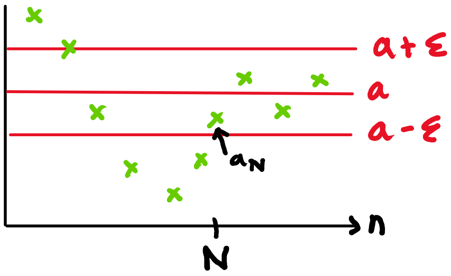
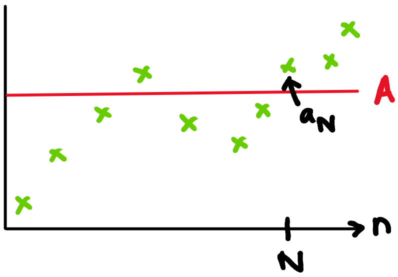
$(a_n) \xrightarrow[\text{converges to}]{\text{tends to}} a$ ← limit $\in \mathbb{R}$

for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n > N$.



$(a_n) \longrightarrow \infty$

for all $A > 0$, there is $N \in \mathbb{N}$ such that $a_n > A$ for all $n > N$.



Use the definition to show $a_n = \frac{n}{n+1}$ converges to 1.

We want to show that for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|a_n - 1| < \varepsilon$ for all $n > N$.

Let $\varepsilon > 0$. Take $N \in \mathbb{N}$ such that $N \geqslant \frac{1}{\varepsilon}$ (N exists since $\mathbb{N}$ are not bounded so for any $x \in \mathbb{R}$, there is $n \in \mathbb{N}$ such that $n \geqslant x$)

For all $n > N$,

$$
\begin{aligned}
|a_n - 1| &= \left| \frac{n}{n+1} - 1 \right| \\
&= \left| \frac{n - (n+1)}{n+1} \right| \\
&= \left| \frac{-1}{n+1} \right| \\
&= \frac{1}{n+1} \\
&< \frac{1}{n} && n > N \text{ so } \frac{1}{n} < \frac{1}{N} \\
&< \frac{1}{N} && \frac{1}{N} \leqslant \varepsilon \text{ so } \frac{1}{\varepsilon} \leqslant N \\
&\leqslant \varepsilon
\end{aligned}
$$

Since $|a_n - 1| < \varepsilon$ for all $n > N$, $(a_n) \longrightarrow 1$.

If $(a_n) \to a$ and $(b_n) \to b$, then

- $(a_n + b_n) \to a+b$ (sum rule)
- $(a_n b_n) \to ab$ (product rule)
- $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$ if $b \neq 0$ (quotient rule)

If $(a_n) \to L$ and $(c_n) \to L$ and there is $N \in \mathbb{N}$ such that $a_n \leq b_n \leq c_n$ for all $n > N$, then $(b_n) \to L$ (sandwich rule).

Allowed to assume $\left(\frac{1}{n}\right) \to 0$.

Find the limits of $a_n = \dfrac{2n^2+3n}{n^3+n^2}$ and $b_n = \dfrac{3n^2 + n\cos n}{2n(n-3)}$

$$a_n = \frac{\frac{2n^2+3n}{n^3}}{\frac{n^3+n^2}{n^3}} = \frac{\frac{2}{n}+\frac{3}{n^2}}{1+\frac{1}{n}}$$

Product rule: $\left(\frac{2}{n}\right) \to 0$, $\left(\frac{3}{n^2}\right) \to 0$

Sum rule: $\left(\frac{2}{n}+\frac{3}{n^2}\right) \to 0$, $\left(1+\frac{1}{n}\right) \to 1$ check nonzero for quotient rule

Quotient rule: $(a_n) \to 0$

$$b_n = \frac{3n^2+n\cos n}{2n^2-6n} = \frac{3+\frac{\cos n}{n}}{2-\frac{6}{n}}$$

Product rule: $\left(\frac{6}{n}\right) \to 0$

Sum rule: $\left(2-\frac{6}{n}\right) \to 2$

Since $-1 \leq \cos n \leq 1$, $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$ and $\left(\frac{1}{n}\right) \to 0$ and $\left(-\frac{1}{n}\right) \to 0$, $\left(\frac{\cos n}{n}\right) \to 0$ by the sandwich rule.

Sum rule: $\left(3+\frac{\cos n}{n}\right) \to 3$

Quotient rule: $(b_n) \to \frac{3}{2}$

$\sum_{k=1}^{\infty} a_k \overset{\text{converges to}}{=} S$ if $(S_n) \to S$ where $S_n = \sum_{k=1}^{n} a_k$ ← partial sums

Not convergent = divergent

Theorem (Sum Rule). *For series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge, then $\sum_{k=1}^{\infty}(a_k + b_k)$ converges.*

Theorem (Null Sequence Test). *The series $\sum_{k=1}^{\infty} a_k$ only converges if $(a_k) \to 0$.*

Theorem (Comparison Test). *For series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, if*

- $a_k, b_k \geq 0$ *for all* $k \in \mathbb{N}$
- $a_k \leq Mb_k$ *for all* $k \in \mathbb{N}$ *and* $M > 0$
- $\sum_{k=1}^{\infty} b_k$ *converges*

then $\sum_{k=1}^{\infty} a_k$ converges. Similarly, if

- $a_k, b_k \geq 0$ *for all* $k \in \mathbb{N}$
- $a_k \geq Mb_k$ *for all* $k \in \mathbb{N}$ *and* $M > 0$
- $\sum_{k=1}^{\infty} b_k$ *diverges*

then $\sum_{k=1}^{\infty} a_k$ diverges.

Theorem (Ratio Test). *For a series $\sum_{k=1}^{\infty} a_k$, if $(|\frac{a_{k+1}}{a_k}|) \to L$, then*

- *if* $L < 1$, $\sum_{k=1}^{\infty} a_k$ *converges*
- *if* $L > 1$, $\sum_{k=1}^{\infty} a_k$ *diverges*

Theorem (Alternating Series Test). *For a sequence (a_k), if*

- $a_k > 0$ *for all* $k \in \mathbb{N}$
- (a_k) *is decreasing*
- $(a_k) \to 0$

then $\sum_{k=1}^{\infty}(-1)^k a_k$ converges.

Which of these are convergent?

$\sum_{n=1}^{\infty} \frac{n^2+n^3}{n^5}$ Sum rule

$\sum_{n=1}^{\infty} \frac{1}{n!}$ Ratio test

$\sum_{n=1}^{\infty} (-1)^n$ Null sequence test

$\sum_{n=1}^{\infty} \frac{3^n}{n}$ Ratio test

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ Alternating series test

$\sum_{n=1}^{\infty} \frac{\sqrt{n}+2}{n^{3/2}+1}$ Comparison test

$$\frac{\sqrt{n}+2}{n^{3/2}+1} = \frac{1+\frac{2}{\sqrt{n}}}{n+\frac{1}{\sqrt{n}}} > \frac{1+\frac{2}{\sqrt{n}}}{2n} > \frac{1}{2n}$$