Assignment 1: Sets, Numbers and Logic

This assignment covers the first four sections of the module. In order to obtain full marks you must have a good understanding of the following topics:

Set theory Venn diagrams, set operations, formal proof of set identities.

Real numbers Rational and irrational numbers.

Proof and reasoning Proof by contradiction, proof by induction.

Functions Injective, surjective and bijective functions, composition, inverse functions.

You are encouraged to complete online quiz 1 before attempting this assignment.

The questions themselves are marked out of a total of 20. There are a further five marks for clarity of exposition. In particular, you should write clearly and concisely, but using full sentences where appropriate, and ensure that you explain the details of any necessary steps in your arguments. The assignment is therefore marked out of 25, and will contribute up to 5% towards your overall mark for this module.

The deadline for this assignment is: **2pm on Friday 1 November 2024 (week 5)**. You should submit your solutions via Tabula.

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1 Let A and B be arbitrary sets, and consider the following identity (the second minimisation law):

$$(A \cup B) \cap (A \cup B') = A$$

(a) Draw Venn diagrams to illustrate each side of this identity. **[2 marks]**

(b) Prove this identity, by showing that each side is a subset of the other. **[4 marks]**

2 Adapt the proof of the irrationality of $\sqrt{2}$ to show that $\sqrt[3]{5}$ is irrational:

(a) State and prove a suitable analogue of the lemma from the notes that an integer n is even if and only if n^2 is even. **[3 marks]**

(b) Use this lemma to prove by contradiction that $\sqrt[3]{5}$ is irrational. **[3 marks]**

3 Prove by induction that:

(a) $n! > 3^n$ for $n \geqslant 7$. **[2 marks]**

(b) The nth derivative of $f(x) = x^2e^x$ is $f^{(n)}(x) = (x^2 + 2nx + n(n-1))e^x$. **[2 marks]**

4 Suppose that A, B and C are sets, and that $f\colon A \to B$ and $g\colon B \to C$ are bijective functions.

(a) Prove that the composite function $g \circ f\colon A \to C$ is also a bijection. **[2 marks]**

(b) Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. **[2 marks]**

Example 2.2 The product of two even integers is even, the product of two odd integers is odd, and the product of an even and odd integer is even.

A consequence of this result is that the square of an even integer is even, and the square of an odd integer is odd.

Also the *converse* of this is true: given that a^2 is even, where $a \in \mathbb{Z}$, then a must be even, whereas if a^2 is an odd perfect square, then a must be odd.

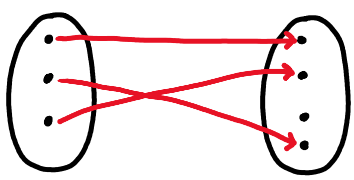
$n! = 1 \times 2 \times \ldots \times n$

$f, f', f'', \ldots, f^{(n)}$

Let $f: X \to Y$ be a function.

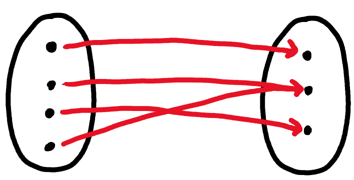
Injective: for all $y \in Y$, there exists at most one $x \in X$ such that $y = f(x)$.

for all $w, x \in X$, if $f(w) = f(x)$, then $w = x$.



X Y

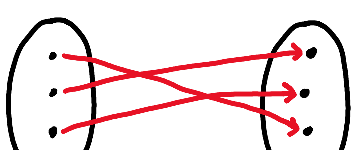
Surjective: for all $y \in Y$, there exists at least one $x \in X$ such that $y = f(x)$.



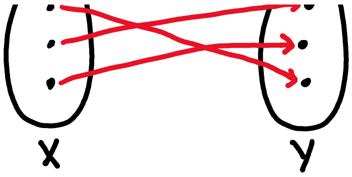
X Y

Bijective: Injective + Surjective

for all $y \in Y$, there exists exactly one $x \in X$ such that $y = f(x)$.



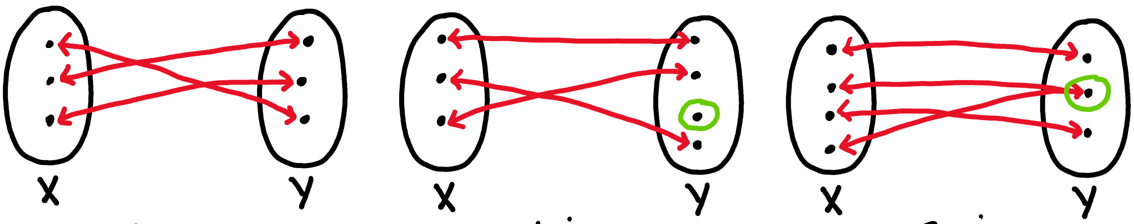
$id_X : X \to X$



$id_X : X \to X$
$id_X(x) = x$ for $x \in X$

Invertible: there exists $f^{-1}: Y \to X$ such that
$f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$

Invertible if and only if bijective



Bij. | Inj. Not surj. | Surj. Not inj.

1. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$
   Not inj, $f(2) = f(-2)$
   Not surj, no $x \in \mathbb{R}$, $f(x) = -1$

2. $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$, $f(x) = x^2$
   Surj.
   Not inj, $f(2) = f(-2)$

3. $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$
   Inj.
   Not surj, no $x \in \mathbb{R}\setminus\{0\}$, $f(x) = 0$

4. $f: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, $f(x) = \frac{1}{x}$
   Bij.

5. $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$, $f(x) = \frac{1}{x}$
   Bij.

6. $f: \mathbb{N} \to \mathbb{Z}$,
   $$f(x) = \begin{cases} \frac{x}{2} & x \text{ even} \\ -\left(\frac{x+1}{2}\right) & x \text{ odd} \end{cases}$$
   Inj.
   Not surj, no $x \in \mathbb{N}$, $f(x) = 0$

7. $f: \mathbb{N} \to \mathbb{Z}$,
   $$f(x) = \begin{cases} \frac{x}{2} & x \text{ even} \\ -\left(\frac{x-1}{2}\right) & x \text{ odd} \end{cases} \qquad f^{-1}(x) = \begin{cases} 2x & x > 0 \\ 1-2x & x \leq 0 \end{cases}$$
   Bij.

8. $f: \mathbb{N} \to \mathbb{N}$,
   $$f(x) = \begin{cases} 1 & x = 1 \\ x-1 & x > 1 \end{cases}$$
   Surj.
   Not inj, $f(1) = f(2)$

Prove if $f: X \to Y$ is invertible, then f is bijective.

Suppose f is invertible. Then, there is $f^{-1}: Y \to X$ such that $(f^{-1} \circ f)(x) = x$ for all $x \in X$ and $(f \circ f^{-1})(y) = y$ for all $y \in Y$.

Injective: Let $f(w) = f(x)$ for $w, x \in X$.

$\Rightarrow f^{-1}(f(w)) = f^{-1}(f(x))$

Injective: Let $f(w)=f(x)$ for $w,x\in X$.

$$\Rightarrow f^{-1}(f(w))=f^{-1}(f(x))$$

$$\Rightarrow (f^{-1}\circ f)(w)=(f^{-1}\circ f)(x)$$

$$\Rightarrow w=x$$ so f is injective.

Surjective: Let $y\in Y$. We want to find $x\in X$ such that $y=f(x)$.

$f^{-1}: Y\longrightarrow X$ so $f^{-1}(y)\in X$. Let $x=f^{-1}(y)$.

$f(x)=f(f^{-1}(y))=(f\circ f^{-1})(y)=y$ so f is surjective.