_2(P \cup \{u\})$. □

This shows that the net $(L_j(P))_P$ is non-decreasing in P. Since it is bounded above by L_0, it converges on some $L_j \leq L_0$. In fact, since $L_1(P) + L_2(P) = L(P)$ for all P, we have $L_1 + L_2 = L_0$.

Given $P \in \mathcal{P}$, let $\omega(P) = \bigwedge \alpha(P) \in M$.

Lemma 3.2. *If $P, Q \in \mathcal{P}$ with $P \subseteq Q$, then $\sigma(\omega(P), \omega(Q)) \leq \frac{1}{2}(L(Q) - L(P))$.*

Proof. Again, it is enough to verify this when $Q = P \cup \{u\}$. Let $u, c, \underline{b}$ be as in the proof of Lemma 3.1. By Lemma 2.7, we have $\sigma(\omega(P), \omega(P \cup \{u\})) \leq \langle a_{i-1}, a_i \rangle_c = \frac{1}{2}(l(\underline{b}) - l(\underline{a})) = \frac{1}{2}(L(P \cup \{u\}) - L(P))$. □

Let $\delta(P) = \frac{1}{2}(L_0 - L(P))$. Thus $\delta : \mathcal{P} \longrightarrow [0, \infty)$ is non-increasing and $\delta \to 0$. By Lemma 2.3, we see that if M is complete, then $\omega(P)$ converges on some $\omega(I) \in M$.

For the remainder of this section, we will assume that M is complete.

Now $\omega(P) \leq a \wedge b$ for all $P \in \mathcal{P}$, and so $\omega \leq a \wedge b$. Also:

Lemma 3.3. *$\sigma(a, \omega) \leq L_2$ and $\sigma(b, \omega) \leq L_1$.*

Proof. Let $P \in \mathcal{P}$. In the above notation, $a = a_0$, $b = a_n$ and $\omega(P) = a_0 \wedge a_1 \wedge \cdots \wedge a_n$. Write $b_i = a_0 \wedge a_1 \wedge \cdots \wedge a_i$. Then

$$\sigma(b_{i-1}, b_i) = \sigma(b_{i-1} \wedge a_{i-1}, b_{i-1} \wedge a_{i-1} \wedge a_i) \leq \sigma(a_{i-1}, a_{i-1} \wedge a_i).$$

Therefore,

$$\sigma(a, \omega(P)) = \sigma(b_0, b_n) \leq \sum_{i=1}^{n} \sigma(b_{i-1}, b_i) \leq \sum_{i=1}^{n} \sigma(a_{i-1}, a_{i-1} \wedge a_i) = L_2(P).$$

Now $\omega(P) \to \omega(I)$ and $L_2(P) \longrightarrow L_2$, and so $\sigma(a, \omega(I)) \leq L_2$. Similarly, $\sigma(b, \omega(I)) \leq L_1$. $\square$

Note that in the degenerate case, where $T = 0$, we get $L_0 = 0$ and $\omega(I) = a = b$.

Now let $\mathcal{I}$ be the set of all closed subintervals of I. Given $J = [t, u] \in \mathcal{I}$, define $\mathcal{P}(J)$ intrinsically to J: in other words, it is the set of all finite subsets of J which contain $\{t, u\}$. We set $T(J) = u - t$. Note that if $J, K \in \mathcal{I}$ with $J \cap K \neq \varnothing$, then $J \cup K \in \mathcal{I}$. In this case, if $P \in \mathcal{P}(J)$ and $Q \in \mathcal{P}(J)$, then $P \cup Q \in \mathcal{P}(J \cup K)$.

Given $J \in \mathcal{I}$ and $P \in \mathcal{P}(J)$, let $L_j(P, J)$ be defined as above, intrinsically to J (i.e. as in the definition of $L_j(P)$ with J replacing I). As before, we see that $(L_j(J, P))_P$ converges on some $L_j(J) \leq L_0(J) \leq T(J)$, where $L_0(J)$ is the rectifiable length of $\alpha(J)$. Now $L_1(J, P) + L_2(J, P) = L_0(J, P)$ for all $P \in \mathcal{P}(J)$, and so $L_1(J) + L_2(J) = L_0(J)$, similarly as before.

We say that $J, K \in \mathcal{I}$ are ***adjacent*** intervals if $J \cap K$ is a singleton.

Lemma 3.4. *If $J, K \in \mathcal{I}$ are adjacent, then $L_j(J \cup K) = L_j(J) + L_j(K)$.*

Proof. Let $P \in \mathcal{P}(J)$ and $Q \in \mathcal{P}(K)$. Then $P \cup Q \in \mathcal{P}(J \cup K)$ and $L_j(J \cup K, P \cup Q) = L_j(J, P) + L_j(K, Q)$. Now pass to limits. $\square$

If $P \in \mathcal{P}(J)$, we can define $\omega(J, P)$ intrinsically to J. As before, $(\omega(J, P))_P$ converges on some $\omega(J) \in M$. If $J = [t, u]$, then $\omega(J) \leq \alpha(t) \wedge \alpha(u)$. Also, $\sigma(\alpha(t), \omega(J)) \leq L_2(J)$ and $\sigma(\alpha(u), \omega(J)) \leq L_1(J)$.

Lemma 3.5. *Let $J, K \in \mathcal{I}$ with $J \cap K \neq \varnothing$. Then $\omega(J \cup K) = \omega(J) \wedge \omega(K)$.*

Proof. Let $P \in \mathcal{P}(J)$ and $Q \in \mathcal{P}(K)$. Now $\omega(J \cup K, P \cup Q) = \bigwedge \alpha(P \cup Q) = \bigwedge(\alpha(P) \cup \alpha(Q)) = (\bigwedge \alpha(P)) \wedge (\bigwedge \alpha(Q)) = \omega(J, P) \wedge \omega(K, Q)$. Now pass to limits. $\square$

Note in particular, it follows that if $J \subseteq K$, the $\omega(J) \leq \omega(K)$. Note also that if $J = \{t\}$, then $\omega(J) = \alpha(t)$.

Given $t \in I$, let $\beta_1(t) = L_1([0, t])$, $\beta_2(t) = L_2([t, T])$, and $\beta(t) = (\beta_1(t), \beta_2(t)) \in [0, \infty)^2$. This defines a map $\beta : I \longrightarrow [0, \infty)^2$. Denoting the l^1 norm by $||.||$, we see that $||\beta(t)|| \leq L_1 + L_2 \leq L_0$. If $t \leq u$, then $\beta_1(t) \leq \beta_1(u)$. In fact, $\beta_1(u) - \beta_1(t) = L_1([t, u])$. Similarly, $\beta_2(t) \geq \beta_2(u)$ and $\beta_2(t) - \beta_2(u) = L_2([t, u])$. From this, we get:

Lemma 3.6. *$\beta : I \longrightarrow [0, \infty)^2$ is 1-lipschitz.*

Proof. Let $t \leq u$ in I. Then $||\beta(t) - \beta(u)|| = |\beta_1(t) - \beta_1(u)| + |\beta_2(t) - \beta_2(u)| = L_1([t,u]) + L_2([t,u]) = L_0([t,u]) \leq T([t,u]) = u - t$. □

Now let $\Omega = \{\underline{x} \in [0,\infty)^2 \mid (\exists v \in I)(\underline{x} \leq \beta(v))\}$. This is starlike about $\underline{0}$.

If $\underline{x} \in \Omega$, then $x_1 \leq \beta_1(v)$, so (by the Intermediate Value Theorem) there is some $t \in [0,v]$ with $\beta_1(t) = x_1$. Similarly, there is some $u \in [v,T]$ with $\beta_2(u) = x_2$. In other words, we have $\underline{x} = (\beta_1(t), \beta_2(u))$ with $t \leq u$. (Conversely, $(\beta_1(t), \beta_2(u)) \in \Omega$ for any $t, u \in I$ with $t \leq u$.)

Now given $\underline{x} \in D$, choose such $t \leq u$, and let $J(\underline{x}) = [t,u] \in \mathcal{I}$. We write $\phi(\underline{x}) = \omega(J(\underline{x}))$. This defines a map $\phi : \Omega \longrightarrow M$ which we will see does not depend on the choices of t, u we have made.

Lemma 3.7. *ϕ is well defined and 1-lipschitz.*

Proof. In fact, the same argument effectively shows both. To this end, suppose that we have $\underline{x}, \underline{x}' \in \Omega$, with $x_1 = x_1'$ and $x_2 \leq x_2'$. Suppose we have chosen $t \leq u, u'$ with $\beta_1(t) = x_1$, $\beta_2(u) = x_2$ and $\beta_2(u') = x_2'$. We can suppose without loss of generality that $u \leq u'$. (This is necessarily the case if $x_2 < x_2'$.) Let $J = [t,u]$ and $K = [u,u']$. Thus, J, K are adjacent intervals and $J \cup K = [t,u']$. By Lemma 3.5, $\omega(J \cup K) = \omega(J) \wedge \omega(K)$. Now $\omega(J), \omega(K) \leq \alpha(u)$, and so by Lemma 2.1, we have $\sigma(\omega(J), \omega(J \cup K)) \leq \sigma(\alpha(u), \omega(K))$. By Lemma 3.2 (applied intrinsically to the interval K), we have $\sigma(\alpha(u), \omega(K)) \leq L_2(K)$. But $L_2(K) = L_2([u,u']) = \beta_2(u) - \beta_2(u') = x_2 - x_2'$. So $\sigma(\omega(J), \omega(J \cup K)) \leq x_2 - x_2'$.

