EC119 Mathematical Analysis 2024–2025 Assignment 1

Assignment 1: Sets, Numbers and Logic

This assignment covers the first four sections of the module. In order to obtain full marks you must have a good understanding of the following topics:

Set theory Venn diagrams, set operations, formal proof of set identities.

Real numbers Rational and irrational numbers.

Proof and reasoning Proof by contradiction, proof by induction.

Functions Injective, surjective and bijective functions, composition, inverse functions.

You are encouraged to complete online quiz 1 before attempting this assignment.

The questions themselves are marked out of a total of 20. There are a further five marks for clarity of exposition. In particular, you should write clearly and concisely, but using full sentences where appropriate, and ensure that you explain the details of any necessary steps in your arguments. The assignment is therefore marked out of 25, and will contribute up to 5% towards your overall mark for this module.

The deadline for this assignment is: **2pm on Friday 1 November 2024 (week 5)**. You should submit your solutions via Tabula.

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1 Let A and B be arbitrary sets, and consider the following identity (the second minimisation law):

$$(A \cup B) \cap (A \cup B') = A$$

(a) Draw Venn diagrams to illustrate each side of this identity. **[2 marks]**

(b) Prove this identity, by showing that each side is a subset of the other. **[4 marks]**

2 Adapt the proof of the irrationality of $\sqrt{2}$ to show that $\sqrt[3]{5}$ is irrational:

(a) State and prove a suitable analogue of the lemma from the notes that an integer n is even if and only if n^2 is even. **[3 marks]**

(b) Use this lemma to prove by contradiction that $\sqrt[3]{5}$ is irrational. **[3 marks]**

3 Prove by induction that:

(a) $n! > 3^n$ for $n \geqslant 7$. **[2 marks]**

(b) The nth derivative of $f(x) = x^2e^x$ is $f^{(n)}(x) = (x^2 + 2nx + n(n-1))e^x$. **[2 marks]**

4 Suppose that A, B and C are sets, and that $f\colon A \to B$ and $g\colon B \to C$ are bijective functions.

(a) Prove that the composite function $g \circ f\colon A \to C$ is also a bijection. **[2 marks]**

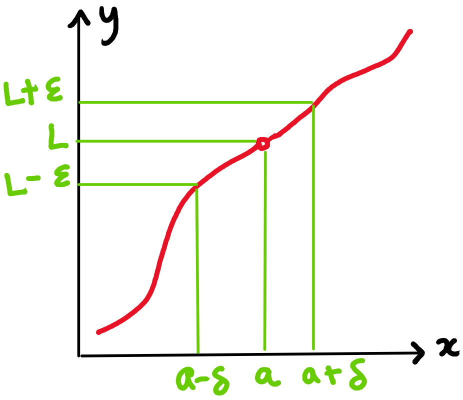
(b) Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. **[2 marks]**

General Feedback:

- So far, so good!
- Need to consider all cases in Q1(b)
- Some confusion on Q2(a)
- Some used $n=0$ for the base case in Q3(b)
- Some just showed $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ have the same domain and codomain in Q4(b)
- Please upload assignments as a single PDF! Clarity marks will be deducted next time.

$f: A \to B$ where $A, B \subseteq \mathbb{R}$: real-valued function.

For a real-valued function f, the limit of $f(x)$ as $x \to a$ is L if for all $\varepsilon > 0$, there is $\delta > 0$ such that whenever $0 < |x - a| < \delta$, we have $|f(x) - L| < \varepsilon$.



This is denoted $f(x) \to L$ as $x \to a$, or $\lim_{x \to a} f(x) = L$.

