Additional Questions

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Week 11

L'Hôpital's Rule

1. Let $f(x) = 5$ and $g(x) = (x+2)^2(x^2+4x+1)$.
   (a) Find the limit as $x \to -2$ of $\frac{f(x)}{g(x)}$.
   (b) Find the limit as $x \to -2$ of $\frac{f'(x)}{g'(x)}$.

   This shows that L'Hôpital's rule can only be applied for the specific indeterminate forms.

2. Use L'Hôpital's rule to evaluate the following limits:
   (a) $\lim_{x\to -4} \frac{\sin(\pi x)}{x^2-16}$
   (b) $\lim_{x\to\infty} \frac{\ln(3x)}{x^2}$
   (c) $\lim_{x\to 0} \frac{\sin(2x)+7x^2-2x}{x^2(x-1)^2}$

3. Use L'Hôpital's rule to evaluate the limit $\lim_{x\to\infty} x\ln(1+\frac{3}{x})$.

4. Use L'Hôpital's rule to evaluate the limit $\lim_{x\to\infty}(e^x+x)^{\frac{1}{x}}$.

Improper Integrals

1. Determine if each of the integrals
   (a) $\int_0^\infty (1+2x)e^{-x}\,dx$
   (b) $\int_{-\infty}^0 (1+2x)e^{-x}\,dx$

   converges or diverges, and find its value if it converges.

2. Find the area of the region between the curve $y = \frac{7}{x^2}$ and the x-axis, bounded by $x = 1$ on the left.

3. Find the area of the region between the curve $y = -\frac{1}{\sqrt{3-x}}$ and the x-axis, bounded by $x = 0$ and $x = 3$.

Differential Equations

1. Use an integrating factor to find the general solution to the first order differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{t} + te^{-t} = 0$ for $t > 0$.

2. A company opened a bank account with £100,000 at the start of 2010, which accrues 5% compound interest continuously. They withdraw £4,000 each year.

   (a) Construct a first order differential equation describing the rate of change of the account balance.

   (b) Find the general solution describing the account balance in terms of the time in years.

   (c) How much was in the bank account at the start of 2020?

   (d) In what year will the account balance reach £150,000?

3. Show that $x^2 - x^{-1}$ is a solution to the second order differential equation $x^2 \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2y$.

4. Suppose that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = f(x)$ for some function $f : \mathbb{R} \to \mathbb{R}$, and $\sin(x) + x^2$ is a solution to the differential equation. Find an expression for $f(x)$.

5. Consider the second order differential equation $ay''(x) + by'(x) + cy(x) = 0$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Show that if e^{mx} and e^{nx} are two solutions to the differential equation, then $e^{mx} + e^{nx}$ is a solution to the differential equation.