Now if $\underline{x} = \underline{x}'$, we see that $\omega(J) = \omega(J \cup K)$. In other words, this is independent of the choice of u: we get the same answer whether we set $J(\underline{x}) = [t,u]$ or $J(\underline{x}) = [t,u']$.

The same argument applies to the first variable, x_1. This shows that $\phi(\underline{x})$ is well defined.

Moreover, we see that $\sigma(\phi(x_1,x_2), \phi(x_1',x_2')) \leq |x_1 - x_1'| + |x_2 - x_2'|$. In other words, ϕ is 1-lipschitz as claimed. □

If $v \in I$, then $\beta(v) = (\beta_1(v), \beta_2(v))$, so we can take $t = u = v$ and $J(\beta(v)) = \{v\}$ in the definition of $\phi(\beta(v))$. Thus $\phi(\beta(v)) = \omega(\{v\}) = \alpha(v)$. This shows that $\phi \circ \beta = \alpha$.

In summary, we have 1-lipschitz maps $\beta : I \longrightarrow \Omega$ and $\phi : \Omega \longrightarrow M$ with $\phi \circ \beta = \alpha$. Note that $\phi(\underline{0}) = \omega$.

Suppose that $\underline{x} \leq \underline{y} \in \Omega$. We can choose $J(\underline{x}) \supseteq J(\underline{y})$, so by Lemma 3.5, we get $\phi(\underline{x}) \leq \phi(\underline{y})$. It follows that if $\underline{x}, \underline{y} \in \Omega$, then $\phi(\underline{x} \wedge \underline{y}) = \phi(\underline{x}) \wedge \phi(\underline{y})$. Moreover, if there is some $\underline{z} \in \Omega$ with $\underline{x}, \underline{y} \leq \underline{z}$, we can take $J(\underline{x}) \subseteq J(\underline{x}) \cap J(\underline{y})$. Thus, by Lemma 3.5, we get $\phi(\underline{x} \wedge \underline{y}) = \phi(\underline{x}) \wedge \phi(\underline{y})$.

We have achieved the objective mentioned at the beginning of this section. We mention a couple of consequences.

We define a map $\xi : [0,T] \times [0,1] \longrightarrow \Omega$ by $\xi(t,\tau) = \tau\beta(t)$. We write $\xi_t(\tau) = \xi(t,\tau)$. Thus, $\xi_t : [0,1] \longrightarrow \Omega$ is a euclidean geodesic, hence also an l^1 geodesic in

Ω from $\underline{0}$ to $\beta(t)$. Let $\gamma_t = \phi \circ \xi_t : [0,1] \longrightarrow M$. This is a geodesic in M from ω to $\alpha(t)$.

Suppose that α is a closed path in M based at $a = b$. Set $p = a = b$. Thus $\omega(I) \leq a \wedge b$, so also $\omega(I) = p$. In this case, γ_0 and γ_T are both constant paths at p. From this we get:

Lemma 3.8. *Any closed rectifiable curve, α, bounds a singular disc in M (a continuous map of the unit disc) of diameter equal to* $\operatorname{diam}(\alpha)$.

(It is not hard to give use this to prove a similar statement for non-rectifiable curves, in particular, showing that M is simply connected.)

We note the following consequence, to be used in Sections 7 and 8.

Lemma 3.9. *Let $x, y \in M$, and let α_1, α_2 be rectifiable paths from x to y. Then α_1 and α_2 are homotopic relative to x, y through rectifiable paths all of length at most* $\max(L(\alpha_1), L(\alpha_2))$.

Proof. Let $r = \max(L(\alpha_1), L(\alpha_2))$. We can parameterise α_1 and the reverse path, $-\alpha_2$, as 1-lipschitz paths $\alpha_1 : [0, r] \longrightarrow M$ and $-\alpha_2 : [r, 2r] \longrightarrow M$. These concatenate to give a closed path, $\alpha := \alpha_1 \cup (-\alpha_2) : [0, 2r] \longrightarrow M$, based at x. Let γ_t be the geodesic from x to $\alpha(t)$ described above. Given $t \in [0, r]$, let δ_t be the concatenation of γ_t with $\alpha|[t, r]$. This connects x to y in M. Also, $L(\delta_t) \leq r$, and $\delta_0 = \alpha_1$ and $\delta_r = \gamma_r$. This therefore gives a homotopy from α_1 to γ_r relative to x, y through paths of length at most r. We similarly get a homotopy from γ_r to α_2. $\square$

4. Modular and locally median spaces

The aim of this section is to show that a connected modular locally median space is median: Proposition 4.2. We won't actually use this result in this form. What we really need is an approximate version which is bit more involved to state and prove: see Proposition 5.6. For expository reasons, we give the simpler version first, which contains most of the essential ingredients.

First we make a few general observations.

Recall that a "square" in a metric space, (M, σ), is a (cyclically ordered) quadruple of points, $a, b, c, d \in M$, satisfying $a.b.c$, $b.c.d$, $c.d.a$ and $d.a.b$. We have observed that this implies $\sigma(a, b) = \sigma(c, d)$ and $\sigma(b, c) = \sigma(d, a)$.

In a median metric space, there is at most one way of "completing a square". In other words, if a, b, c, d and a, b, c, e are both squares, then $d = e$. This follows since $bde = bd(ace) = (bda)(bdc)e = ace = e$, and similarly $bde = d$.

What we really need is a variation on this statement. To state it, consider the following hypothesis applied to a metric space, M, for any $r \geq 0$:

$\Pi_0(r)$: Suppose $x, y, z \in M$ and $m, m' \in \operatorname{Med}(x, y, z)$, with $\sigma(m, m') \leq r$. Then $m = m'$.

(Thus a modular space is median if and only if $\Pi_0(r)$ holds for all $r \geq 0$.)

Lemma 4.1. *Suppose M is modular and satisfies $\Pi_0(2t)$ for some $t \geq 0$. Let $a, b, c, d, e \in M$ with $\sigma(a, b) = \sigma(b, c) = \sigma(c, d) = \sigma(d, a) = \sigma(c, e) = \sigma(e, a) = t$, and $\sigma(a, c) = \sigma(b, d) = \sigma(b, e) = 2t$. Then $d = e$.*

Proof. Let $u = \sigma(d, e)/2$. We want to show that $u = 0$.

Let $p \in \mathrm{Med}(a, d, e)$ and $q \in \mathrm{Med}(c, d, e)$. Then $\sigma(p, d) = \sigma(p, e) = \sigma(q, d) = \sigma(q, e) = u$, and $\sigma(a, p) = \sigma(c, q) = t - u$. Now $2u \geq \sigma(p, q) \geq \sigma(a, c) - \sigma(a, p) - \sigma(c, q) = 2t - 2(t - u) = 2u$, and so $\sigma(p, q) = 2u$. Also, $2t - u = \sigma(b, a) + \sigma(a, p) \geq \sigma(b, p) \geq \sigma(b, d) - \sigma(p, d) = 2t - u$, so $\sigma(b, p) = 2t - u$. Similarly, $\sigma(b, q) = 2t - u$, so $p, q \in \mathrm{Med}(b, d, e)$. Since $\sigma(p, q) = 2u \leq 2t$, we get $p = q$, so $u = 0$. $\square$

The following definition will be useful in our discussion. We say that a quintuple, (y_1, y_2, y_3, m, m'), is a ***bad quintuple*** of ***size*** $r > 0$ if $\sigma(y_i, m) = \sigma(y_i, m') = r$ for all i, and $\sigma(y_i, y_j) = \sigma(m, m') = 2r$ for all $i \neq j$.

Such quintuples arise in the following way when the uniqueness of the median fails. Suppose $x_1, x_2, x_3 \in M$ and $m, m' \in \mathrm{Med}(x_1, x_2, x_3)$ are distinct. Let $r = \sigma(m, m')/2$. Choose $y_i \in \mathrm{Med}(x_i, m, m')$. If $i \neq j$, $x_i.m.x_j$, so by interpolation we have $x_i.y_i.m.y_j.x_j$. In particular, $y_i.m.y_j$. Similarly, $y_i.m'.y_j$, so $m, m' \in \mathrm{Med}(y_1, y_2, y_3)$. We also have $m.y_i.m'$ for all i. Thus m, y_i, m', y_j is a square for all $i \neq j$. Thus $\sigma(m, y_i) = \sigma(m', y_j)$. It follows that $\sigma(m, y_i) = r$ for all i. In other words, (y_1, y_2, y_3, m, m') is a bad quintuple of size r.

Proposition 4.2. *Let (M, σ) be a connected modular metric space which is uniformly locally median. Then M is median.*

Proof. Let M be ϵ-locally median for $\epsilon > 0$. First recall that by Lemma 2.5, any two points of M have a midpoint. (This is all we need of connectedness here.)

Let $x_1, x_2, x_3 \in M$ and let $m, m' \in \mathrm{Med}(x_1, x_2, x_3)$. Let $t = \sigma(m, m')/4$, and choose $y_i \in \mathrm{Med}(x_i, m, m')$. From the discussion above, we see that if $m \neq m'$, then (y_1, y_2, y_3, m, m') is bad quintuple of size $2t$.

Note that $\mathrm{diam}\{y_1, y_2, y_3\} = 4t = \sigma(m, m')$. Since M is ϵ-locally median, we see that if $\sigma(m, m') \leq \eta$, then $m = m'$. It follows that $\Pi_0(\epsilon)$ holds.