If f and g are real-valued functions with $f(x) \to L$ and $g(x) \to M$ as $x \to a$, then

- $f(x) + g(x) \to L + M$ as $x \to a$ (sum rule)
- $f(x)g(x) \to LM$ as $x \to a$ (product rule)
- $\dfrac{f(x)}{g(x)} \to \dfrac{L}{M}$ as $x \to a$ if $M \neq 0$ (quotient rule)

If f, g and h are real-valued functions with $f(x) \to L$ and $h(x) \to L$ as $x \to a$, and $f(x) \leq g(x) \leq h(x)$, then $g(x) \to L$ as $x \to a$ (sandwich rule)

If f, g and h are real-valued functions with $f(x) \to L$ and $h(x) \to L$ as $x \to a$, and $f(x) \leq g(x) \leq h(x)$, then $g(x) \to L$ as $x \to a$ (sandwich rule)

Find $\lim\limits_{x \to 1} \frac{x^2 - 3x + 2}{x - 1}$, $\lim\limits_{x \to 2} \frac{x - 2}{x^2 - 4}$ and $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right)$.

1. $$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x-2)}{x-1}$$

$$= \lim_{x \to 1} (x - 2)$$

$$= \left(\lim_{x \to 1} x\right) - 2 \qquad \text{(sum rule)}$$

$$= -1$$

2. $$\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{x-2}{(x+2)(x-2)}$$

$$= \lim_{x \to 2} \frac{1}{x+2}$$

$$= \frac{1}{\lim\limits_{x \to 2} (x+2)} \qquad \text{(quotient rule)}$$

$$= \frac{1}{\left(\lim\limits_{x \to 2} x\right) + 2} \qquad \text{(sum rule)}$$

$$= \frac{1}{4}$$

3. $$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

Since $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$, $-x \leq x \sin\left(\frac{1}{x}\right) \leq x$ and $\lim\limits_{x \to 0} x = \lim\limits_{x \to 0} (-x) = 0$, by the sandwich rule, $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

To find a limit as $x \to \infty$, let $t = \frac{1}{x}$ (so $x = \frac{1}{t}$) and find the limit as $t \to 0$.

Find $\lim_{x\to\infty} \frac{5x^3+2x^2-7}{x^4+3x}$

Let $t=\frac{1}{x}$, so $x=\frac{1}{t}$. Then,

$$\lim_{x\to\infty} \frac{5x^3+2x^2-7}{x^4+3x} = \lim_{t\to 0} \frac{5\left(\frac{1}{t}\right)^3+2\left(\frac{1}{t}\right)^2-7}{\left(\frac{1}{t}\right)^4+3\left(\frac{1}{t}\right)}$$

$$= \lim_{t\to 0} \frac{\frac{5+2t-7t^3}{t^3}}{\frac{1+3t^3}{t^4}}$$

$$= \lim_{t\to 0} \frac{5t+2t^2-7t^4}{1+3t^3}$$

$= 0$ by the quotient and sum rules.

For a real-valued function f, $f(x)$ is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$.

We say $f(x)$ is continuous on an interval if $f(x)$ is continuous at every point in the interval.

See notes for basic results on continuity.

Intermediate Value Theorem

For $f:[a,b]\longrightarrow \mathbb{R}$ where $f(a)=\alpha$ and $f(b)=\beta$, if $f(x)$ is continuous on $[a,b]$, then for any $\gamma\in(\alpha,\beta)$, there is $c\in(a,b)$ such that $\gamma=f(c)$.

Root of f : value of x such that $f(x)=0$

Show that $f(x)=3x^3+x^2-6x+1$ has three real roots in $[-2,2]$.

$f(-2)=-7$
$f(-1)=5$
$f(0)=1$
$f(1)=-1$
$f(2)=17$

Since f is a polynomial, f is continuous on $\mathbb{R}$.

Since f is continuous on $[-2,-1]$, by the IVT, for every $\gamma\in(-7,5)$, there is $c\in(-2,-1)$ such that $\gamma=f(c)$.

Take $\gamma=0$, then there is a root in $(-2,-1)$.

Use the same argument on $[0,1]$ and $[1,2]$ for the other two roots.