Types of proof:

- Direct proof
- Induction
- Contradiction
- Exhaustion

Theorem 2.4 $\sqrt{2}$ *is not a rational number.*

Proof Suppose that $\sqrt{2}$ is rational. In that case, it can be written in the form $\sqrt{2} = \frac{p}{q}$, where p and q are integers which have no factors in common – that is, $\frac{p}{q}$ is a proper fraction – and $q \neq 0$.
Squaring both sides:

$$2 = \frac{p^2}{q^2} \implies p^2 = 2q^2$$

Thus p^2 is an even integer, and by Example 2.2 this means that p also must be even. (If p were odd, then p^2 would be odd.)
So we can write $p = 2r$ where r is some integer. Therefore

$$2q^2 = (2r)^2 = 4r^2$$

and so $q^2 = 2r^2$.
Hence q is also even, for the same reason that p is.
This means that both p *and* q are even, which contradicts the hypothesis that $\frac{p}{q}$ was in its simplest form. So the only option open to us is to conclude that *it is not possible* to express $\sqrt{2}$ as a rational number. □

Prove $\sqrt[3]{2}$ is not rational.

Lemma: n^3 even if and only if n even

] n^2 even $\Leftrightarrow$ n even

Suppose n even. Let $n = 2k$ for $k \in \mathbb{Z}$.

$n^3 = (2k)^3 = 8k^3 = 2(\underbrace{4k^3}_{\in \mathbb{Z}})$ $\therefore n^3$ even.

Suppose n odd. Let $n = 2k+1$ for $k \in \mathbb{Z}$.

$n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1$
$= 2(\underbrace{4k^3 + 6k^2 + 3k}_{\in \mathbb{Z}}) + 1$ $\therefore n^3$ odd.

Theorem 2.4 $\sqrt{2}$ *is not a rational number.*

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Suppose $\sqrt[3]{2}$ is rational.

$\sqrt[3]{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$

and p and q have no common factors.

$2 = \frac{p^3}{q^3} \Rightarrow p^3 = 2q^3$

p^3 even $\xRightarrow{\text{(lemma)}}$ p even

Let $p = 2r$ for $r \in \mathbb{Z}$

$(2r)^3 = 2q^3 \Rightarrow 8r^3 = 2q^3 \Rightarrow q^3 = 4r^3$
$= 2(2r^3)$

q^3 even $\xRightarrow{\text{(lemma)}}$ q even

p and q have 2 as a common factor ↯

$\therefore$ $\sqrt[3]{2}$ is not rational factor 4

Hence, $\sqrt[3]{2}$ is not rational.

Definition 3.1 (Principle of mathematical induction) Suppose we have a variable proposition $P(n)$ which depends on some natural number n. Then if $P(1)$ is true, and if $P(k) \Rightarrow P(k+1)$ for some $k \in \mathbb{N}$, then $P(n)$ is true for all $n \in \mathbb{N}$.

1. Prove base case (usually $n=1$)
2. Inductive step
   Assume $n=k$ is true to prove $n=k+1$ is true

Prove 9^n-1 is divisible by 8 for all $n \in \mathbb{N}$.

Let $P(n)$ be the statement above.

1. Base case:

   $P(1)$: $9^1-1=8$ $\therefore$ $P(1)$ is true.

2. Inductive step:

   Suppose $P(k)$ is true for $k \in \mathbb{N}$.

   Let $9^k-1=8r$ for $r \in \mathbb{Z}$.

$$
\begin{aligned}
9^{k+1}-1 &= 9(9^k)-1 \\
&= 8(9^k)+9^k-1 \\
&= 8(9^k)+8r \\
&= 8(\underbrace{9^k+r}_{\in \mathbb{Z}}) \quad \therefore P(k+1) \text{ is true}
\end{aligned}
$$

3. Conclusion

   Since $P(1)$ is true and $P(k) \Rightarrow P(k+1)$ for $k \in \mathbb{N}$, $P(n)$ is true by induction for every $n \in \mathbb{N}$.