Now let $r_n = 2^n \epsilon$. We claim inductively that $\Pi_0(r_n)$ holds for all $n \in \mathbb{N}$. We have already verified this for $n = 0$, so we suppose it holds for a given n. Let $m, m' \in \mathrm{Med}(x_1, x_2, x_3)$ with $\sigma(m, m') \leq r_{n+1} = 2r_n$. Let y_1, y_2, y_3 and t be as above. Note that $t \leq r_n/2$.

Let z, z' be midpoints of y_3, m and y_3, m' respectively. Thus $\sigma(z, y_3) = \sigma(z', y_3) = \sigma(z, m) = \sigma(z', m') = t$, and $\sigma(z, z') = 2t$. Let $w_1 \in \mathrm{Med}(y_1, z, z')$ and $w_2 \in \mathrm{Med}(y_2, z, z')$. Now $\sigma(z, y_1) = 3t$, so $\sigma(z, w_1) = t$ and $\sigma(y_1, w_1) = 2t$. Similarly we get $\sigma(z', w_1) = \sigma(z, w_2) = \sigma(z', w_2) = t$ and $\sigma(y, w_2) = 2t$. Now $\sigma(w_1, w_2) \leq 2t \leq r_n$. So applying Lemma 4.1 (with $(a, b, c, d, e) = (z, y_3, z', w_1, w_2)$), we get $w_1 = w_2$. Set $w = w_1 = w_2$.

Now $\sigma(w,y_1) = \sigma(w,y_2) = \sigma(w,y_3)$, so $w \in \mathrm{Med}(y_1,y_2,y_3)$. But $\sigma(w,m) \leq \sigma(w,z) + \sigma(z,m) \leq 2t \leq r_n$. Therefore, by the inductive hypothesis, we get $w = m$.

Similarly, $w = m'$, so $m = m'$. We have therefore verified $\Pi_0(r_{n+1})$.

It follows that $\Pi_0(r_n)$ holds for all n, and so M is median. □

In fact, we can obtain a stronger result from this argument, as we note in Section 9.

5. Approximate medians

In this section we rework the results of Section 4, with weaker hypotheses and conclusions. These essentially require that distances are measured up to an arbitrarily small additive error. We first give some definitions and conventions, which will be used again later.

Let $\eta > 0$. Given $t, u \in \mathbb{R}$, write $t \sim_\eta u$ to mean that $|t - u| \leq \eta$, and write $t \leq_\eta u$ to mean that $t \leq u + \eta$. Clearly $t \sim_\eta u \sim_\eta v \Rightarrow t \sim_{2\eta} v$, and $t \leq_\eta u \leq_\eta v \Rightarrow t \leq_{2\eta} v$.

Let (M, σ) be a metric space. Given $a, b, x \in M$, write $a.x.b_{\sim\eta}$ to mean $\sigma(a,x) + \sigma(b,x) \leq_\eta \sigma(a,b)$. This is equivalent to saying that $\langle a,b \rangle_x \leq \eta/2$. The following generalisation of Lemma 2.1 is easily verified:

Lemma 5.1. *Suppose $a, b, c, d \in M$ with $a.b.c_{\sim\eta}$ and $b.c.d_{\sim\eta}$. Then $\sigma(b,c) \leq_\eta \sigma(a,d)$.*

Write $I_\eta(a,b) = \{x \in M \mid a.x.b_{\sim\eta}\}$, and $\mathrm{Med}_\eta(a,b,c) = I_\eta(a,b) \cap I_\eta(b,c) \cap I_\eta(a,b)$. We refer to an element, m, of $\mathrm{Med}_\eta(a,b,c)$ as an η-***median*** of a,b,c. Note that $\sigma(c,m) \sim_\eta \langle a,b \rangle_c$.

Definition. We say that M is ***almost modular*** if $\mathrm{Med}_\eta(a,b,c) \neq \varnothing$ for all $a,b,c \in M$ and all $\eta > 0$.

Definition. We say that M is ***almost median*** if it is almost modular, and for all $a,b,c \in M$ and all $\theta > 0$, there is some $\eta > 0$ such that if $m, m' \in \mathrm{Med}_\eta(a,b,c)$, then $\sigma(m,m') \leq \theta$.

Back to a general metric space, M, we say that a path, α, joining $a, b \in M$ is η-***taut*** if $L(\alpha) \leq_\eta \sigma(a,b)$. (Thus, a 0-taut path is geodesic.) Recall that M is ***path-metric*** space if every two points are connected by an η-taut path for all $\eta > 0$.

Similarly, we say that a sequence, $\underline{a} = a_0, a_1, \ldots, a_n$, is η-***taut*** if $l(\underline{a}) \leq_\eta \sigma(a_0, a_n)$. In this case, we write $a_0.a_1.\cdots.a_{n\sim\eta}$. (This is consistent with the earlier notation when $n = 2$.) We again have a rule of "interpolation". For example if $a.b.d_{\sim\eta}$ and $b.c.d_{\sim\eta}$ hold, then so does $a.b.c.d_{\sim 2\eta}$.

We say that c is an η-***midpoint*** of $a, b \in M$ if $\sigma(a,c) \sim_\eta \sigma(a,b)/2$ and $\sigma(b,c) \sim_\eta \sigma(a,b)/2$. If M is a path-metric space, then η-midpoints always exist for all $\eta > 0$.

We now make a few observations.

First, any median metric space is almost median. In fact:

Lemma 5.2. *Let M be a median metric space. Let $x, y, z \in M$ and $m, m' \in \mathrm{Med}_\eta(x, y, z)$ for some $\eta > 0$. The $\sigma(m, m') \leq 4\eta$.*

Proof. Let $d = xyz$. It is enough to show that $\sigma(d, m) \leq 2\eta$. To this end, set $e = xym$. Now $\sigma(m, e) = \langle x, y\rangle_m \leq \eta/2$. Also, $\sigma(z, m) \sim_\eta \langle x, y\rangle_z$ and $\sigma(z, d) = \langle x, y\rangle_z$. Thus $\sigma(z, d) \sim_\eta \sigma(z, m)$, so $\sigma(z, d) \sim_{3\eta/2} \sigma(z, e)$. Now $zde = zd(xye) = (zdx)(zdy)e = dde = d$. In other words, $z.d.e$ holds, that is $\sigma(z, e) = \sigma(z, d) + \sigma(d, e)$. Thus, $\sigma(d, e) \leq 3\eta/2$, so $\sigma(d, m) \leq \sigma(d, e)+\sigma(e, m) \leq (3\eta/2)+(\eta/2) = 2\eta$. Similarly, $\sigma(d, m') \leq 2\eta$, so $\sigma(m, m') \leq 4\eta$. □

We note that this argument also works in an η-locally median space. If we assume that $\eta \leq \epsilon$ and that $\mathrm{diam}(x, y, z) \leq \epsilon$, then (after shrinking ϵ by some fixed factor) we can assume that the relevant median identity holds, and we again deduce that $\sigma(m, m') \leq 4\eta$. (Of course, this can also be verified by a direct argument from first principles, but it is a little involved.)

We also have the following converse.

Lemma 5.3. *A complete almost median metric space is median.*

Proof. Let M be complete and almost median. Let $x, y, z \in M$. Let $(\theta_i)_{i\in\mathcal{N}}$ be any sequence of positive numbers tending to 0. Given i, let η_i be the constant η given by the almost median hypothesis for $\theta = \theta_i$. Let $m_i \in \mathrm{Med}_{\eta_i}(x, y, z)$. Now $\sigma(m_i, m_j) \leq \max(\theta_i, \theta_j)$, and so $(m_i)_i$ is Cauchy. Let m be its limit in M. We see that $m \in \mathrm{Med}(x, y, z)$.

Now suppose $m' \in \mathrm{Med}(x, y, z)$. From the almost median hypothesis again, we see that $\sigma(m, m') \leq \theta$ for all $\theta > 0$. Therefore $m = m'$.

We have shown that $\mathrm{Med}(x, y, z) = \{m\}$, so M is median. □

Next, we consider the following hypothesis on a metric space, M, for constants $H, r \geq 0$.

$\Pi_H(r)$: Suppose $x, y, z \in M$ and $m, m' \in \mathrm{Med}_\eta(x, y, z)$ for some $\eta > 0$, and with $\sigma(m, m') \leq r$. Then $\sigma(m, m') \leq H\eta$.

(This accords with the property defined in Section 4 when $H = 0$.)

Note that in this assertion we only need to consider those $\eta > 0$ satsifying $\eta < r/H$ (otherwise, certainly $\sigma(m, m') \leq r \leq H\eta$).

We have the following variation on Lemma 4.1.

Lemma 5.4. *Given $H \geq 0$, there is some $K = K(H) \geq 0$ with the following property. Suppose M is almost modular and satisfies $\Pi_H(3t)$ for some $t \geq 0$. Suppose $a, b, c, d, e \in M$ with $\sigma(a, b) \sim_\eta t$, $\sigma(b, c) \sim_\eta t$, $\sigma(c, d) \sim_\eta t$, $\sigma(d, a) \sim_\eta t$, $\sigma(c, e) \sim_\eta t$, $\sigma(e, a) \sim_\eta t$, $\sigma(a, c) \sim_\eta 2t$, $\sigma(b, d) \sim_\eta 2t$ and $\sigma(b, e) \sim_\eta 2t$. Then $\sigma(d, e) \leq K\eta$.*

In other words, a, b, c, d, e are assumed to satisfy the same conditions as in Lemma 4.1, except up to an additive error of at most η.

