Practice problems from the taster lecture: Counting primes

1. (a) Let a and b be natural numbers, and let $m = 4a + 1$, $n = 4b + 1$. Show that there exists $c \in \mathbb{N}$ such that $mn = 4c + 1$.

   (b) Let $a_1, \ldots, a_k$ be natural numbers, and for $1 \leq i \leq k$, let $m_i = 4a_i + 1$. Show that there exists $c \in \mathbb{N}$ such that $\prod_{i=1}^{k} m_i = 4c + 1$.

2. Show that there are infinitely many primes of the form $4a + 3$. (Hints: Use the idea from Euclid's proof that there are infnitely many primes, and Question 1 above.)

3. Show that there are infinitely many primes of the form $6a + 5$. (Hints: Use the ideas from Questions 1 and 2 above.)

   The final question below is challenging, and based on the bonus section from the lecture notes – please do read that before attempting! We often do this at Warwick – add an interesting section and/or question to the notes that is non-examinable, but is fun and challenging to try in your own time.

4. Let $\mathcal{P} := \{p_1, \ldots, p_t\}$ be a set of t primes, and let $N(\mathcal{P})$ be the set of all positive integers n for which all prime divisors of n lie in $\mathcal{P}$. Let X be any subset of $N(\mathcal{P})$ of size $t + 1$. Show that there exists a non-empty subset A of X such that $\prod_{a \in A} a$ is a perfect square. (Hint: Use the ideas from Section 2 of the taster lecture notes. You may use the fact that any set of size $t + 1$ has precisely 2^{t+1} subsets, including the empty set.)