The Mean Value Theorem and Taylor's Theorem

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Week 10

Class Content

Theorem (Mean Value Theorem). *For a function $f : [a, b] \to \mathbb{R}$, if f is continuous on $[a, b]$ and differentiable on (a, b), then there is $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(or, rearranging, $f(b) = f(a) + (b - a)f'(c)$).

Question 1

Use the mean value theorem on the function $f : [n, n+1] \to \mathbb{R}$ where $f(x) = \ln(x)$ to show that for all $n \in \mathbb{N}$, $n \ln(1 + \frac{1}{n}) < 1$.

Theorem (Taylor's Theorem, or the nth Mean Value Theorem). *For $f : [a, b] \to \mathbb{R}$, if f is continuous on $[a, b]$, $f^{(k)}$ exists and is continuous on $[a, b]$ for all $k \in \{1, \dots, n-1\}$ and $f^{(n)}$ exists on (a, b), then there is $c \in (a, b)$ such that*

$$\begin{aligned} f(b) &= f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \underbrace{\frac{(b-a)^n}{n!}f^{(n)}(c)}_{R_n} \\ &= \left(\sum_{k=0}^{n-1} \frac{(b-a)^k}{k!} f^{(k)}(a)\right) + R_n \end{aligned}$$

where $R_n = \frac{(b-a)^n}{n!}f^{(n)}(c)$ is known as the Lagrange form of the remainder.

Definition. *For $f : [a, b] \to \mathbb{R}$ where f is continuous on $[a, b]$ and $f^{(n)}$ exists and is continuous on $[a, b]$ for all $n \in \mathbb{N}$, the* **Taylor series** *of f about $x = c$ for $c \in [a, b]$ is the series*

$$\sum_{n=0}^{\infty} \frac{(x-c)^n}{n!} f^{(n)}(c)$$

The range of values for which the series converges, i.e.

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-c)^n}{n!} f^{(n)}(c)$$

is the **interval of convergence**.

Question 2

1. Find the Taylor series about $x = -1$ of $\frac{1}{x^2}$.
2. Determine the interval of convergence for the Taylor series.

1 Additional Questions

1. Use the mean value theorem to show that for all $x \in \mathbb{R}_{>0}$, $1 + 2x < e^{2x} < (1 - 2x)^{-1}$.

2. Use the mean value theorem to show that for all $x, y \in (\frac{\pi}{4}, \frac{\pi}{3})$ with $x \leq y$, $\cos^2 y - \cos^2 x \leq \frac{3(x-y)}{4}$.

3. Let $f(x) = \sqrt{x}$.

   (a) Find the Taylor series of $f(x)$ about $x = 1$ up to and including the term in $(x - 1)^4$.

   (b) Use this to approximate the value of $\sqrt{1.5}$ to three decimal places.

4. Use the Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

   about $x = 0$ to find:

   (a) The Taylor series of e^{x^2-1} about $x = 0$.

   (b) The Taylor series of e^x about $x = -1$.

   (c) The Taylor series of $e^{\sin x}$ about $x = 0$ up to and including the term in x^4.