Proof. First note that we can always assume that $\eta < 2t/(K-2)$ (Otherwise, we would certainly have $\sigma(d,e) \leq 24+2\eta \leq 2t+\eta \leq 2(K-2)\eta+2\eta = K\eta$ as required.) This means that if we take K at least some fixed constant, K_0 (independently of H), then all our constructions will keep us within a set of diameter at most $3t$.

We now follow the proof of Lemma 4.1. We set $u = \sigma(d,e)/2$ as before. All the inequalities hold up to an additive error bounded by some multiple of η. In particular there are fixed constants, $h, k \geq 0$, such that $p, q \in \mathrm{Med}_{k\eta}(b, d, e)$ and $\sigma(p,q) \sim_{h\eta} 2u$. Now, by the observation of the previous paragraph, we can arrange that $\sigma(p,q) \leq 3t$, and so applying $\Pi_H(3t)$, we get $\sigma(p,q) \leq Hk\eta$. Thus $\sigma(d,e) = 2u \leq \sigma(p,q) + h\eta \leq (Hk+h)\eta$. We set $K = \max(K_0, Hk+h)$. □

Of course, it would not be hard to explicitly derive a formula for $K(H)$.

Lemma 5.5. *Let M be an almost modular path-metric space which ϵ-locally median for some $\epsilon > 0$. Then for any $r \geq 0$, there is some $H \geq 0$ such that $\Pi_H(r)$ holds.*

Proof. Given $n \in \mathbb{N}$, let $r_n = (\frac{4}{3})^n \epsilon/2$. We claim that there are constants $H_n \geq 0$ such that M satisfies $\Pi_{H_n}(r_n)$ for all n. In other words, if $x_1, x_2, x_3 \in M$ and $m, m' \in \mathrm{Med}_\eta(x_1, x_2, x_3)$ with $\sigma(m,m') \leq r_n$, then $\sigma(m,m') \leq H_n\eta$. As noted earlier, we can always assume that $\eta < r_n/H_n$. This means that provided we take H_n large enough we can all distances involved in our construction are appropriately bounded.

We first reduce to sets of controlled diameter, similarly as with the proof of Proposition 4.2. Let $t = \sigma(m,m')/4$. We take $y_i \in \mathrm{Med}_\eta(x_i, m, m')$. This time we get $\sigma(y_i, m) \sim_{k\eta} t$, $\sigma(y_i, m') \sim_{k\eta} t$ and $\sigma(y_i, y_j) \sim_{k\eta} 2t$ for $i \neq j$, and $m, m' \in \mathrm{Med}_{k\eta}(y_1, y_2, y_3)$, where k is some fixed constant. In other words, (y_1, y_2, y_3, m, m') is a bad quintuple of size $2t$, up to an additive error of at most $k\eta$.

We next describe the inductive argument. Suppose that we have found H_n so that $\Pi_{H_n}(r_n)$ holds. We now assume that $\sigma(m,m') \leq r_{n+1} = 4r_n/3$. Thus, $3t \leq r_n$, so $\Pi_{H_n}(3t)$ holds. Let $K_n = K(H_n)$ be the constant given by Lemma 5.4.

We now follow the proof of Proposition 4.2, allowing for additive errors. Specifically, we let z, z' be η-midpoints of y_3, m and y_3, m' respectively. Let $w_1 \in \mathrm{Med}_\eta(y_1, z, z')$ and $w_2 \in \mathrm{Med}_\eta(y_2, z, z')$. We now get the same relations between distances, up to tome additive constant $k'\eta$, where k' is some fixed positive constant. Now M satisfies $\Pi_{H_n}(3t)$ and so by Lemma 5.3, we get $\sigma(w_1, w_2) \leq K_n k'\eta$. Set $w = w_1$. Continuing the argument, we get the same relations up to some additive constant $K'_n\eta$ (where K'_n depends only K_n, hence only on H_n). In particular, $\sigma(w,m) \leq 2t + K'_n\eta$. Now, we can assume that $\eta \leq r_{n+1}/H_{n+1}$, and so if H_{n+1} is at least some constant $H'_n := 1/4K'_n$, we get $\sigma(w,m) \leq 2t + \frac{1}{4}r_{n+1} = 2t + \frac{1}{3}r_n \leq r_n$. By $\Pi_{H_n}(r_n)$ again, we get $\sigma(w,m) \leq K_n K'_n\eta$. Similarly, $\sigma(w,m) \leq K_n K'_n\eta$ and so $\sigma(m,m') \leq H_{n+1}\eta$, on setting $H_{n+1} = \max(H'_n, 2K_nK'_n)$. We have therefore verified $\Pi_{H_{n+1}}(r_{n+1})$.

Finally, for the case $n = 0$, we use the fact that M is ϵ-locally median. We can assume that $\mathrm{diam}\{y_1, y_2, y_3\} \leq \epsilon$. Since $m, m' \in \mathrm{Med}_{k\eta}(y_1, y_2, y_3)$, it follows by Lemma 5.2 and the subsequent discussion, that $\sigma(m, m') \leq 4k\eta$.

It follows by induction that $\Pi_{H_n}(r_n)$ holds for all n. □

Proposition 5.6. *Let M be an almost modular path-metric space which is ϵ-locally median for some $\epsilon > 0$. Then M is almost median.*

Proof. Given $x, y, z \in M$ and $\theta > 0$, let $r = \mathrm{diam}\{x, y, z\}$ and let H be the constant given by Lemma 5.5. Set $\eta = \theta/H$. □

Thus, if we assume it addition, that M is complete, then it follows by Lemma 5.3 that M is median.

6. Retracting sequences

We give a construction, related to that of Section 3, to pull back a sequence of points towards a given basepoint. The main result we are aiming at is Lemma 6.2, though we first give the argument under a slightly stronger hypotheses, where the situation is more transparent.

Let M be a metric space with basepoint, $p \in M$, and let $r > 0$. Consider the following hypotheses on M:

$\nabla_0(p)$: For all $x, y \in M$, $\mathrm{Med}(p, x, y) \neq \varnothing$.

$\nabla_0(p, r)$: if $x, y \in M$ and $\sigma(x, y) \leq r$, then $\mathrm{Med}(p, x, y) \neq \varnothing$.

Lemma 6.1. *If M is a geodesic metric space satisfying $\nabla_0(p, r)$ for some $p \in M$ and some $r > 0$, then it satisfies $\nabla_0(p)$.*

Proof. By iteration, it is enough to show that $\nabla_0(p, r)$ implies $\nabla_0(p, 3r/2)$.

To this end, let $a_1, a_2 \in M$ with $\sigma(a_1, a_2) \leq 3r/2$. Since M is geodesic, we can find $a'_1, a'_2 \in M$ with $\sigma(a_i, a'_i) \leq r/2$, $\sigma(a'_1, a'_2) \leq r/2$ and with $a_1.a'_1.a'_2.a_2$. Choose some $b_i \in \mathrm{Med}(p, a_i, a'_i)$ and $b \in \mathrm{Med}(p, a'_1, a'_2)$. Thus, $a_1.b_1.a'_1.b.a'_2.b_2.a_2$. It follows that $\rho(b, b_i) \leq r$, and so we can choose $c_i \in \mathrm{Med}(b, b_i)$. Thus, $a_1.b_1.c_1.b.c_2.b_2.a_2$ and $a'_i.b.c_i.p$. Now $a'_i.b.c_i$ and $b.c_i.b_i$, so by Lemma 2.2, we have $\sigma(b, c_i) \leq \sigma(a'_i, b_i) \leq \sigma(a'_i, a_i) \leq r/2$, and so $\sigma(c_1, c_2) \leq r$. Thus, there is some $d \in \mathrm{Med}(p, c_1, c_2)$. Now $a_i.b_i.c_i.d.p$ amd $a_1.b_1.c_1.d.c_2.b_2.a_2$. In particular, $a_i.d.p$ and $a_1.d.a_2$, so $d \in \mathrm{Med}(p, a_1, a_2)$ as required. □

(In fact, one can show directly that $\nabla_0(p, r)$ implies $\nabla_0(p)$, by cutting a geodesic from a_1 to a_2 into sufficiently small segments, and applying a similar argument with a greater number of steps.)

What we really need is a similar result for approximate medians. Consider the following two hypotheses:

$\nabla(p)$: For all $x, y \in M$ and $\eta > 0$, $\mathrm{Med}_\eta(p, x, y) \neq \varnothing$.

$\nabla(p, r)$: If $x, y \in M$ and $\sigma(x, y) \leq r$, the $\mathrm{Med}_\eta(p, x, y) \neq \varnothing$ for all $\eta > 0$.

Lemma 6.2. *If M is a path-metric space satisfying $\nabla(p, r)$ for some $r > 0$, then it satisfies $\nabla(p)$.*

Proof. We follow the same argument as Lemma 6.1, allowing for arbitrarily small additive constants.

