Recap (IVT): For $f: [a,b] \to \mathbb{R}$, if f is continuous on $[a,b]$ with $\alpha = f(a)$ and $\beta = f(b)$, then for all $\gamma \in (\alpha, \beta)$, there is $c \in (a,b)$ such that $\gamma = f(c)$.

Taking $\gamma = 0$, then if $f(a)$ and $f(b)$ have different signs, then there is a root of f in (a,b).

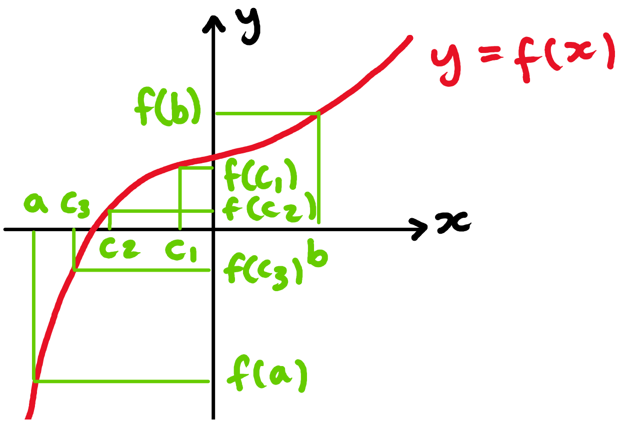
The Bisection Method

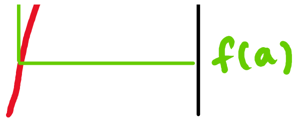
Let $c_1 = \frac{a+b}{2}$ (the midpoint of (a,b)) and consider $f(c_1)$

- If $f(c_1) = 0$, then c_1 is a root so we are done.
- If $f(a)$ and $f(c_1)$ have different signs, then by IVT, the root is in (a, c_1).
- Otherwise, $f(b)$ and $f(c_1)$ have different signs, so by IVT, the root is in (c_1, b).

Either way, we have a new smaller interval. Let c_2 be the midpoint of this, and repeat. We can stop either when we find the root, or when we know the root to a suitable degree of accuracy.

e.g. to find a root that is 2.4 to one decimal place, we must show it is in $(2.35, 2.45)$.





Let $f:\mathbb{R}\to\mathbb{R}$ where $f(x)=3x^7-5x^6+4x^2-3$.

1. Find $[a,b]$ where f is continuous and $f(a)$ and $f(b)$ have different signs.

2. Prove that f has a root in (a,b).

3. Use the bisection method to find this root to one decimal place.

1. $f(0)=-3$
   $f(1)=-1$
   $f(2)=77$ so $f(1)$ and $f(2)$ have different signs.

   Since f is a polynomial, f is continuous on $\mathbb{R}$ so f is continuous on $[1,2]$.

2. (Use IVT)

3. Let $c_1=\dfrac{1+2}{2}=1\cdot5$ so $f(c_1)=0.305$ (3dp)

   Since $f(1)$ and $f(c_1)$ have different signs, the root is in $(1,1\cdot5)$.

   Let $c_2=\dfrac{1+1\cdot5}{2}=1\cdot25$ so $f(c_2)=-1\cdot518$ (3dp)

   Since $f(c_1)$ and $f(c_2)$ have different signs, the root is in $(1\cdot25,1\cdot5)$.

   $\vdots$

   The root is in $(1\cdot46875, 1\cdot5)$ so to one decimal place, the root is $1\cdot5$.

The Newton Raphson Method

do not need to

The Newton-Raphson Method

do not need to prove on homework

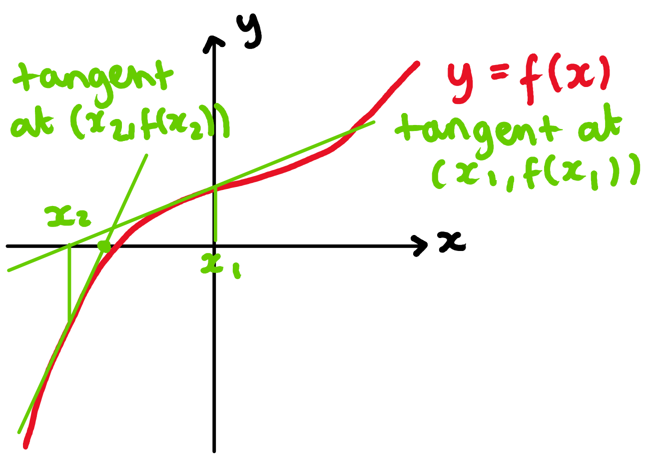
(same setup as above except f is differentiable on (a,b))

Choose $x_1 \in (a,b)$ and use the recurrence relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

to get a sequence which (hopefully) converges to a root.

Geometrically, $(x_{n+1}, 0)$ is the point where the tangent to $y=f(x)$ at $(x_n, f(x_n))$ crosses the x-axis.



Let $f: \mathbb{R}_{>0} \to \mathbb{R}$ where $f(x) = \ln(xe^x + 1) - x^4$.
Use the Newton-Raphson method to find the root of f in $[1,2]$.

$$f'(x) = \frac{(x+1)e^x}{xe^x+1} - 4x^3$$

Let $x_1 = 1.5$. Then,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.2494\ldots$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.1298\ldots$$

$$x_4 = 1.1005\ldots$$

$$x_5 = 1.0988\ldots$$

$$x_6 = 1.0988\ldots$$

Hence to three decimal places, the root is 1.099.