The Mean Value Theorem and Taylor's Theorem

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Week 10

Class Content

Theorem (Mean Value Theorem)**.** *For a function $f : [a, b] \to \mathbb{R}$, if f is continuous on $[a, b]$ and differentiable on (a, b), then there is $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(or, rearranging, $f(b) = f(a) + (b - a)f'(c)$).

Question 1

Use the mean value theorem on the function $f : [n, n+1] \to \mathbb{R}$ where $f(x) = \ln(x)$ to show that for all $n \in \mathbb{N}$, $n \ln(1 + \frac{1}{n}) < 1$.

Solution Since f is differentiable on $\mathbb{R}_{>0}$, f is continuous on $[n, n+1]$ and differentiable on $(n, n+1)$. By the mean value theorem, there is $c \in (n, n+1)$ such that $f(n+1) = f(n) + f'(c)$. Since $f'(c) = \frac{1}{c}$, this implies that

$$\begin{aligned} \ln(n+1) &= \ln(n) + \frac{1}{c} \\ \implies \ln(n+1) - \ln(n) &= \frac{1}{c} \\ \implies \ln\left(1 + \frac{1}{n}\right) &= \frac{1}{c} \end{aligned}$$

Since $n < c < n+1$, $\frac{1}{n+1} < \frac{1}{c} < \frac{1}{n}$, so

$$\begin{aligned} \ln\left(1 + \frac{1}{n}\right) &< \frac{1}{n} \\ \implies n \ln\left(1 + \frac{1}{n}\right) &< 1 \end{aligned}$$

Theorem (Taylor's Theorem, or the nth Mean Value Theorem)**.** *For $f : [a, b] \to \mathbb{R}$, if f is continuous on $[a, b]$, $f^{(k)}$ exists and is continuous on $[a, b]$ for all $k \in \{1, \ldots, n-1\}$ and $f^{(n)}$ exists on (a, b), then there is $c \in (a, b)$ such that*

$$\begin{aligned} f(b) &= f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \underbrace{\frac{(b-a)^n}{n!}f^{(n)}(c)}_{R_n} \\ &= \left(\sum_{k=0}^{n-1} \frac{(b-a)^k}{k!} f^{(k)}(a)\right) + R_n \end{aligned}$$

where $R_n = \frac{(b-a)^n}{n!} f^{(n)}(c)$ is known as the Lagrange form of the remainder.

Definition. *For $f : [a, b] \to \mathbb{R}$ where f is continuous on $[a, b]$ and $f^{(n)}$ exists and is continuous on $[a, b]$ for all $n \in \mathbb{N}$, the* **Taylor series** *of f about $x = c$ for $c \in [a, b]$ is the series*

$$\sum_{n=0}^{\infty} \frac{(x-c)^n}{n!} f^{(n)}(c)$$

The range of values for which the series converges, i.e.

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-c)^n}{n!} f^{(n)}(c)$$

is the **interval of convergence**.

Question 2

1. Find the Taylor series about $x = -1$ of $\frac{1}{x^2}$.
2. Determine the interval of convergence for the Taylor series.

Solution

1. Finding the derivatives,

$$\begin{aligned}
f(x) &= x^{-2} & f(-1) &= 1 \\
f'(x) &= -2x^{-3} & f'(-1) &= 2 \\
f''(x) &= 6x^{-4} & f''(-1) &= 6 \\
&\vdots & &\vdots \\
f^{(n)}(x) &= (-2)(-3)\dots(-(n+1))x^{-(n+2)} & f^{(n)}(-1) &= (n+1)! \\
&= (-1)^{n-2}(n+1)!x^{-(n+2)} \\
&= (-1)^n(n+1)!x^{-(n+2)}
\end{aligned}$$

By Taylor's theorem,

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} f^{(n)}(-1) \\
&= \sum_{n=0}^{\infty} \frac{(x+1)^n(n+1)!}{n!} \\
&= \sum_{n=0}^{\infty} (n+1)(x+1)^n
\end{aligned}$$

2. Let $a_n = (n+1)(x+1)^n$, so

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+2)(x+1)^{n+1}}{(n+1)(x+1)^n}\right| \\
&= \frac{n+2}{n+1}|x+1| \\
&= \frac{1+\frac{2}{n}}{1+\frac{1}{n}}|x+1|
\end{aligned}$$

By the sum, product and quotient rules, $(|\frac{a_{n+1}}{a_n}|) \to |x+1|$. By the ratio test, this implies that the series converges if $|x+1| < 1$ and diverges if $|x+1| > 1$. When $|x+1| = 1$, either $x = 0$ or $x = -2$. If $x = 0$, then

$$\sum_{n=0}^{\infty} (n+1)(x+1)^n = \sum_{n=0}^{\infty} (n+1)$$

diverges, and if $x = -2$ then

$$\sum_{n=0}^{\infty}(n+1)(x+1)^n = \sum_{n=0}^{\infty}(-1)^n(n+1)$$

diverges. Hence, the Taylor series converges when $|x+1| < 1$, which implies that $-1 < x+1 < 1$, so the interval of convergence is $(-2, 0)$.

