Mean Value Theorem (MVT): For $f:[a,b] \to \mathbb{R}$, if f is continuous on $[a,b]$ and differentiable on (a,b), there is $c \in (a,b)$ such that

$$\underbrace{f'(c)}_{\text{Gradient of } f \text{ at } (c, f(c))} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{Gradient of chord between } (a,f(a)) \text{ and } (b,f(b))}$$

Rearranging, $f(b) = f(a) + (b-a)f'(c)$.

Use the mean value theorem on $f:[n, n+1] \to \mathbb{R}$ where $n \in \mathbb{N}$ and $f(x) = \ln x$ to show that for all $n \in \mathbb{N}$, $n \ln\left(1+\frac{1}{n}\right) < 1$.

We know that $\ln x$ is differentiable on $\mathbb{R}_{>0}$, so f is continuous on $[n, n+1]$ and differentiable on $(n, n+1)$.

By the mean value theorem, there is $c \in (n, n+1)$ such that

$$\begin{aligned} f(n+1) &= f(n) + ((n+1)-n)f'(c) \\ &= f(n) + \underbrace{f'(c)}_{\text{want to bound this using } n<c<n+1} \end{aligned}$$

Since $f'(c) = \frac{1}{c}$,

$$\begin{aligned} \ln(n+1) &= \ln(n) + \frac{1}{c} \\ \ln(n+1) - \ln(n) &= \frac{1}{c} \\ \ln\left(\frac{n+1}{n}\right) &= \frac{1}{c} \\ \ln\left(1+\frac{1}{n}\right) &= \frac{1}{c} \end{aligned}$$

Since $n < c < n+1$, $\frac{1}{n+1} < \frac{1}{c} < \frac{1}{n}$ so

$$\ln\left(1+\frac{1}{n}\right) < \frac{1}{n} \Rightarrow n\ln\left(1+\frac{1}{n}\right) < 1$$

Taylor's Theorem (n^{th} MVT): For $f:[a,b]\to\mathbb{R}$, if
- f is continuous on $[a,b]$
- $f^{(k)}$ exists and is continuous on $[a,b]$ for $k\in\{1,\dots,n-1\}$
- $f^{(n)}$ exists on (a,b)

then there is $c\in(a,b)$ such that

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\dots+\frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)$$

$$+\underbrace{\frac{(b-a)^n}{n!}f^{(n)}(c)}_{R_n}$$

R_n = Lagrange form of the remainder

$$=\left(\sum_{k=0}^{n-1}\frac{(b-a)^k}{k!}f^{(k)}(a)\right)+R_n$$

For $f:[a,b]\to\mathbb{R}$ where f satisfies the conditions above for all $n\in\mathbb{N}$, the Taylor series at $x=c$ for $c\in[a,b]$ is the series

$$\sum_{n=0}^{\infty}\frac{(x-c)^n}{n!}f^{(n)}(c)=f(c)+(x-c)f'(c)+\frac{(x-c)^2}{2!}f''(c)+\dots$$

The range of values of x where the series converges to $f(x)$ is the interval of convergence.

1. Find the Taylor series at $x=-1$ of $f(x)=\frac{1}{x^2}$.

2. Determine the interval of convergence.

1. Find the derivatives:

$f(x)=x^{-2}$	$f(-1)=1=1!$
$f'(x)=-2x^{-3}$	$f'(-1)=2=2!$
$f''(x)=6x^{-4}$	$f''(-1)=6=3!$
$f^{(3)}(x)=-24x^{-5}$	$f^{(3)}(-1)=24=4!$

$f^{(3)}(x) = -24x^{-5}$ $\qquad$ $f^{(3)}(-1) = 24 = 4!$

$\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

$f^{(n)}(x) = (-1)^n (n+1)!\, x^{-(n+2)}$ $\qquad$ $f^{(n)}(-1) = (n+1)!$

The Taylor series of f at $x=-1$ is

$$\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} f^{(n)}(-1) = \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}(n+1)!$$

$$= \sum_{n=0}^{\infty} (x+1)^n (n+1) = 1 + 2(x+1) + 3(x+1)^2 + 4(x+1)^3 + \dots$$

2. Let $a_n = (x+1)^n(n+1)$ so we can use the ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+1)^{n+1}}{(x+1)^n}\frac{(n+2)}{(n+1)}\right|$$

$$= \left|\frac{n+2}{n+1}(x+1)\right|$$

$$= \frac{n+2}{n+1}\underbrace{|x+1|}$$

$x \in \mathbb{R}$ so this is a constant

$$= \frac{1+\frac{2}{n}}{1+\frac{1}{n}}|x+1|$$

By the sum, quotient and product rules, $\left(\left|\frac{a_{n+1}}{a_n}\right|\right) \to |x+1|$.

By the ratio test, the series $\sum_{n=0}^{\infty} a_n$

- converges when $|x+1| < 1$ $\qquad$ $-2 < x < 0$
- diverges when $|x+1| > 1$ $\qquad$ $x < -2,\ x > 0$

When $|x+1| = 1$, $x = 0$ or $x = -2$.

If $x=0$, $\sum^{\infty} a_n = \sum^{\infty} (n+1)$ which diverges

If $x=0$, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (n+1)$ which diverges
If $x=-2$, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (-1)^n (n+1)$ which diverges

by null sequence test

Hence, the interval of convergence is $(-2, 0)$.