It is enough to show that $\nabla(p, r)$ implies $\nabla(p, 5r/4)$. To this end, suppose $\eta > 0$ and let $\zeta = \frac{1}{6}\min(\eta, r)$. Given $a_1, a_2 \in M$ with $\sigma(a_1, a_2) \leq 5r/4$, we can find $a_1', a_2' \in M$ with $\sigma(a_i, a_i') \leq (r - \zeta)/2$, $\sigma(a_1', a_2') \leq (r - \zeta)/2$ and with $\sigma(a_1, a_2) \leq \sigma(a_1, a_1') + \sigma(a_1', a_2') + \sigma(a_2', a_2) + \zeta$. We can proceed to choose b_i, c_i, d as before, except with Med_ζ replacing Med. (Note that using Lemma 5.1, we get $\sigma(b, c_i) \leq \sigma(a_i', b_i) + \zeta \leq \sigma(a_i', a_i) + 2\zeta \leq r/2$, so $\sigma(c_1, c_2) \leq r$.) This time we get $d \in \mathrm{Med}_{6\zeta}(p, a_1, a_2) \subseteq \mathrm{Med}_\eta(p, a_1, a_2)$ as required. □

We finish this section with the following observation:

Lemma 6.3. *Suppose that (M, σ) is a path-metric space satisfying $\nabla(p)$. Then for any $R > 0$, the metric σ restricted to $N(x, R)$ is an intrinsic path-metric on $N(x, h)$.*

Proof. Let $x_1, x_2 \in N(p, R)$. Choose $\eta > 0$ with $\eta < (R - \max(\sigma(p, x), \sigma(p, y)))/2$. Choose any $y \in \mathrm{Med}_\eta(p, x_1, x_2)$. Note that $\sigma(p, y) + \sigma(y, x_i) \leq \sigma(p, x_i) + \eta < R - \eta$. Let α_i be a path in M from y to x_i with $L(\alpha_i) < \sigma(y, x_i) + \eta$. Then $\sigma(p, y) + L(\alpha_i) < R$, so α is contained in $N(p, r)$. Now $\alpha_1 \cup \alpha_2$ connects x_1 to x_2 in $N(p, r)$, and $L(\alpha_1 \cup \alpha_2) \leq \sigma(y, x_1) + \sigma(y, x_2) + 2\eta \leq \sigma(x_1, x_2) + 2\eta$. Since $\eta > 0$ is arbitrarily small, the statement follows. □

7. Path lifting

Suppose (X, ρ) and (M, σ) are path-metric spaces. Let $f : X \longrightarrow M$ be a local isometry. In other words, for all $x \in X$, there is some $r > 0$ such that $f|N(x, r)$ is an isometry from $N(x, r)$ to $N(f(x), r)$. We write g_x for the local inverse isometry.

It is easily checked that if α is a rectifiable path in X, then $f\alpha$ is rectifiable in M, and that $L(f\alpha) = L(\alpha)$. It follows that f is 1-lipschitz.

Moreover, since f is locally injective, if α, α' are two paths in M based at the same point and with $f\alpha = f\alpha'$, then $\alpha = \alpha'$. We refer to such a path, if it exists, as the ***lift*** of β to X. We write $\alpha = \tilde{\beta}$.

Let $p \in X$ be some basepoint. Given $x \in X$, write $h(x) = \rho(p, x)$. We suppose that there is some $R > 0$ such that the following two conditions hold.

$\Phi(R)$: $h(x) < R$ for all $x \in X$, and

$\Lambda(R)$: if β is a path in M based at $f(p)$ and with $L(\beta) < R$, then there is a (unique) path α in X based at p with $f\alpha = \beta$.

Lemma 7.1. *Let $x \in X$, and suppose β is a path in M based at $f(x)$ and with $L(\beta) < R - h(x)$. Then there is a unique path α in X based at x with $f\alpha = \beta$.*

Proof. Let $\eta = R - h(x) - L(\beta) > 0$. Since ρ is a path-metric, there is some path γ in X from p to x with $L(\gamma) < h(x) + \eta$. Let $\delta = f\gamma$. Then $L(\delta) = L(\gamma)$. Now $\delta \cup \beta$ is a path in M based at $f(p)$ with $L(\delta \cup \beta) < R$. By $\Lambda(R)$, this lifts to a path in M based at p. By uniqueness of lifts, this has the form $\gamma \cup \alpha$, where α is a path based at x, and with $f\alpha = \beta$ as required. $\square$

The following is standard "homotopy lifting" proceedure.

Suppose $x \in X$ and let $I = [0, L] \subseteq \mathbb{R}$ be a real interval. Suppose we have a continuous family of paths $\beta_\theta : I \longrightarrow M$, all based at $f(x)$, parameterised by $\theta \in [0,1]$. In other words, the map $[(t,\theta) \mapsto \beta_\theta(t)] : I \times [0,1] \longrightarrow M$ is continuous, and $\beta_\theta(0) = f(x)$ for all θ. Suppose also that $L(\beta_\theta) < R - h(x)$ for all θ. By Lemma 7.1, β_θ lifts to a unique path $\alpha_\theta : I \longrightarrow X$ in X based at x. Now the map $[(t,\theta) \mapsto \alpha_\theta(t)] : I \times [0,1] \longrightarrow X$ is also continuous. (Locally, it has the form $g_y \circ [(t,\theta) \mapsto \beta_\theta(t)]$, where g_y is a local isometry from M to X near some point $y \in M$.) Suppose also that each β_θ has the same terminal point, $b := \beta_\theta(L)$. Then each α_θ has the same terminal point (since the map $[\theta \mapsto \alpha_\theta(L)]$ is locally constant, hence constant). We deduce:

Lemma 7.2. *Suppose $x \in X$. Suppose that β_0 and β_1 are paths in M from $f(x)$ to the same point $b \in M$, with $L(\beta_0), L(\beta_1) < R - h(x)$. Let α_0, α_1 be respectively the lifts of β_0, β_1 to X based at x. Suppose that β_0, β_1 are homotopic in M relative to their endpoints $f(x), b$, through paths all of length less than $R - h(x)$. Then α_0, α_1 have the same terminal point, y, in X, with $f(y) = b$.*

We now make some additional assumptions, namely that X satisfies Property $\nabla(p)$ defined in Section 6, and that M is ϵ-locally median for some $\epsilon > 0$. Although not strictly necessary, in view of Lemma 2.8, there is no loss in assuming that any two points of M a distance at most ϵ apart are connected by a geodesic.

Lemma 7.3. *Suppose $x_1, x_2 \in X$ with $\rho(x_1, x_2) < \epsilon/2$. Then $\rho(x_1, x_2) = \sigma(fx_1, fx_2)$.*

Proof. Let $\eta = \min(R - h(x_1), R - h(x_2), \epsilon/2) > 0$. By $\nabla(p)$, there is some $y \in \mathrm{Med}_\eta(p, x_1, x_2)$. Let $r = \max(\rho(y, x_1), \rho(y, x_2))$. Thus $r \le \rho(x_1, x_2) + \eta \le 2(\epsilon/2) = \epsilon$. Also $h(y) + r < R$ (since for some i, $h(y) + r = h(y) + \rho(y, x_i) \le h(x_i) + \eta < R$). Let α_i be a path in X from y to x_i, with $L(\alpha_i) < \rho(y, x_i) + \eta \le r + \eta$.

Let $a_i = fx_i$ and $b = fy$. Since f is 1-lipschitz, we have $\mathrm{diam}\{a_1, a_2, b\} < \epsilon$ and $L(f\alpha_i) < r$. Since M is ϵ-locally median, there is unique median, $c \in \mathrm{Med}(a_1, a_2, b)$. Let $\beta, \gamma_1, \gamma_2$ be geodesics in M from b to c, c to a_1 and c to a_2 respectively. Thus, $L(\beta \cup \gamma_i) = \sigma(b, a_i) < r$, and $L(\gamma_1 \cup \gamma_2) = \sigma(a, b)$.

Since $r < R - h(y)$, Lemma 7.1 tells us $\beta \cup \gamma_i$ lifts to a path $\tilde{\beta} \cup \tilde{\gamma}_1$ in X based at y. Let z be the terminal point of $\tilde{\beta}$ (so $fz = c$).

Now Proposition 3.9 (and the subsequent discussion) tells us that $f\alpha_i$ and $\beta \cup \gamma_i$ are homotopic relative to their endpoints, b, a_i, through paths of length at most $r < R - h(y)$ in M. Therefore, by Lemma 7.2, α_i and $\tilde{\beta} \cup \tilde{\gamma}_i$ have the same terminal points in X. In particular, $\tilde{\gamma}_i$ connects z to x_i in X. Thus, $\tilde{\gamma}_1 \cup \tilde{\gamma}_2$ connects x_1 to x_2 in X. Moreover, $L(\tilde{\gamma}_1 \cup \tilde{\gamma}_2) = L(\gamma_1 \cup \gamma_2) = \sigma(a, b)$. We see that $\rho(x, y) \leq \sigma(a, b)$.

Since f is 1-lipschitz, we also have $\sigma(a, b) \leq \rho(x, y)$, and so $\rho(x, y) = \sigma(a, b)$ as required. $\square$

Note that the argument also shows that x_1 and x_2 are connected by a geodesic (namely $\tilde{\gamma}_1 \cup \tilde{\gamma}_2$) in X.

8. Construction of Y

In this section, we aim to construct a map $f : Y \longrightarrow M$ of the type described in Section 1. It will be constructed as an increasing union of spaces, X_n for $n \in \mathbb{N}$. We will start from some such map, X, and explain how it can be extended to a larger space X'. The spaces, X_n, then arise as an iteration of this process.

So to begin we suppose we have a map, $f : X \longrightarrow M$, between path-metric spaces, (X, ρ) and (M, σ). We have a basepoint, $p \in X$, and write $h(x) = \rho(p, x)$ for $x \in X$. We assume that f is a local isometry. We suppose that for some $R > 0$, X satisfies the following.