1 Additional Questions

1. Use the mean value theorem to show that for all $x \in \mathbb{R}_{>0}$, $1 + 2x < e^{2x} < (1-2x)^{-1}$.

Solution Let $x \in \mathbb{R}_{>0}$ and define $f : [0, x] \to \mathbb{R}$ where $f(y) = e^{2y}$. Since f is differentiable on $\mathbb{R}$, f is continuous on $[0, x]$ and differentiable on $(0, x)$. By the mean value theorem, there is $c \in (0, x)$ such that $f(x) = f(0) + xf'(c)$, so $e^{2x} = 1 + 2xe^{2c}$. Since $0 < c < x$, $2e^c > 2e^0 = 2$ and so $e^{2x} > 1 + 2x$.

Let $g : [0, x] \to \mathbb{R}$ where $g(y) = e^{-2y}$. Since f is differentiable on $\mathbb{R}$, f is continuous on $[0, x]$ and differentiable on $(0, x)$. By the mean value theorem, there is $c \in (0, x)$ such that $f(x) = f(0) + xf'(c)$, so $e^{-2x} = 1 - 2xe^{2c}$. Since $0 < c < x$, $-2e^{-2c} > -2e^0 = -2$ and so $e^{-2x} > 1 - 2x$.

Combining these inequalities, since $e^{-2x} > 1-2x$ if and only if $e^{2x} < (1-2x)^{-1}$, this implies that $1 + 2x < e^{2x} < (1-2x)^{-1}$.

2. Use the mean value theorem to show that for all $x, y \in (\frac{\pi}{4}, \frac{\pi}{3})$ with $x \leq y$, $\cos^2 y - \cos^2 x \leq \frac{3(x-y)}{4}$.

Solution Let $x, y \in (\frac{\pi}{4}, \frac{\pi}{3})$ with $x \leq y$ and define $f : [x, y] \to \mathbb{R}$ where $f(z) = \cos^2 z$. Since f is differentiable on $\mathbb{R}$, f is continuous on $[x, y]$ and differentiable on (x, y).

By the mean value theorem, there is $c \in (x, y)$ such that $f(y) = f(x) + (y-x)f'(c)$. Since $f'(c) = -\sin 2c$, this implies that $\cos^2 y = \cos^2 x - (y-x)\sin 2c$ and hence $\cos^2 y - \cos^2 x = (x-y)\sin 2c$.

Since $\frac{\pi}{4} < c < \frac{\pi}{3}$, $\frac{\sqrt{3}}{2} < \sin 2c < 1$. Since $\frac{3}{4} < \frac{\sqrt{3}}{2}$, $\frac{3}{4} < \sin 2c$. Since $x - y \leq 0$, $(x-y)\sin 2c \leq \frac{3(x-y)}{4}$. Hence, $\cos^2 y - \cos^2 x \leq \frac{3(x-y)}{4}$.

3. Let $f(x) = \sqrt{x}$.

(a) Find the Taylor series of $f(x)$ about $x = 1$ up to and including the term in $(x-1)^4$.

(b) Use this to approximate the value of $\sqrt{1.5}$ to three decimal places.

Solution

(a) Finding the derivatives,

$$f(x) = x^{\frac{1}{2}} \qquad f(1) = 1$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \qquad f'(1) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} \qquad f''(1) = -\frac{1}{4}$$

$$f^{(3)}(x) = \frac{3}{8}x^{-\frac{5}{2}} \qquad f^{(3)}(1) = \frac{3}{8}$$

$$f^{(4)}(x) = -\frac{15}{16}x^{-\frac{7}{2}} \qquad f^{(4)}(1) = -\frac{15}{16}$$

By Taylor's theorem,

$$\sqrt{x} = 1 + \frac{x-1}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{16} - \frac{5(x-1)^4}{128} + R_5$$

(b) Using the Taylor series,

$$\sqrt{1.5} \approx 1 + \frac{1}{4} - \frac{1}{32} + \frac{1}{128} - \frac{5}{2048} = 1.22412109375$$

so to three decimal places, $\sqrt{1.5} \approx 1.224$.

4. Use the Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

about $x = 0$ to find:

(a) The Taylor series of e^{x^2-1} about $x = 0$.
(b) The Taylor series of e^x about $x = -1$.
(c) The Taylor series of $e^{\sin x}$ about $x = 0$ up to and including the term in x^4.

Solution

(a) Since $e^{x^2-1} = \frac{e^{x^2}}{e}$,

$$e^{x^2-1} = \frac{1}{e}\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{e(n!)}$$

(b) Since $e^x = \frac{e^{x+1}}{e}$,

$$e^x = \frac{1}{e}\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} = \sum_{n=0}^{\infty} \frac{(x+1)^n}{e(n!)}$$

(c) By expanding the Taylor series,

$$\begin{aligned}
\sin x &= x - \frac{x^3}{6} + \dots \\
e^{\sin x} &= \sum_{n=0}^{\infty} \frac{\sin^n x}{n!} \\
&= 1 + \sin x + \frac{\sin^2 x}{2} + \frac{\sin^3 x}{6} + \frac{\sin^4 x}{24} + \dots \\
&= 1 + \left(x - \frac{x^3}{6}\right) + \frac{1}{2}\left(x - \frac{x^3}{6}\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{6}\right)^3 + \frac{1}{24}\left(x - \frac{x^3}{6}\right)^4 + \dots \\
&= 1 + \left(x - \frac{x^3}{6}\right) + \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \dots\right) + \frac{1}{6}\left(x^3 + \dots\right) + \frac{1}{24}\left(x^4 + \dots\right) \\
&= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \dots
\end{aligned}$$

Hence, up to the term in x^4, the Taylor series of $e^{\sin x}$ is $1 + x + \frac{x^2}{2} - \frac{x^4}{8}$.