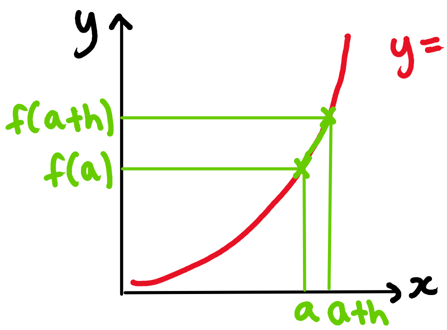
For a real valued function f and $a \in \mathbb{R}$, $f(x)$ is differentiable at $x=a$ if

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \quad \left(= \frac{f(a+h)-f(a)}{(a+h)-a} \right)$$

exists. Then, we say the limit is the derivative of $f(x)$ at $x=a$, denoted $f'(a)$.

We say f is differentiable on an interval if f is differentiable at every point in the interval.



$$\text{Gradient} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(a+h)-f(a)}{(a+h)-a}$$

If $f(x)$ is differentiable at $x=a$, then $f(x)$ is continuous at $x=a$.

Recap: $f(x)$ is continuous at $x=a$ if $\lim_{x \to a} f(x) = f(a)$.

Suppose $f(x)$ is differentiable at $x=a$. Then,

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \text{ exists.}$$

By the product rule,

$$\lim_{\underbrace{h \to 0}_{x \to a}} (\underbrace{f(a+h)}_{f(x)} - f(a)) = \left(\overbrace{\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}^{f'(a)} \right) \left(\overbrace{\lim_{h \to 0} h}^{0} \right) = 0$$

Let $x = a+h$ so $h = x-a$. Then, $h \to 0$ if and only if $(x-a) \to 0$, which happens if and only if $x \to a$ by the sum rule.

$$\lim_{x \to a} (f(x)-f(a)) = \lim_{h \to 0} (f(a+h)-f(a)) = 0$$

By the sum rule,

$$\lim f(x) = \overbrace{\lim (f(x)-f(a))}^{0} + \overbrace{\lim f(a)}^{f(a)} = f(a)$$

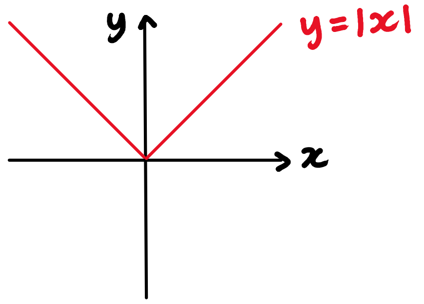
by the sum rule,

$$\lim_{x\to a} f(x) = \lim_{x\to a} \underbrace{(f(x)-f(a))}_{=0} + \lim_{x\to a} \underbrace{f(a)}_{=f(a)} = f(a)$$

so by definition, $f(x)$ is continuous at $x=a$.

Prove that $f:\mathbb{R}\to\mathbb{R}$ where $f(x)=|x|$ is continuous but not differentiable at $x=0$.

Recap: $\lim_{x\to a} f(x) = L$ if for all $\varepsilon>0$, there is $\delta>0$ such that whenever $0<|x-a|<\delta$, $|f(x)-L|<\varepsilon$.



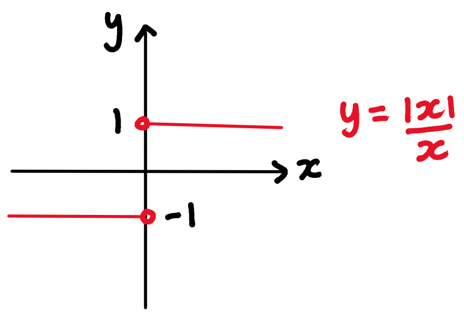
We want to show $\lim_{x\to 0} f(x) = f(0) = 0$

so we want to show that for all $\varepsilon>0$, there is $\delta>0$ such that whenever $0<|x|<\delta$, $|f(x)|<\varepsilon$.

Let $\varepsilon>0$. Take $\delta=\varepsilon$.

If $0<|x|<\delta$, then
$$\begin{aligned} |f(x)| &= ||x|| \\ &= |x| \\ &< \delta \\ &= \varepsilon \end{aligned}$$

so $\lim_{x\to 0} f(x) = 0$ so $f(x)$ is continuous at $x=0$.



$$\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

does not exist.

Theorem. *If f and g are real-valued functions that are differentiable at $x=a$, then*

- *$f(x)+g(x)$ is differentiable at $x=a$ with derivative $f'(a)+g'(a)$ (sum rule)*
- *$f(x)g(x)$ is differentiable at $x=a$ with derivative $f'(a)g(a)+f(a)g'(a)$ (product rule)*
- *$\frac{f(x)}{g(x)}$ is differentiable at $x=a$ with derivative $\frac{f'(a)g(a)-f(a)g'(a)}{(g(a))^2}$ if $g(a)\neq 0$ (quotient rule)*

Theorem (Chain Rule)**.** *If $f: A\to B$ is differentiable at $x=a$ and $g: B\to C$ is differentiable at $x=f(a)$, then $g\circ f$ is differentiable at $x=a$ with $(g\circ f)'(a) = g'(f(a))f'(a)$.*

Theorem (Leibniz' Theorem)**.** *If f and g are real-valued functions that are differentiable at $x=a$, then if $h(a)=f(a)g(a)$*

$$h^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n-k)}(a)$$

1 0th row binomial Find:

$k=0 \quad (k)$

$\uparrow$

binomial coefficient (nCk)

1	0th row
1 1	1st row
$\binom{2}{0} \rightarrow$ 1 2 1	2nd row
1 3 3 1	3rd row
1 4 6 4 1	⋮
1 5 10 10 5 1	
⋮	

$\binom{5}{1}$ $\uparrow$ (5) $\qquad$ $\binom{4}{3}$ $\uparrow$ (4)

Find:

1. $f'(x)$ where $f(x) = \dfrac{6x^2}{2-x}$
2. $f'(x)$ where $f(x) = x^x$
3. $f'(x)$ where $f(x) = \ln(xe^x + 1) - x^4$
4. $f^{(5)}(x)$ where $f(x) = x^2 \sin x$