$\nabla(p)$: $(\forall x, y \in X)(\forall \eta > 0) \operatorname{Med}_\eta(p, x, y) \neq \varnothing$,

$\Phi(R)$: $(\forall x \in X)(h(x) < R)$, and

$\Lambda(R)$: any path of length less than R based at $f(p)$ in M lifts to a path based at p in X.

We also assume that M is ϵ-locally median for some $\epsilon > 0$. As we have observed in Section 2, there is no loss in supposing that any two points of M distance at most ϵ apart are connected by a geodesic. (This will simplify the discussion a little.)

Let $\lambda = \epsilon/25$.

We aim to construct a path-metric space, (X', ρ'), and an isometric embedding $\iota : X \longrightarrow X'$, with $\iota X = N(\iota p, R)$, and a map $f' : X' \longrightarrow M$ with $f = f' \circ \iota$. Moreover, X' and f' will satisfy the same conditions as X and f, with ιp replacing p and $R' := R + \lambda$ replacing R.

We begin by writing $P = \{(x, a) \in X \times M \mid \sigma(a, fx) < \lambda\}$. Define a relation, $\sim$, on P by $(x, a) \sim (y, b)$ if $a = b$ and $\rho(x, y) < 2\lambda$.

Lemma 8.1. *$\sim$ is an equivalence relation on P.*

Proof. Suppose $(x,a) \sim (y,a) \sim (z,a)$. Then $\rho(x,z) \leq \rho(x,y) + \rho(y,z) < 2\lambda + 2\lambda = 4\lambda < \epsilon/2$. By Lemma 7.3, $\rho(x,z) = \sigma(fx,fz) \leq \sigma(a,fx) + \sigma(a,fz) < \lambda + \lambda = 2\lambda$, so $(x,a) \sim (z,a)$. □

Let $X' = P/\sim$. Write $[[x,a]] \in X'$ for the $\sim$-class of (x,a). Define a map $\iota : X \longrightarrow X'$ by $\iota(x) = [[x,fx]]$.

Lemma 8.2. *ι is injective.*

Proof. Suppose $\iota x = \iota y$. Then $(x,fx) \sim (y,fy)$ so $fx = fy$. Also $\rho(x,y) < 2\lambda < \epsilon/2$, so by Lemma 7.3, $\rho(x,y) = \sigma(fx,fy) = 0$, so $x = y$. □

Define $f' : X' \longrightarrow M$ by $f'([[x,a]]) = a$. Thus $f' \circ \iota = f$. Given $v \in X'$, let $\Xi(v) = \{x \in X \mid v = [[x, f'v]]\} \subseteq \{x \in X \mid \sigma(x, f'v) < \lambda\}$. Note that if $x, y \in \Xi(v)$, then $\rho(x,y) < 2\lambda$.

Definition. Let I be a compact real interval. We say that a map $\alpha : I \longrightarrow X'$ is an ***admissible path*** if $f' \circ \alpha : I \longrightarrow M$ is continuous, and for any $t \in I$, there is a neighbourhood, J, of t in I such that for all $u \in J$, there exist $x \in \Xi(\alpha(t))$ and $y \in \Xi(\alpha(u))$ with $\rho(x,y) < 3\lambda$.

In fact, we can assume this holds for all such x, y. For suppose $x' \in \Xi(\alpha(t))$ and $y' \in \Xi(\alpha(u))$. Since $f' \circ \alpha$ is continuous, we can choose such a J to ensure that $\sigma(f'(\alpha(t)), f'(\alpha(u))) < \lambda$. Now

$$\rho(x',y') \leq \rho(x',x) + \rho(x,y) + \rho(y,y') < 2\lambda + 3\lambda + 2\lambda = 7\lambda < \epsilon/2.$$

Thus, by Lemma 7.3, we have

$$\begin{aligned}\rho(x',y') &= \sigma(fx',fy') \\ &\leq \sigma(fx', f'(\alpha(t))) + \sigma(f'(\alpha(t)), f'(\alpha(u))) + \sigma(f'(\alpha(u)), fy') \\ &< \lambda + \lambda + \lambda = 3\lambda.\end{aligned}$$

Note that essentially the same argument shows that (we can assume that) if $w \in J$, and $z \in \Xi(w)$, then also $\rho(y,z) < 3\lambda$.

Note also that the concatenation of two admissible paths is an admissible path.

Given an admissible path α, we write $L'(\alpha) = L(f'\alpha) \in [0,\infty]$. If β is a (continuous) path in X, then $\iota\beta$ is an admissible path, and $f'\iota\beta = f\beta$, so $L'(\iota\beta) = L(\beta)$.

Let $v = [[x,a]] \in X'$. Let $b = fx$, so $\sigma(b,a) < \lambda$. Let $U = \{w \in X' \mid x \in \Xi(w)\} \subseteq X'$ and $V = N(b,\lambda) \subseteq M$. We see that $f'U \subseteq V$. We define a map $\phi : V \longrightarrow X'$ by $\phi(c) = [[x,c]]$. In other words, $f'\phi(c) = c$ and $\phi f(w) = [[x, f'w]] = w$. Thus $f'|U$ and ϕ are inverse maps. In particular $f'U = V$ and $\phi V = U$. Note that $f'\iota x = fx = b$, and $\phi b = \iota x$. Also $\phi f' v = v$.

If β is a continuous path in V, then $\phi\beta$ is an admissible path contained in U. By definition, $L'(\phi\beta) = L(\beta)$. In particular, we can take β to be a geodesic in M from b to $f'v$. Then $\phi\beta$ is an admissible path from ιx to v, with $L'(\phi\beta) = \sigma(b, f'v) < \lambda$.

In other words, if $v \in X'$ and $x \in \Xi(v)$, then there is an admissible path, α, from ιx to v in X' with $L'(\alpha) = \sigma(fx, f'v) < \lambda$.

Let $p' = \iota p \in X'$ and $R' = R + \lambda$.

Lemma 8.3. *Let $v \in X'$. Then there is an admissible path, γ, from p' to v in X' with $L'(\gamma) < R'$.*

Proof. Let $x \in \Xi(v)$. By $\Phi(R)$ in X, there is a path, δ, from p to x in X with $L(\delta) < R$. Thus $\iota\delta$ connects p' to ιx in X', and $L'(\iota\delta) = L(f\delta) = L(\delta)$. By the discussion above, there is an admissible path, α, in X' from ιx to v with $L'(\alpha) < \lambda$. Let $\gamma = (\iota\delta) \cup \alpha$. $\square$

It immediately follows that any two points of X' are connected by an admissible path. Given $v, w \in X'$, let $\rho'(v, w)$ be the infimum of $L'(\gamma)$ as γ varies over all admissible paths from v to w. Thus ρ' is a pseudometric on X'. In fact, we will see shortly that it is a path-metric.

We write $h'(v) = \rho'(p', v)$ for $v \in X'$. By Lemma 8.3, $h'(v) < R'$ for all $v \in X'$.

Given $v \in X'$, choose $x \in \Xi(v)$, and let U, V and $\phi : V \longrightarrow U$ be as constructed above.

A path α in U is admissible if and only if $f'\alpha$ is a continuous path in V. By definition, $L'(\alpha) = L(f'\alpha)$. We have observed that there is an admissible path, α, in X' from ιx to $f'v$ with $L'(\alpha) < \lambda$. It follows that $\rho'(\iota x, w) = \sigma(fx, f'w)$ for all $w \in U$. In particular, we see that $U = N(\iota x, \lambda)$; that is, the open λ-neighbourhood of ιx in (X', ρ').

Write $b = fx$, and let $\theta = (\lambda - \rho(\iota x, v))/2 = (\lambda - \sigma(b, f'v))/2$. Now $N(v, \theta) \subseteq U$ and $N(b, \theta) \subseteq V$. By the above discussion regarding admissible paths, we see that $f'|N(v, \theta)$ and $\phi|N(b, \theta)$ are inverse isometries from $N(v, \theta) \subseteq X'$ to $N(b, \theta) \subseteq M$. It follows that ρ' is indeed a metric on X'. Moreover, we have:

Lemma 8.4. *$f' : (X', \rho') \longrightarrow (M, \sigma)$ is a local isometry.*

In particular, f' is 1-lipschitz and locally injective. It follows that lifts of paths to X' are unique: if α, β are continuous paths in (X', ρ') based at the same point, with $f'\alpha = f'\beta$, then $\alpha = \beta$.

Since being admissible is a local property, we see that α is an admissible path in X' if and only if it is continuous with respect to the metric ρ'. Moreover, $L'(\alpha) = L(\alpha)$; that is, its rectifiable length in (X', ρ'). Directly from the definition of ρ' we therefore see:

Lemma 8.5. *ρ' is a path-metric on X'.*

Henceforth a "path" in X' will be assumed to be continuous, or equivalently, admissible.

Next, we show that (X', ρ') satisfies $\Lambda(R')$:

Lemma 8.6. *Let β be a path in M based at $f'(p') = fp$, with $L(\beta) < R'$. Then there is a (unique) path, α in X', based at p' with $f'\alpha = \beta$.*

Proof. We can write $\beta = \beta_1 \cup \beta_2$ with $L(\beta_1) < R$ and $L(\beta_2) < \lambda$. By $\Lambda(R)$ in X, we can lift β_1 to a path, γ, in X, based at p. Let $\alpha_1 = \iota\gamma$. Then $f'\alpha_1 = f'\iota\gamma = f\gamma = \beta_1$. Let $x \in X$ be the terminal point of γ_1. Then $v := \iota x$ is the terminal point of α_1, and $f'v = fx$ is the initial point of β_2. Certainly, $x \in \Xi(v)$. Let $U = N(v, \lambda) \subseteq X'$, $V = N(f'v, \lambda) \subseteq M$, and let $\phi : V \longrightarrow U$ be the inverse of $f'|U$ as constructed above. Since $L(\beta_2) < \lambda$, $\beta_2 \subseteq V$. Now $\alpha_2 := \phi\beta_2$ is a path based at v, with $f'\alpha_2 = \beta_2$. Let $\alpha = \alpha_1 \cup \alpha_2$. Then $f\alpha = \beta$ as required. □

Now $f = f' \circ \iota$, and f and f' are both local isometries (by hypothesis, and by Lemma 8.4, respectively). It follows that ι is a local isometry. Since ρ is a path-metric (by hypothesis), it follows that ι is 1-lipschitz. In particular, $\iota X \subseteq N(p', R) \subseteq X'$. In fact:

Lemma 8.7. $\iota X = N(p', R)$.

Proof. Let $v \in N(p', R)$. Let γ be a path in X' from p' to v with $L(\gamma) < R$. Let $\beta = f'\gamma$. Then $L(\beta) = L(\gamma) < R$. By $\Lambda(R)$ in X, we can lift γ to a path, α based at p in X. Now $f'(\iota\alpha) = f\gamma = \beta = f'\gamma$. Since f' is locally injective, we have $\iota\alpha = \gamma$. In particular, $\iota x = v$, where $x \in X$ is the terminal point of α. This shows that $v \in \iota X$. □

Since ρ' is a path-metric (by Lemma 8.5), we could have chosen γ in the above proof to satisfy $L(\gamma) < h'(v) + \eta$ for any given $\eta > 0$. Then $L(\alpha) < h'(v) + \eta$, so $h(x) < h'(v)$. Letting $\eta > 0$, we get $h(x) \leq h'(v)$. But ι is 1-lipschitz, and so $h(x) = h'(v)$. In other words, we have shown:

Lemma 8.8. *If $x \in X$, then $h'(\iota x) = h(x)$.*

We can now simplify notation.

We identify X with ιX in X' via ι. Thus, ι is just inclusion. We can then write $p' = p$ and $X = N(p, R) \subseteq X'$. By Lemma 8.8, we have $h'|X = h$, so we just write this as h. We still have two metrics, ρ and ρ' on X. Since the inclusion, $(X, \rho) \hookrightarrow (X', \rho')$ is 1-lipschitz, we have $\rho' \leq \rho$ on X. Indeed, ρ is the path-metric on X induced by the restriction of ρ' to X. In particular, the path, α, in X is continuous with respect to ρ if and only if it is continuous with respect to ρ', and its rectifiable length is the same measured in either. We can unambiguously write it as $L(\alpha)$. (We will eventually see that in fact $\rho = \rho'$.) We can now write $f = f' : X' \longrightarrow M$, so that $f|X$ is the original map we started out with.

We will need the following:

Lemma 8.9. *Let $x, y \in X$ with $\rho'(x, y) < 3\lambda$. Then $\rho'(x, y) = \rho(x, y)$.*

Proof. Let α be a path in X' from x to y with $L(\alpha) < 4\lambda$. Now α is admissible, so we can cover it with subpaths, β, such that if $w, w' \in \beta$, then there exist $z, z' \in X$, with $\rho'(z, w) < \lambda$, $\rho'(z', w') < \lambda$ and $\rho(z, z') < 3\lambda$. (Take $z \in \Xi(w)$ and $z' \in \Xi(w')$.) By the Heine-Borel Theorem, we can therefore find a sequence of points, $x = v_0, v_1, \ldots, v_n = y$ along α and $x_i \in X$ with $\rho'(x_i, v_i) < \lambda$ and

$\rho(x_{i-1}, x_i) < 3\lambda$ for all i. Moreover, we can set $x_0 = x$ and $x_n = y$. We claim that $\rho(x, x_i) < 5\lambda$ for all i. This certainly holds for $i = 0$, so we assume inductively that it holds for a given i.

Now $\rho(x, x_{i+1}) \leq \rho(x, x_i) + \rho(x_i, x_{i+1}) \leq 5\lambda + 3\lambda = 8\lambda < \epsilon/2$. Therefore, by Lemma 7.3, we have $\rho(x, x_{i+1}) = \sigma(fx, fx_{i+1}) \leq \sigma(fx, fv_{i+1}) + \sigma(fv_{i+1}, fx_{i+1}) \leq \rho'(x, v_{i+1}) + \rho'(v_{i+1}, x_{i+1}) < 4\lambda + \lambda = 5\lambda$. The claim now follows by induction.

In particular, we get $\rho(x, y) = \rho(x, x_n) < 5\lambda < \epsilon/2$. By Lemma 7.3 again, we get $\rho(x, y) = \sigma(fx, fy) \leq \rho'(x, y)$. Since $\rho' \leq \rho$, we get $\rho(x, y) = \rho'(x, y)$. □

Now, given $a, b, c \in X'$, write $I'_\eta(a, b)$ and $\mathrm{Med}'_\eta(a, b, c) = I'_\eta(a, b) \cap I'_\eta(b, c) \cap I'_\eta(c, a)$ for the sets, I_η and Med_η, defined with respect to the metric ρ'.

We claim that (X', ρ') satisfies $\nabla(p)$. In other words:

Lemma 8.10. *If $x_1, x_2 \in X'$ and $\eta > 0$, then* $\mathrm{Med}'_\eta(p, x_1, x_2) \neq \varnothing$.

Proof. By Lemma 6.2, it is enough to verify $\nabla(p, \lambda)$. In other words, we can assume that $\rho'(x_1, x_2) \leq \lambda$. Let $\zeta = \min(R' - h(x_1), R' - h(x_2), \eta/2, \lambda) > 0$. Since ρ is a path-metric on X' (Lemma 8.5) and $X = N(p, R)$ (Lemma 8.7) we can find $w_i \in X \cap I'_\zeta(p, x_i)$ with $\rho'(x_i, w_i) < \lambda$. Now $\rho'(w_1, w_2) < 3\lambda$, so by Lemma 8.9, $\rho(w_1, w_2) = \rho'(w_1, w_2) < 3\lambda$. By $\nabla(p)$ in X, there is some $y \in \mathrm{Med}_\zeta(p, w_1, w_2)$ (defined with respect to the metric ρ). Now,

$$\begin{aligned}\rho'(p, y) + \rho'(y, x_i) &\leq \rho'(p, y) + \rho'(y, w_i) + \rho'(w_i, x_i) \\ &\leq \rho(p, y) + \rho(y, w_i) + \rho'(w_i, x_i) \\ &\leq \rho(p, w_i) + \rho'(w_i, x_i) + \zeta \\ &= \rho'(p, w_i) + \rho'(w_i, x_i) + \zeta \\ &\leq \rho'(p, x_i) + 2\zeta.\end{aligned}$$

(The penultimate equality uses Lemma 8.8.) In other words, $y \in I'_{2\zeta}(p, x_i)$. Note that the ρ'-diameter of $\{x_1, x_2, y\}$ is at most $4\lambda + \zeta \leq 5\lambda < \epsilon/2$. (We can now forget about w_i.)

We now proceed as in the proof of Lemma 7.3. (Here ρ' replaces ρ, R' replaces R and ζ replaces η.) As before, we let $a_i = fx_i$, $b = fy$, and $c \in \mathrm{Med}(b, a_1, a_2)$. Let $\beta, \gamma_1, \gamma_2$ be geodesics in M from b to c, from c to a_1 and from c to a_2 respectively. Let $\tilde{\beta}, \tilde{\gamma}_1, \tilde{\gamma}_2$ be their lifts to X'. Let $z \in X'$ be the terminal point of $\tilde{\beta}$. Then $\tilde{\gamma}_i$ connects z to x_i.

Now,

$$\begin{aligned}\rho'(x_1, z) + \rho'(x_2, z) &\leq L(\tilde{\gamma}_1) + L(\tilde{\gamma}_2) = L(\gamma_1) + L(\gamma_2) \\ &= \sigma(a_1, c) + \sigma(a_2, c) = \sigma(a_1, a_2) \leq \rho'(x_1, x_2).\end{aligned}$$

Therefore $z \in I'(x_1, x_2)$.

Also,

$$\begin{aligned}\rho'(p,z)+\rho'(z,x_i) &= h(z)+L(\tilde{\gamma}_i)=h(z)+L(\gamma_1)\\ &\leq h(y)+\rho'(y,z)+L(\gamma_i)\leq h(y)+L(\beta)+L(\gamma_i)\\ &\leq h(y)+\sigma(b,a_i)\leq h(y)+\rho'(y,x_i)\\ &\leq \rho'(p,x_i)+2\zeta,\end{aligned}$$

the last step following by the earlier observation that $y \in I'_{2\eta}(p,x_i)$. Thus, $z \in I'_{2\zeta}(p,x_i)$.

Since $2\zeta \leq \eta$, we have shown that $z \in \mathrm{Med}'_\eta(p,x_1,x_2)$. This verifies $\nabla(p,\lambda)$. $\square$

Lemma 8.11. *The metrics ρ and ρ' on X are equal.*

Proof. By Lemma 8.7, $X = N(p,R)$. By Lemma 8.10, we can now apply Lemma 6.3 to (X',ρ'). This shows that the restriction of ρ' to X is an intrinsic path-metric on X. But we have already noted that ρ is the path-metric induced by ρ' restricted to X. Therefore $\rho=\rho'$. $\square$

We can now inductively apply this process to give the following:

Lemma 8.12. *Let (M,σ) be an ϵ-locally median path-metric space, and let $q \in M$. Then there is a path-metric space (Y,ρ), with basepoint $p \in X$, satisfying $\nabla(p)$, and a local isometry, $f : Y \longrightarrow M$, with $f(p)=q$ and such that every path in M based at q lifts to a path in Y based at p.*

Proof. Let $\lambda = \epsilon/25$. Let $X_1 = N(q,\lambda) \subseteq M$, let ρ_1 be metric σ restricted to X, let $p=q$, and let $f_1 : X_1 \longrightarrow M$ be the inclusion map. Clearly X_1 satisfies $\nabla(p,\lambda)$. Also any two points of X_1 are connected by a geodesic, so σ_1 is an intrinsic geodesic metric on X_1. Also, X_1 satisfies $\Phi(\lambda)$ and $\Lambda(\lambda)$.

We now proceed to construct (X_n,ρ_n) and $f_n : X_n \longrightarrow M$, satisfying the earlier conditions with $R=n\lambda$. For this, we inductively set $X_{n+1}=X'_n$ by the above construction. We have isometric embeddings $X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots$, such that $f_n|X_m = f_m$ whenever $m \leq n$. We now let $Y=\bigcup_{n=1}^\infty X_n$, so that $X_n = N(p,n\lambda)$ in Y. The maps f_n combine to give us the required map $f : Y \longrightarrow M$. $\square$

Let us now suppose, in addition, that M is simply connected. We can construct an inverse map $g : M \longrightarrow Y$ by the standard procedure of lifting paths. Given $a \in M$, let β be a path from q to a, let α be its lift to Y, and let $g(a)$ be the terminal point of α. Since M is simply connected, this is independent of the choice of β. Since f is a local isometry, so is g. Since ρ and σ are both path-metrics, f is an isometry. It follows that M satisfies $\nabla(q)$. Since this holds for all $q \in M$, we get:

Proposition 8.13. *For any $\epsilon > 0$, any ϵ-locally median simply connected path-metric space is almost modular.*

Finally, if M is also complete, Lemma 5.3 tells us that M is median.

This proves Theorem 1.1.

9. Locally median spaces

The hypothesis of our main result, Theorem 1.1 made use of a uniform constant $\epsilon > 0$ to define "ϵ-locally median". We suspect this is not necessary. To give some discussion of this, we introduce some terminology.

Let M be a metric space. We say that at triple, $(a, b, c) \in M$ is ***good*** if $\#\operatorname{Med}(a, b, c) = 1$. Otherwise it is ***bad***. A subset, $A \subseteq M$ is ***good*** if (a, b, c) is good for all $a, b, c \in A$. We say that M is ***locally median*** if every point has a good neighbourhood.

We conjecture that a complete simply connected locally median path-metric space is median.

One can make some general observations regarding this, and show it to be the case under various additional hypotheses. Indeed it would seem that any counterexample would have to have some bizarre pathologies.

First we note:

Proposition 9.1. *A complete connected locally median modular space is median.*

Proof. This essentially follows from the argument of Lemma 4.2. Suppose, for contradiction that M is not median. Then we can certainly find some bad quintuple of size $r > 0$, say. The proof of Lemma 4.2 gives us another bad quintuple of size at most $r/2$ within an r-neighbourhood of the original. We can now iterate this process to give a sequence of bad quintuples of size $r/2^n$, for $n \in \mathbb{N}$. By completeness, these must converge on some point of M, and therefore eventually lie in a good neighbourhood of that point, giving a contradiction. □

Here is another result:

Lemma 9.2. *Suppose that M is a complete simply connected path-metric space. Suppose that for every bounded set $A \subseteq M$, there is some $\epsilon > 0$ such that every triple in A of diameter at most ϵ is good. Then M is median.*

Proof. Let $q \in M$. Given $r > 0$, let $\epsilon(r) > 0$ be some such constant for $A = N(q, r)$. We can assume that $\epsilon(r)$ is non-increasing in r. We can now find a sequence $(R_n)_{n \in \mathbb{N}}$, with $R_0 = 0$, with $R_n \to \infty$, and such that $R_{n+1} \leq R_n + \epsilon(R_{n+1})/25$ for all $n \in \mathbb{N}$.

We can now construct an increasing sequence of spaces $X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots$ exactly as in Section 8, such that X_n satisfies $\nabla(p)$, $\Lambda(R_n)$ and $\Phi(R_n)$ for all n. (The extension from X_n to X_{n+1} only makes reference to the subset $N(q, R_{n+1}) \subseteq M$. In particular we just need the fact that all triples of diameter at most $\epsilon(R_{n+1})$ in this set are good.) We now let $Y = \bigcup_{n=1}^{\infty} X_n$, and complete the argument as before. □

This applies, when X is locally compact. It is well known that in a complete locally compact path-metric space, every closed bounded subset is compact.

Proposition 9.3. *A complete simply connected locally compact locally median path-metric space is median.*

Proof. From the above observation, it is easily seen that the hypotheses of Lemma 9.2 hold. □

(In fact, some of the argument can be simplified a little in this case, since any complete locally compact path-metric space is geodesic.)

One can also generalise to the case where M satisfies a certain locally simply-connected hypothesis:

(USC): For all $\epsilon > 0$, there is some $\delta = \delta(\epsilon) > 0$, such that any two paths of length less than δ connecting a pair of points, $a, b \in M$, are homotopic relative to a, b through paths of length at most ϵ.

Recall that Lemma 3.9 tells us that a complete median metric space satisfies (USC) taking $\delta = \epsilon$.

We aim to show:

Proposition 9.4. *Suppose that M is a complete simply connected locally median path-metric space satisfying (USC). Then M is median.*

To this end, suppose that M satisfies the hypotheses of Proposition 9.4, but is not median. We first note:

Lemma 9.5. *Let $t, u > 0$. Suppose (a, b, c) is a bad triple in M with $\operatorname{diam}\{a, b, c\} < \delta(t)/16$. Then there is a bad triple, (a', b', c'), in M, with $\operatorname{diam}\{a', b', c'\} < u$ and $\sigma(a, a') < t$.*

Proof. Suppose not. This means that all triples in $N(a, t)$ of diameter at most u are good. This allows us to carry out the construction of Section 8 inside $N(a, t)$, as follows.

Let $R = t/2$. Let $\delta = \delta(R)$. We can suppose $\delta < t$. Let $q \in N(a, \delta/4)$. Thus, $N(a, \delta/8) \subseteq N(q, \delta/2)$. We can now construct a map $f : X \longrightarrow M$, with basepoint $p \in X$, with $f(p) = q$, and satisfying $\nabla(p)$, $\Lambda(R)$ and $\Phi(R)$.

We now lift $N(q, \delta/2)$ to X. In other we construct a left inverse, $g : N(q, \delta/2) \longrightarrow X$, to f as follows. Given $d \in N(q, \delta/2)$, let β be a path from q to d with $L(\beta) < \delta$. Let α be its lift to X, based at p, and let $g(d)$ be the terminal point of α. This is well defined independently of the choice of β (since by the choice of δ, any two such paths are homotopic in M through paths of length less than R, and these all lift to X, cf. Lemma 7.2). It now follows that $f|N(p, \delta/4)$ is an isometry from $N(p, \delta/4)$ to $N(q, \delta/4)$. In particular, $N(q, \delta/4)$ satisfies $\nabla(q)$. Since this applies to any $q \in N(q, \delta/4)$, we see that any if (x, y, z) is any triple in $N(a, \delta/8)$ and $\eta > 0$, then $\operatorname{Med}_\eta(x, y, z) \neq \varnothing$.

Moreover, all triples in $N(a, \delta)$ are good. It follows that $N(a, \delta/16)$ is good. In particular, we get the contradiction that (a, b, c) is a good triple. □

Proof of Proposition 9.4. Suppose for contradiction that M satisfies the hypotheses but not the conclusion. By Theorem 1.1, M contains bad triples of arbitrarily small diameter. Let $\delta_n = \delta(1/2^n)$, as given by (USC). Let a_0, b_0, c_0 be a bad triple with $\operatorname{diam}\{a_0, b_0, c_0\} < \delta_0/16$. Given $n \in \mathbb{N}$, we construct a bad triple, (a_n, b_n, c_n) with $\operatorname{diam}\{a_n, b_n, c_n\} < \delta_n/16$ inductively as follows. By Lemma 9.5, there is a bad triple, $(a_{n+1}, b_{n+1}, c_{n+1})$ with $\operatorname{diam}\{a_{n+1}, b_{n+1}, c_{n+1}\} < \delta_{n+1}/16$, and with $\sigma(a_n, a_{n+1}) < 1/2^n$. Since M is complete, $a_n \to a$, for some $a \in M$. For all sufficiently large n, (a_n, b_n, c_n) lies in a good neighbourhood of a, hence is good, giving a contradiction. $\square$

A related result is given in [M2], where M is assumed to admit a reversible "conical geodesic bicombing": see Theorem 4.1 thereof. Here "conical" is a weak convexity hypothesis. Proposition 9.4 here would enable us to remove the "conical" hypothesis: any space with a geodesic bicombing would satisfy (USC). Of course, the resulting proof is much more involved.

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Mathematics Institute, University of Warwick, Coventry, CV4 7AL, Great Britain